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The Equation of Motion of a Dislocation*

J. D. ESHELBY
Department of Physics, University of Illinois, Urbana, Illinois
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The elastic field surrounding an arbitrarily moving screw dislocation is found, and a useful analogy with two-dimensional electromagnetic fields is pointed out. These results are applied to a screw dislocation accelerating from rest and approaching the velocity of sound asymptotically. The applied stress needed to maintain this motion is found on the assumption that the Peierls condition is satisfied near the center of the dislocation. A general integral equation of motion is derived for a simplified dislocation model, and the kind of behavior it predicts is illustrated.

I. INTRODUCTION

An extensive literature exists dealing with the properties of dislocations at rest (see the recent review article by Nabarro[1]). There has been much less discussion of their motion, perhaps because it is still uncertain whether it is governed by friction or inertia. Though at present most authors seem to suppose that dissipative processes are dominant, some still adopt the dynamic point of view initiated by Frank.[2] It therefore seems worth while to investigate the dynamical behavior of dislocations on the assumption that dissipative effects are negligible.

Ideally, we should discuss the change in shape with time of a dislocation loop in an applied stress field. Here we only consider the plane problem of an infinite straight dislocation. As a further simplification we shall suppose it is a pure screw dislocation. The problem is then one in antiplane strain, and only a single velocity of sound is involved instead of the two which would appear if there were an edge component. (We assume the medium is isotropic). Frank has shown[3] that uniformly moving screw dislocations behave in a manner reminiscent of particles in the special theory of relativity (the velocity of transverse elastic waves taking the place of that of light), and we shall sometimes find it convenient to use relativistic terminology.

In Secs. II and III we find the elastic field surrounding an arbitrarily moving screw dislocation and show how it is related to the electromagnetic field of a moving line-charge. This is illustrated (Sec. IV) by the case of motion from rest with constant proper acceleration, and the applied field necessary to maintain this particular motion is found by requiring the Peierls condition to be satisfied. In Sec. V a general equation of motion is derived for a simplified model of a dislocation, and the kind of behavior it predicts is exemplified in Sec. VI.

II. THE DISPLACEMENT ROUND A MOVING SCREW DISLOCATION[4,5]

The expression[6]

$$w=\tfrac{1}{4}blkH_1^{(2)}(kr)\sin\theta\, e^{i\omega t},\quad k=\omega/c, \tag{1}$$

is the time-dependent part of the displacement field around a screw dislocation oscillating along the x axis, the position of its center being

$$\xi=le^{i\omega t}. \tag{2}$$

As a solution of the wave equation, $c^2\nabla^2 w-\partial^2 w/\partial t^2=0$, Eq. (1) is characterized by the boundary conditions

$$\lim_{\epsilon\to 0} w(x;\, y=\pm\epsilon)=\pm\tfrac{1}{2}bl\delta(x)e^{i\omega t},$$

so that it represents a region of alternating slip across the x axis concentrated at the origin, the product of area and amplitude of slip being bl, in the limit when $l=\text{const}b^{-1}\to 0$.

Let us apply to both (1) and (2) the operator

$$\frac{1}{2\pi i}\int_C(\cdots)\frac{d\omega}{\omega}, \tag{3}$$

where C is the real ω-axis indented below the origin. From (2) we get

$$\xi=lH(t),\quad \text{where}\quad H(t)=\begin{matrix}1\\0\end{matrix}\quad \text{for}\quad t\gtrless 0.$$

To find the associated displacement we replace the Bessel function in (1) by the integral representation

$$H_1^{(2)}(kr)=-\frac{2}{\pi}\int_1^\infty \frac{e^{-ikrv}}{\sqrt{(v^2-1)}}vdv,$$

apply Eq. (3), and interchange the order of integration. Since

$$\frac{1}{2\pi}\int_C e^{i\omega z}d\omega=\delta(z),$$

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1 F. R. N. Nabarro, Advances in Phys. **1**, 271 (1952).

