to be found in a natural source. At^{215} and At^{216} may also exist but confirmatory evidence would be desirable.† Recently, Peppard *et al.*[19] have reported the occurrence of minute amounts of the $4n+1$ series in uranium and thorium ores. These experiments imply the existence of the 0.02-second At^{217} in these ores, since it is a member of the $4n+1$ decay chain. The results reported in the present paper add another isotope of astatine to this list and supply the first case in which the half-life of the astatine is sufficiently long that a chemical identification (as, for example, by extraction into tributyl phosphate or volatilization at low temperature) of the activity was possible.

It is worthy of note that other authors have published predictions similar to those discussed above on the alpha-branching of AcK and that these predictions are in substantial agreement with the experimental results reported here. Vigneron[20] predicted an alpha-particle energy of 5.45±0.15 Mev for AcK and an α/β^- branching ratio in the range of 1/300 to 1/2700. He also estimated an alpha-energy of 6.20±0.15 Mev for At^{219} and an alpha half-life of 15 seconds to 15 minutes. Karlik[21] predicted an alpha-particle energy of 5.55 Mev and an α/β^- branching ratio of 2×10^{-3} for AcK. Feather[22] predicted a branching ratio of a few per thousand. Jentschke[23] predicted an alpha branching ratio of 4×10^{-5} to 6×10^{-4} for AcK and indicated α/β^- branching at At^{219} and beta-emission at Bi^{215}.

We wish to thank Dr. F. Hagemann of the Argonne National Laboratory for making the Ac^{227} source available to us and Dr. J. Hollander of this laboratory for helpful discussions on this problem.

† *Note added in proof:* P. Avignon [J. phys. et radium **11**, 521 (1950)] has confirmed the work of Karlik and Bernert on the alpha-particle group assigned to At^{215} using a considerably larger source. Avignon's revised alpha-particle energy of 8.04 Mev is closer to the 8.00-Mev energy assigned by Meinke, Ghiorso, and Seaborg [Phys. Rev. **81**, 782 (1951)] to the At^{215} occurring in the decay chain of Pa^{227} and perhaps can be identified with it. However, this cannot be regarded as established.

[19] Peppard, Mason, Gray, and Mech, J. Am. Chem. Soc. **74**, 6081 (1952).

[20] L. Vigneron, Compt. rend. **225**, 1067 (1947).

[21] B. Karlik, Acta Phys. Austria **2**, 182 (1948).

[22] N. Feather, Repts. Prog. Phys. **11**, 19 (1948).

[23] W. Jentschke, Phys. Rev. **77**, 99 (1950).

PHYSICAL REVIEW VOLUME 90, NUMBER 2 APRIL 15, 1953

A Generalized Method of Field Quantization

H. S. Green
University of Adelaide, Adelaide, South Australia

(Received October 10, 1952)

A method of field quantization is investigated which is more general than the usual methods of quantization in accordance with Bose or Fermi statistics, though these are included in the scheme. The commutation properties and matrix representations of the quantized field amplitudes are determined, and the energy levels of the field are derived in the usual way. It is shown that spin-half fields can be quantized in such a way that an arbitrary finite number of particles can exist in each eigenstate. With the generalized statistics, the interchange of two particles of the same kind may or may not be physically significant, according to the type of interaction by means of which they are created or annihilated. Physical consequences of the assumption that there are particles which obey the generalized statistics are briefly examined.

1. INTRODUCTION

It is commonly acknowledged that quantized field theories, even in their recently developed state, involve mathematical inconsistencies (renormalization and regularization procedures) which make it unlikely that they have reached their final formulation. It is nevertheless difficult to see how they can be modified without infringing a set of rules which are dictated by the requirements of physical verisimilitude. For this reason, any possible means of relaxation of the present rigid structure of field theory deserves to be fully explored.