2 F. C. Frank, *Report of a Conference on the Strength of Solids* (Physical Society, London, 1948), p. 46. [*Added in proof:* See also N. F. Mott, Phil. Mag. **43**, 1151 (1952); Fisher, Hart, and Pry, Phys. Rev. **87**, 958 (1952).]

3 F. C. Frank, Proc. Phys. Soc. (London) **A62**, 131 (1949).

4 F. R. N. Nabarro, Phil. Mag. **42**, 1224 (1951).

5 J. D. Eshelby, Phil. Trans. Roy. Soc. **A244**, 87 (1951).

6 J. D. Eshelby, Proc. Roy. Soc. (London) **A197**, 396 (1949).

we find

$$w=\frac{bl}{2\pi r}\sin\theta\int_1^\infty \frac{\delta(v-ct/r)}{\sqrt{(v^2-1)}}vdv$$

$$=\frac{bl}{2\pi r}\sin\theta\frac{ct}{\sqrt{(c^2t^2-r^2)}}H(ct-r).$$

If the jump occurs along the line $\theta=\theta_0$ instead of $\theta=0$, we have only to replace θ by $\theta-\theta_0$. A continuous motion in which the position of the center of the dislocation is given by

$$x=\xi(t),\quad y=\eta(t)$$

can be looked upon as a series of jumps from (ξ,η) to $(\xi+\dot{\xi}dt, \eta+\dot{\eta}dt)$ in successive intervals of time dt. Hence the displacement produced by the moving dislocation is

$$w=\frac{b}{2\pi}\int_{-\infty}^{\tau_0}\frac{c(t-\tau)\{(y-\eta)\dot{\xi}-(x-\xi)\dot{\eta}\}}{(x-\xi)^2+(y-\eta)^2}\frac{d\tau}{s}. \tag{4}$$

Here ξ and η are functions of τ, while

$$s^2=c^2(t-\tau)^2-(x-\xi)^2-(y-\eta)^2,$$

and τ_0 is that root of $s^2=0$ which is less than t. τ_0 is unique if the velocity of the dislocation is always less than c, which we assume is the case.

III. AN ELECTROMAGNETIC ANALOGY

Physically more important than w itself are its time-derivative $\dot{w}$ and the stresses $p_{zx}=\mu\partial w/\partial x$, $p_{zy}=\mu\partial w/\partial y$. ($\mu$ is the shear modulus.) They are rather hard to derive from (4) by differentiation, since this yields an infinite term from the variation of the upper limit plus a divergent integral. We may take an arbitrary value for τ_0 and follow the differentiation by an integration by parts. When τ_0 approaches its correct value the term from the variation of the upper limit cancels the integrated part, leaving a finite result.

For the time-derivative of (1) we may write

$$\dot{w}=-\frac{\partial}{\partial y}\{\tfrac{1}{4}ibl\omega H_0^{(2)}(kr)e^{i\omega t}\},$$

and repeating for the zero-order Bessel function the argument leading from (1) to (4), we find, for a dislocation moving along the x axis,

$$\dot{w}=-\frac{\partial I}{\partial y},\quad\text{where}\quad I=\frac{bc}{2\pi}\int_{-\infty}^{\tau_0}\dot{\xi}\frac{d\tau}{s}.$$

This suggests that we might be able to derive the velocity and stresses from a set of potentials involving simpler integrals than that for w. The substitution

$$r=c(t-\tau),\quad z^2=r^2-(x-\xi)^2-y^2$$

changes I into the form

$$I=\frac{b}{4\pi c}\int_{-\infty}^{\infty}\left[\frac{\dot{\xi}}{r-(x-\xi)\dot{\xi}/c}\right]_{\tau=t-r/c}dx.$$

If we identify the velocities of light and sound, this is seen to be the x component of the vector potential of an electrical line-charge moving in the same way as the dislocation.