One procedure of field theory which has long remained unchallenged is the method of quantization, in accordance with either Bose or Fermi statistics. This has been authoritatively described by Pauli.[1] One modification of this formalism which has been seriously considered is Dirac's introduction of the indefinite metric[2]; the only application of this device, however, which lends itself to a consistent physical interpretation, is Gupta's treatment of the longitudinal electromagnetic field.[3]

In this paper, a generalization of the existing methods of field quantization is investigated, which has nothing to do with the metric, but involves a departure from Bose and Fermi statistics which, however, remain as special and indeed the simplest examples. To some physical particles, the application of ordinary quantum statistics seems unquestionable, for example, of Fermi statistics to the electron. To others, such as the proton, there is still room for doubt, and to the mesons no

[1] W. Pauli, Revs. Modern Phys. **13**, 203 (1941).

[2] P. A. M. Dirac, Proc. Roy. Soc. (London) **A180**, 1 (1942); see also W. Pauli, Revs. Modern Phys. **15**, 175 (1943).

[3] S. N. Gupta, Proc. Phys. Soc. (London) **A63**, 681 (1950); see also K. Bleuler, Helv. Phys. Acta **23**, 567 (1950).

existing theory seems to be properly applicable. It is, of course, quite possible that all physical particles obey the ordinary quantum statistics, and that the source of our modern difficulties should be sought in other directions. However, the existence of schemes of quantization more general than those normally employed is an interesting fact whose possible physical significance cannot be ignored.

In the interaction representation, any field $\Psi(x)$ can be expanded in the form

$$\Psi(x)=\sum_k\{b_{k+}\varphi_{k+}(x)+b_{k-}^*\varphi_{k-}(x)\}, \qquad (1)$$

where $\varphi_{k+}(x)$ and $\varphi_{k-}(x)$ comprise a complete set of *ortho*-normal functions of which the former contains only negative, and the latter only positive frequencies. One may take, for example,

$$\varphi_{k\pm}=V^{-\frac{1}{2}}\exp(\mp ip_k{}^\alpha x_\alpha)\Psi_{k\pm}, \quad (p_k{}^4>0) \qquad (2)$$

where V is the volume of the 3-dimensional region considered, $p_k{}^\alpha$ are the possible values of the energy-momentum 4-vector,[4] and $\Psi_{k\pm}$ is a suitably normalized scalar, spinor, or vector. The vacuum will be the state of lowest energy provided its state vector $\mathbf{\Psi}_0$ satisfies

$$b_{k+}\mathbf{\Psi}_0=b_{k-}\mathbf{\Psi}_0=0; \qquad (3)$$

with this notation b_r ($r=k+$ or $k-$) always annihilates, and $b_r{}^*$ always creates, a particle of positive energy. A scheme of quantization will be considered satisfactory if it ensures

$$\partial_\alpha\Psi(x)=i[P_\alpha, \Psi(x)], \qquad (4)$$

where P_α is the total energy-momentum 4-vector for the field.

2. SPIN-HALF FIELDS

It will be assumed that the energy-momentum 4-vector of a spin-half field is obtained in the form

$$P^\alpha=\sum_r p_r{}^\alpha[b_r{}^*, b_r]. \qquad (5)$$

This is, for example, the form to which the energy-momentum of the electron field naturally reduces, when expressed in the usual way (apart from a factor $\frac{1}{2}$, which is for convenience absorbed into the operators b_r and $b_r{}^*$). Now, the necessary condition under which (4) is satisfied is

$$[b_r, [b_s{}^*, b_t]]=\delta_{rs}b_t; \qquad (6)$$

for $s=t$; it will be assumed to hold in general. This, with the relation obtained by taking the Hermitean conjugate, implies

$$[P^\alpha, b_r]=-p_r{}^\alpha b_r, \quad [P^\alpha, b_r{}^*]=p_r{}^\alpha b_r{}^* \qquad (7)$$

—which are equivalent to (4), on account of (1) and (2). The relation (6) will be supplemented by

$$[b_r, [b_s, b_t]]=0. \qquad (8)$$

[4] Greek affixes run from 1 to 4 and are used consistently with a metric tensor $g_{\alpha\beta}=(-1, -1, -1, 1)$.