There is in fact a close resemblance between antiplane strain elastic problems and electromagnetic fields in which all quantities are independent of the z coordinate. Consider the case where $E_z=H_x=H_y=0$, and make the identification

$$\dot{w}=H_z/\sqrt{\rho},\quad p_{zx}=-E_y\sqrt{\mu},\quad p_{zy}=E_x\sqrt{\mu}. \tag{5}$$

(We use Heaviside units.) Then all Maxwell's equations are satisfied identically except $\operatorname{curl}\mathbf{E}+(1/c)(\partial\mathbf{H}/\partial t)=0$ which becomes

$$\partial p_{zx}/\partial x+\partial p_{zy}/\partial y=\rho\partial^2 w/\partial t^2,\quad\text{or}$$

$$c^2\nabla^2 w-\partial^2 w/\partial t^2=0, \tag{6}$$

the elastic equilibrium or wave equation. If there is a moving line-charge with linear density λ, Gauss's relation for the conservation of charge states that

$$\oint\mathbf{E}\cdot d\mathbf{s}=\oint(lE_x+mE_y)ds=\lambda\quad\text{or}\quad 0, \tag{7}$$

according as the circuit does or does not enclose the charge. Using (5) this becomes

$$\oint\left(l\frac{\partial w}{\partial y}-m\frac{\partial w}{\partial x}\right)ds=\oint\frac{\partial w}{\partial s}ds=\frac{\lambda}{\sqrt{\mu}}\quad\text{or}\quad 0, \tag{8}$$

showing that the elastic counterpart of the charge is a dislocation with a Burgers vector of magnitude $\lambda/\sqrt{\mu}$ directed along the z axis.

With the aid of this analogy we can complete the scheme for deriving the velocity and stresses from a set of potentials:

$$p_{zy}=-\mu^{\frac{1}{2}}\left(\frac{\partial\varphi}{\partial x}+\frac{1}{c}\frac{\partial A_x}{\partial t}\right),\quad p_{zx}=\mu^{\frac{1}{2}}\left(\frac{\partial\varphi}{\partial y}+\frac{1}{c}\frac{\partial A_y}{\partial t}\right),$$

$$\dot{w}=\rho^{-\frac{1}{2}}\left(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right), \tag{9a}$$

where

$$\{\varphi, A_x, A_y\}=\frac{b\mu^{\frac{1}{2}}}{2\pi c}\int_{-\infty}^{\tau_0}\{c,\dot{\xi}(\tau),\dot{\eta}(\tau)\}\frac{d\tau}{s}. \tag{9b}$$

It is easy to verify that if we make the translation expressed by (5), the electromagnetic energy density and Poynting's vector become the elastic energy density and the elastic energy-flow vector:[7]

$$S_x=p_{zx}\dot{w},\quad S_y=p_{zy}\dot{w}. \tag{10}$$

The electromagnetic momentum density becomes not the ordinary elastic momentum density $\rho\dot{w}$ but the quasi momentum used, for example, in discussing the

[7] A. E. H. Love, *Mathematical Theory of Elasticity* (Cambridge University Press, Cambridge, 1927).

collisions of phonons with one another and with electrons. For a uniformly moving screw dislocation this quasi momentum agrees with the effective momentum introduced by Frank.[3] The Maxwell tensor becomes the "Maxwell tensor of elasticity."[5] The Lorentz force on the line-charge becomes

$$F_x=b(p_{zy}+\dot{w}v_y),\quad F_y=-b(p_{zx}+\dot{w}v_y),$$

where v is the velocity of the dislocation and $\dot{w}$, p_{zx}, p_{zy} refer to an applied stress-field. The terms independent of v are the orthodox expressions for the force on a stationary dislocation. Nabarro[8] has shown that the terms in v are physically significant. He has also pointed out that in the scattering of sound waves by a screw dislocation, the force and quasi momentum are related in the same way as ordinary force and momentum.