It follows from (6) and (8) that

$$[b_r{}^*, [b_s, b_t]]=\delta_{rs}b_t-\delta_{rt}b_s, \quad [b_r{}^*, [b_s{}^*, b_t{}^*]]=0. \qquad (9)$$

All the above relations are satisfied if one assumes

$$\{b_r, b_s\}=0, \quad \{b_r{}^*, b_s\}=\tfrac{1}{2}\delta_{rs}, \qquad (10a)$$

in the usual way; however, there is an infinite number of other ways in which they can be satisfied as well. For example, if one has

$$\begin{aligned} &b_rb_sb_t+b_tb_sb_r=0,\\ &b_r{}^*b_sb_t+b_tb_sb_r{}^*=\delta_{rs}b_t,\\ &b_rb_s{}^*b_t+b_tb_s{}^*b_r=\delta_{rs}b_t+\delta_{ts}b_r, \end{aligned} \qquad (10b)$$

they are satisfied identically; but the relations (10a) are incompatible with (10b). In fact, (10a) and (10b) determine completely different representations of matrices which satisfy (6). It is important that

$$[[b_r{}^*, b_r], [b_s{}^*, b_s]]=0 \qquad (11)$$

is a consequence only of (6); thus, whatever additional assumptions are made, the various terms in the summation (5) can be diagonalized simultaneously and the energy levels obtained. The fact that the b's neither commute nor anticommute in general does not add an intolerable complication to the theory.

To obtain the energy levels of the system, it is necessary to select a definite representation for the b's. It will be shown that any one of them, b_r say, has an irreducible matrix representation with $k+1$ rows and columns, where k is any integer. The representative of b_r in this representation, which makes the energy diagonal, is

$$(b_r)_{mn}=\delta_{m,\,n-1}\{\tfrac{1}{2}m(k-m+1)\}^{\frac{1}{2}}, \quad (1\leq m, n\leq k+1). \qquad (12)$$

This gives

$$[b_r{}^*, b_r]_{mn}=\delta_{mn}(n-1-\tfrac{1}{2}k), \qquad (13)$$

and

$$\begin{aligned}[b_r, [b_r{}^*, b_r]]_{mn}&=\delta_{m,\,n-1}\{\tfrac{1}{2}m(k-m+1)\}^{\frac{1}{2}}\\ &=(b_r)_{mn}.\end{aligned} \qquad (14)$$

Thus (6) is satisfied for $r=s=t$. It is clear from (13) and (5) that the separation of neighboring energy levels corresponding to any eigenstate r is $p_r{}^4$, and one may therefore interpret the $(j+1)$th energy level in the usual way as corresponding to a state in which j similar particles are present. If $\mathbf{\Psi}_0$ is the normalized state vector for the vacuum,

$$\mathbf{\Psi}_{j(r)}=\{2^j(k-j)!\}^{\frac{1}{2}}(j!k!)^{-\frac{1}{2}}(b_r{}^*)^j\mathbf{\Psi}_0 \qquad (15)$$

is the normalized state vector for a situation in which j particles are present in the rth eigenstate. It follows from (12) that $(b_r{}^*)^{k+1}=0$; so that it is impossible to have more than k particles in the same eigenstate.

According to (5), the "zero-point" energy of the vacuum would be $-\frac{1}{2}k\sum_r p_r{}^4$; this has to be subtracted from P^4 to obtain the observable energy. Thus,

$$P_{\mathrm{obs}}{}^{\alpha}=\sum_r p_r{}^{\alpha}([b_r{}^*, b_r]+\tfrac{1}{2}k). \tag{16}$$