If there is a charge density σ in the electromagnetic problem, the equation $\mathrm{div}\mathbf{E}=\sigma$ becomes

$$\partial e_{zy}/\partial x-\partial e_{zx}/\partial y=\mu^{-\frac{1}{2}}\sigma, \tag{11}$$

in terms of the strains $e_{zx}=p_{zx}/\mu$, $e_{zy}=p_{zy}/\mu$. A continuous distribution of σ corresponds to a state of antiplane self-stress. A general state of self-stress can be specified[5] by a tensor S_{ij}; for antiplane strain the nonvanishing components are

$$S_{zx}=-\mu^{-\frac{1}{2}}\partial\sigma/\partial y,\quad S_{zy}=\mu^{\frac{1}{2}}\partial\sigma/\partial x.$$

A varying S_{ij} formally represents a state of plastic flow.

In place of (5) we might have taken an electromagnetic field in which $E_x=E_y=H_z=0$ and made the identification,

$$\dot{w}=-E_z/\sqrt{\rho},\quad p_{zx}=-H_y\sqrt{\mu},\quad p_{zy}=H_x\sqrt{\mu}. \tag{12}$$

In place of (6) and (11) we should have

$$\partial p_{zx}/\partial x+\partial p_{zy}/\partial y+i_z/c=\rho\partial^2 w/\partial t^2,\ \partial e_{zy}/\partial x-\partial e_{zx}/\partial y=0,$$

where i_z, the current in the z direction, has to be equated to c times the body force (which must be parallel to the z axis to preserve the antiplane strain character). In contradistinction to (5) the correspondence (12) is adapted to problems with body force but no self-stress. We can, however, simulate the stress field of a dislocation with the help of a double layer of force along the x axis extending from the center of the dislocation to infinity. The electromagnetic field of the corresponding double current sheet can be derived from a vector potential $(0, 0, A_z)$. However, A_z is equal to w and the analogy is too good to be useful.

A possible third analogy is suggested by the static case. It is natural to identify the displacement of a stationary screw dislocation with the scalar potential of a current-carrying wire; (p_{zx}, p_{zy})—which transforms like a vector for rotation about the z axis—is then proportional to (H_x, H_y). However, this analogy cannot be extended to the time-dependent case since the equilibrium equation $\mathrm{div}(p_{zx}, p_{zy})=\rho\partial^2 w/\partial t^2$ must come from one of Maxwell's curl equations, which alone contain time derivatives. Hence, the identification of the stress with $\mathbf{E}$ or $\mathbf{H}$ must always have the crosswise character of Eqs. (5) and (12).

[8] F. R. N. Nabarro, Proc. Roy. Soc. (London) **A209**, 278 (1951).

IV. A DISLOCATION WITH CONSTANT PROPER ACCELERATION

As an example we consider a case of accelerated motion. Since a uniform acceleration would finally bring the velocity above c, we consider a dislocation at rest for negative t and thereafter moving with constant acceleration in its own rest system:

$$\begin{aligned}&\xi=x_0,\quad \eta=0 &&\text{for}\quad t<0;\\ &\xi^2-c^2t^2=x_0^2,\quad \eta=0 &&\text{for}\quad t>0.\end{aligned} \tag{13}$$

For $t>0$ there are the simple relations

$$\dot{\xi}=c^2t/\xi,\quad \beta\equiv\sqrt{(1-\dot{\xi}^2/c^2)}=x_0/\xi,\quad \partial^2\xi/\partial t^2=\beta^3c^3/x_0.$$

The dislocation starts with an acceleration c^2/x_0, conveniently specified by its initial x coordinate x_0, and approaches the velocity c asymptotically.

It is convenient to use the abbreviation

$$R^2=x^2-c^2t^2,$$

where x, t refer to the point and time of observation. Then at any time points near the dislocation line are distinguished by small x_0-R and small y.

Choosing units in which $c=1$, we have to evaluate (9b) with

$$\begin{aligned}&s^2=\tau^2-2t\tau+s_0^2,\quad \dot{\xi}=\dot{\eta}=0 &&\text{for}\quad \tau<0;\\ &s^2=2x\xi-2t\tau-R^2-y^2-x_0^2,\quad \dot{\xi}=\tau/\xi,\quad \dot{\eta}=0 &&\text{for}\quad \tau>0;\end{aligned}$$

where

$$s_0=\sqrt{(2xx_0-R^2-y^2-x_0^2)}$$

is the interval between $(x_0, 0, 0)$ and (x, y, t). The integral for $\tau<0$ is elementary; we omit an infinite constant. The integrals for $\tau>0$ yield to the substitution

$$\tan^2\psi=s^2/\{(R-x_0)^2+y^2\},$$

giving

$$\begin{aligned}\varphi&=b\mu^{\frac{1}{2}}/2\pi(x_0/R)^{\frac{1}{2}}(2/kR)\{tk'\tan\psi-x[\Delta\tan\psi\\&\qquad+\tfrac{1}{2}(1+k'^2)F-E]\}+(b/2\pi)\ln(t+s_0),\\ A_x&=b\mu^{\frac{1}{2}}/2\pi(x_z/R)^{\frac{1}{2}}(2/kR)\{xk'\tan\psi-t[\Delta\tan\psi\\&\qquad+\tfrac{1}{2}(1+k'^2)F-E]\},\quad A_y=0,\end{aligned} \tag{14}$$

where with the usual notation for elliptic integrals

$$\Delta=\sqrt{(1-k^2\sin^2\psi)},\ F(\psi,k)=\int_0^\psi\frac{d\psi}{\Delta},\ E(\psi,k)=\int_0^\psi\Delta d\psi$$

and

$$\tan\psi=s_0k/k'\sqrt{(4x_0R)},\ k^2=1-k'^2=4x_0R/\{(R+x_0)^2+y^2\}.$$

The displacement can be expressed in terms of elliptic integrals of the first and third kinds. Equation (14) is only valid when

$$c^2t^2-(x-x_0)^2-y^2>0,\quad t>0,\ x>ct,$$

but this is all we shall need.

Near the center of the dislocation p_{zy} is given approximately by

$$p_{zy}=\frac{\mu b}{2\pi}\left\{\frac{x'}{\beta r'^2}-\frac{1}{2x_0}\ln\frac{8x_0}{r'}\left[\frac{2x_0}{s_0^2}(t-s_0)\right]-\frac{t-s_0}{x_0s_0}-\frac{s_0}{2x_0(t+s_0)}\right\}, \quad (15)$$

where now

$$s_0=\surd\{t^2-[x-\xi(t)]^2\}$$

is the interval from $(x_0, 0, 0)$ to the center of the dislocation at time t and

$$x'=x-\xi(t), \quad r'^2=x'^2/\beta^2(t)+y^2.$$

r' is thus the radial distance from the center of the dislocation measured in its instantaneous rest-coordinates. Equation (14) is subject to the limitations

$$r'\ll x_0, \quad s_0\gg r'. \quad (16)$$

We shall be interested in values of r' of the order of a few lattice spacings. The first condition then means that the velocity acquired by the dislocation in the time that sound travels one lattice spacing shall be small compared with the velocity of sound. Otherwise expressed, the proper acceleration of the dislocation must be small compared with the acceleration (about 10^{18} cm/sec^2) of an atom oscillating with an amplitude of one lattice spacing at the frequency of a lattice vibration near the Debye limit. The second condition requires that the diameter of the disturbed region, $(x-x_0)^2+y^2\leqslant t^2$, which spreads from the starting position of the dislocation shall be large compared with the lattice spacing. Since $s_0\sim t$ for small t, this means that t must be large compared with the period of a lattice vibration.