The simultaneous representation of many b_r's will now be considered. It is clear that an irreducible representation in $(k+1)^p$ dimensions should exist for a set of p different annihilation operators. The author has not, however, succeeded in obtaining a simple formula for the matrix elements of the b's in this representation. For this reason, only a *reducible* representation in 2^{pk} dimensions will be discussed. Let $\sigma_r{}^{(j)}$ and $\tau_r{}^{(j)}$ $(j=1\cdots k,\ r=1\cdots p)$ be a set of Hermitean operators satisfying the following commutation and anticommutation rules:

$$\begin{aligned}[][\sigma_r{}^{(i)}, \sigma_s{}^{(j)}]&=[\sigma_r{}^{(i)}, \tau_s{}^{(j)}]\\ &=[\tau_r{}^{(i)}, \tau_s{}^{(j)}]=0, \quad (i\neq j),\\ \{\sigma_r{}^{(j)}, \sigma_s{}^{(j)}\}&=\{\tau_r{}^{(j)}, \tau_s{}^{(j)}\}=\delta_{rs}, \quad \{\sigma_r{}^{(j)}, \tau_s{}^{(j)}\}=0.\end{aligned} \tag{17}$$

These operators can obviously be expressed as direct products of the members of pk sets of Pauli spin operators, and may therefore be represented without difficulty in 2^{pk} dimensions. Also, one can show that if

$$b_r=\tfrac{1}{2}\sum_{j=1}^{j=k}(\sigma_r{}^{(j)}+i\tau_r{}^{(j)}), \tag{18}$$

the commutation relations (6) and (8) are satisfied. For

$$\begin{aligned}[][b_r, b_s]&=\tfrac{1}{4}\sum_j[\sigma_r{}^{(j)}+i\tau_r{}^{(j)}, \sigma_s{}^{(j)}+i\tau_s{}^{(j)}],\\ [b_r{}^*, b_s]&=\tfrac{1}{4}\sum_j[\sigma_r{}^{(j)}-i\tau_r{}^{(j)}, \sigma_s{}^{(j)}+i\tau_s{}^{(j)}],\end{aligned} \tag{19}$$

and

$$\begin{aligned}&[b_r, [b_s{}^*, b_t]]\\ &\quad=\tfrac{1}{8}\sum_j[\sigma_r{}^{(j)}+i\tau_r{}^{(j)}, [\sigma_s{}^{(j)}-i\tau_s{}^{(j)}, \sigma_t{}^{(j)}+i\tau_t{}^{(j)}]]\\ &\quad=\tfrac{1}{2}\sum_j(\sigma_t{}^{(j)}+i\tau^{(j)})\delta_{rs},\end{aligned} \tag{20}$$

etc. The energy and momentum are diagonal if all the $i\sigma_r{}^{(j)}\tau_r{}^{(j)}$ are diagonal.

For the valuation of expectation values it is necessary to supplement the definition (3) of the vacuum by

$$b_r b_s{}^*\Psi_0=\tfrac{1}{2}k\delta_{rs}\Psi_0, \tag{21}$$

which can be proved in the representations discussed, but is obviously not a general relation. From (21), (3), and (6) it follows that

$$b_r b_s{}^* b_t{}^*\Psi_0=\{\tfrac{1}{2}k\delta_{rs}b_t{}^*+(\tfrac{1}{2}k-1)\delta_{rt}b_s{}^*\}\Psi_0. \tag{22}$$

With the help of these and more complicated formulas which can be deduced from them, it is possible to eliminate the b's from any function of both the b's and b^*'s when applied to the vacuum state vector.

It is natural to interpret $b_r{}^*b_s{}^*\Psi_0$ as the state vector for the situation with two particles in the eigenstates r and s, respectively. However, except in the simplest case of Fermi statistics ($k=1$), this state vector is not equivalent to $b_s{}^*b_r{}^*\Psi_0$, which corresponds to a situation in which the two particles are interchanged. Thus *a new state usually results from the interchange of two particles*, in this theory. There are, however, always certain symmetry properties which are strongly analogous to those possessed by fermions. To illustrate these, the relations (10b), which correspond to $k=2$, will be considered. It follows from (10b) that, if r, s, t, and u are all different eigenstates,

$$\begin{aligned}b_r{}^*b_s{}^*b_t{}^*\Psi_0&=-b_t{}^*b_s{}^*b_r{}^*\Psi_0;\\ b_r{}^*b_s{}^*b_t{}^*b_u{}^*\Psi_0&=-b_t{}^*b_s{}^*b_r{}^*b_u{}^*\Psi_0\\ &=-b_r{}^*b_u{}^*b_t{}^*b_s{}^*\Psi_0\\ &=b_t{}^*b_u{}^*b_r{}^*b_s{}^*\Psi_0.\end{aligned} \tag{23b}$$