To find the applied stress necessary to produce the motion (13), we have to introduce nonlinearity into the problem. Following Nabarro[8] we shall try to satisfy the Peierls law relating stress and displacement at the slip plane. The Peierls law appropriate to a screw dislocation with Burgers vector b and separation a between atomic planes parallel to the slip plane is

$$p_{zy}(y=\tfrac{1}{2}a)=-\mu/2\pi b/a\,\sin 4\pi/bw(y=\tfrac{1}{2}a). \quad (17)$$

This is satisfied by the elastic solution for a screw dislocation which has been moving uniformly for all time with velocity v:

$$w=\frac{b}{2\pi}\tan^{-1}\frac{y\surd(1-v^2/c^2)}{(x-vt)}.$$

Now, the first term in (15) is simply the stress produced by a uniformly moving dislocation which at time t happens to coincide in position and velocity with the accelerated dislocation. For the accelerated dislocation the condition (17) would be satisfied if we could impress on every point of the material a displacement equal to the difference between the displacements of the uniformly moving and accelerated dislocations. The nonlinear behavior is confined to the neighborhood of the center of the dislocation, and so we may hope that it will be enough if the impressed displacement is correct at least in this region. On the planes $y=\pm\frac{1}{2}a$ the last three terms of (15) are slowly varying functions of x' near $x'=0$, in view of (16). Hence the impressed displacement required can be produced by a uniform applied stress,

$$p_{zy}{}^A(t)=\frac{\mu b}{2\pi x_0}\left\{\tfrac{1}{2}\ln\left(\frac{32x_0^2}{as_0}\frac{t-s_0}{s_0}\right)+\frac{s_0}{2(t+s_0)}+\frac{t-s_0}{s_0}\right\}, \quad (18)$$

equal and opposite to the last three terms in (15) taken at the point $x'=0$, $y=\frac{1}{2}a$.

When (18) is multiplied through by b, it becomes a relation between the force on the dislocation and its acceleration. Formally, b and a are independent parameters corresponding to charge and diameter of charge in the associated electromagnetic problem. It is, therefore, natural to interpret the logarithmic term, which diverges when a approaches zero, as the effective mass of the dislocation. In ordinary units,

$$F_x=bp_{zy}{}^A=(1-\dot{\xi}^2/c^2)^{-\frac{3}{2}}(\rho b^2/4\pi)\{\ln f(t)\}\partial^2\xi/\partial t^2+g(t),$$

where $f(t)$ is the argument of the logarithm in (18) and $g(t)$ is b times the second and third terms. This has the form of the relativistic equation of motion of a particle with a slowly varying rest mass $(\rho b^2/4\pi)\ln f(t)$ and a radiation reaction term $g(t)$. It has already been suggested by Frank[3] that a rest mass $(\rho b^2/4\pi)\ln(r_1/a)$ can be ascribed to a screw dislocation, where r_1 is a not very well defined length. In the present case, $f(t)=8ct/a$ for small t and r_1 is, not unreasonably, of the order of the disturbed region surrounding the dislocation. As t increases, r_1 is a complicated function of the two lengths x_0 and ct. For large t, the dislocation takes up a position at a distance x_0 behind the leading edge of the disturbed region, and r_1 becomes $16x_0$. The second term in (18) comes from the part of the integral for φ with $\tau<0$, and may be associated with the discontinuity in $\partial^2\xi/\partial t^2$ at $t=0$. It approaches zero as t increases. The last term in (18) increases as $\surd t$; we may perhaps connect it with the fact that the accelerating dislocation is continually catching up the radiation it has already emitted.