Thus, the particles always divide into two groups in such a way that interchange of two particles in the same group does not produce an essentially different state, but interchange of two particles in different groups gives a new state. The particles divide themselves as equally as possible between the two groups, and particles in the same eigenstate always belong to different groups. These conclusions are easily generalized. When $k>2$, there are k groups of particles and it is possible to distinguish only between members of different groups. Not more than one particle in a given eigenstate may exist in one group.

An observation will now be made which somewhat counteracts the foregoing conclusions. It may happen that, owing to the type of interaction by which they are created, the particles will be distributed with equal probability between the several groups, so that the distinction between particles in different groups will never be physically realized. In electron field theory, the interaction energy involves creation operators always in combinations like $[b_r{}^*, b_s{}^*]$ or $[b_r, b_s{}^*]$. But, assuming $k=2$ in a theory of this type, the operator $[b_r{}^*, b_s{}^*]$ distributes the two particles created with equal probability between the two groups; and since

$$[[b_r, b_s{}^*], [b_t{}^*, b_u{}^*]]=\delta_{ru}[b_s{}^*, b_t{}^*]-\delta_{rt}[b_s{}^*, b_u{}^*], \tag{24}$$

the application of the operator $[b_r, b_s{}^*]$ will not interfere with this equidistribution. It follows that if the state vector is initially either symmetrical or antisymmetrical with respect to the interchange of any two creation operators, it will remain so. It might be thought that under such circumstances the method of quantization must reduce to quantization in accordance with Fermi statistics. That this is not so may be seen from the existence of a nonvanishing state vector,

$$[b_r{}^*, b_s{}^*][b_r{}^*, b_t{}^*]\Psi_0=\{b_t{}^*, b_s{}^*\}(b_r{}^*)^2\Psi_0, \tag{25}$$

with two particles in the rth eigenstate.

For $k>2$ it is also possible to devise interactions, involving symmetrical combinations of k creation or annihilation operators, such that the principle of "indistinguishability of similar particles" is preserved.

3. FIELDS OF INTEGRAL SPIN

A completely parallel development is possible for fields describing particles with integral spin. One then assumes that the energy-momentum 4-vector can be reduced to the form

$$P^\alpha = \sum_r p_r^\alpha \{b_r^*, b_r\}, \tag{26}$$

with the anticommutator instead of the commutator. The fundamental requirement (4) then leads to

$$[b_r, \{b_s^*, b_t\}] = \delta_{rs} b_t \tag{27}$$

instead of (6), to which one adds

$$[b_r, \{b_s, b_t\}] = 0 \tag{28}$$

in place of (8). These relations are satisfied by the usual commutation rules,

$$[b_r, b_s^*] = \tfrac{1}{2}\delta_{rs}, \tag{29a}$$

but also by an infinite number of alternative schemes, for instance, by

$$\begin{aligned} &b_r b_s b_t - b_t b_s b_r = 0,\\ &b_r b_s^* b_t - b_t b_s^* b_r = \delta_{rs} b_t - \delta_{ts} b_r,\\ &b_r b_s b_t^* - b_t^* b_s b_r = \delta_{st} b_r, \end{aligned} \tag{29b}$$

which correspond to (10b).

If one identifies the suffixes in (27), these relations reduce to

$$b_r^2 b_r^* - b_r^* b_r^2 = b_r. \tag{30}$$

The matrix representation of two operators b_r and b_r^* which satisfy this relation, which arises in the quantization of the simple harmonic oscillator, has been discussed by Yang and Wigner.[5] Yang's method can be adapted to obtain all the irreducible representations for any one of the b_r's; these may be compared with the corresponding representations for spin-half fields, obtained in the previous section. To appreciate the degree of degeneracy arising in some of these representations, however, one has to construct the corresponding representations for a set of different b_r's. Consider the set of operators $b_r^{(j)}$ ($j=1\cdots k$, $r=1\cdots p$) which satisfy

$$\begin{aligned} &\{b_r^{(i)}, b_s^{(j)}\} = \{b_r^{(i)*}, b_s^{(j)}\} = 0, \quad (i\neq j),\\ &[b_r^{(j)}, b_s^{(j)}] = 0, \quad [b_r^{(j)}, b_s^{(j)*}] = \tfrac{1}{2}\delta_{rs}. \end{aligned} \tag{31}$$

These can evidently be expressed as direct products of a set of operators of the type satisfying (29a) and a set of Pauli spin operators, the matrix representations of which are well known. Then, if one writes

$$b_r = \sum_{j=1}^{j=k} b_r^{(j)}, \tag{32}$$

both (27) and (28) are satisfied. For one has

$$\begin{aligned} &\{b_r, b_s\} = 2\sum_j b_r^{(j)} b_s^{(j)},\\ &\{b_r, b_s^*\} = 2\sum_j b_r^{(j)} b_s^{(j)*}, \end{aligned} \tag{33}$$

[5] L. M. Yang, Phys. Rev. **84**, 788 (1951); E. P. Wigner, Phys. Rev. **77**, 711 (1950); see also C. R. Putnam, Phys. Rev. **83**, 1047 (1951).

etc. It is readily verified that

$$[\{b_r^*, b_r\}, \{b_s^*, b_s\}] = 0 \tag{34}$$

is a consequence of (27), so that it is always possible to diagonalize simultaneously the various terms in the summation of (26). Also

$$\{b_r^*, b_r\} = \sum_j \{b_r^{(j)*}, b_r^{(j)}\}, \tag{35}$$

so the energy levels of any eigenstate are obtained by superimposing the energy levels of k similar bosons. It is evident that nearly all of these levels are very degenerate for $k>1$, though the degree of degeneracy is magnified by the fact that a reducible representation has been chosen.

The "zero-point" energy of the vacuum is $\tfrac{1}{2}k\sum_r p_r^4$, so the observable energy and momentum of the field must be given by

$$P_{\text{obs}}^\alpha = \sum_r p_r^\alpha (\{b_r^*, b_r\} - \tfrac{1}{2}k) \tag{36}$$

instead of (16); but the vacuum state vector continues to satisfy (21). There are again k different groups of particles, and a new state is generally obtained by interchanging two particles belonging to different groups. The only way in which the fields with integral spin differ from those considered in the last section, in this respect, is that any number of particles can be found in the same group and the same eigenstate.

4. FURTHER CONSEQUENCES

The vacuum expectation values of simple products of the field variables differ in the generalized theory from the ordinary values at most by a constant factor. Thus, it follows immediately from (1), (3), and (21) that

$$\langle \Psi(x)\bar{\Psi}(x')\rangle_0 = kS^{(+)}(x-x'), \tag{37}$$

where $S^{(+)}$ is the ordinary Green's function with positive frequencies for the spin-half field. However, the vacuum expectation value of a product of four field variables differs more radically from the usual value, owing to the fact that the statistics of the particles are different, and more than one particle can be created in the same eigenstate. The interaction of fields whose particles satisfy the generalized statistics clearly requires a protracted study.

Another field in which the generalization of the statistics may be expected to lead to new results is in the quantum-statistical mechanics of ideal gases at low temperatures. If any physical particles are neither bosons nor fermions in the ordinary sense, the statistical thermodynamics of assemblies of such particles will be intermediate between those predicted on the basis of quantum and classical statistics, respectively, and has other interesting features which are at present under investigation.