It is clear that, if a is of the same order as b and x_0/b is reasonably large, the stress necessary to maintain the motion is given approximately by

$$p_{zy}{}^A/\mu=(b/4\pi x_0)\ln(x_0/b), \quad (19)$$

except at the beginning of the motion and in the extreme relativistic region. If $p_{zy}{}^A/\mu=10^{-4}$, (19) gives $x_0\sim 10^4 b$. In covering a distance $10^4 b$, the dislocation acquires 87 percent of the velocity of sound, in agreement with the calculations of Frank[9] and Leibfried and Dietze.[10]

V. THE GENERAL EQUATION OF MOTION

In principle the rectilinear motion of a dislocation in a given applied stress field could be found as follows. We generalize the Peierls-Nabarro equation to include time-dependent states. This will give $p_{zy}(x,t)$ on $y=\pm\frac{1}{2}a$ as an integral involving a general discontinuity $\delta w(x,t)$ in displacement across the slip plane. The stress so found at any point is to be equated to a prescribed function of δw at the same point. Finally, we have to find a solution of the resulting integral equation which has the character of a moving dislocation superimposed on the required applied stress field.

The generalized Peierls-Nabarro equation is easily set up. From the results of Sec. II it follows that, if $\delta w = H(x)H(t)$, then

$$p_{zy} = -\mu\surd(c^2t^2-x^2)H(ct-|x|)/2\pi ctx$$

at the slip plane. A general $\delta w(x,t)$ can be written as

$$\delta w(x,t)=\int_{-\infty}^{\infty}dx'\int_{-\infty}^{\infty}d\tau\frac{\partial^2\delta w(x',\tau)}{\partial x'\partial\tau}H(x-x')H(t-\tau),$$

and the corresponding stress will be

$$p_{zy}=-\frac{\mu}{2\pi}\int_{-\infty}^{\infty}dx'\int_{-\infty}^{\infty}d\tau\frac{\partial^2\delta w(x',\tau)}{\partial x'\partial\tau}\times\frac{\surd\{c^2(t-\tau)^2-(x-x')^2\}H\{c(t-\tau)-|x-x'|\}}{(x-x')c(t-\tau)}. \quad (20)$$

Integrating by parts with respect to τ, we have the generalized Peierls-Nabarro equation

$$f[\delta w(x,t)]=-\frac{\mu}{2\pi}\int_{-\infty}^{\infty}dx'\int_{-\infty}^{t-|x-x'|/c}d\tau\frac{\partial\delta w(x',\tau)}{\partial x'}\times\frac{x-x'}{c(t-\tau)^2\surd\{c^2(t-\tau)^2-(x-x')^2\}}, \quad (21)$$

where f is a function with period b reducing to $\mu\delta w/a$ for small δw.

The possibility of finding interesting solutions of (21) seems remote. Part of the difficulty lies in the fact that the solution would give us more than we need, the shape of the dislocation, specified by $\delta w(x,t)$, instead of merely its position, defined for example as the point where $\partial\delta w/\partial x$ has a maximum or minimum. As a first step we shall try to find an approximate equation of motion in the following way. The shape of the dislocation will be assumed to be independent of its motion, i.e.,

$$\delta w(x,t)=\delta w[x-\xi(t)].$$

We find the rate of flow $W(t)$ of energy into the slip plane. If W is known (for example if there are no dissipative processes and no accumulation of energy in the gap between the elastic half-planes of the Peierls model, so that $W=0$) this gives an integral equation connecting ξ with the applied stress. According to (10),

$$W=\int_{-\infty}^{\infty}(p_{zy}{}^A+p_{zy})\delta\dot{w}dx, \quad (22)$$

where $p_{zy}{}^A$ refers to the applied field and p_{zy} to the field of the dislocation. We assume that $p_{zy}{}^A$ depends on t but not on x. The first term in the integral is simply $bp_{zy}{}^A$, since we must have

$$\int_{-\infty}^{\infty}\frac{\partial\delta w}{\partial x}dx=-b$$

if the dislocation is to have strength b. In principle p_{zy} could be found from (20), but the following method makes it clearer what choice of δw will lead to a simple result. Returning to the electromagnetic analogy, suppose that there is a charge distribution,

$$\sigma=\eta\{x-\xi(t)\}\delta(y),$$

which is confined to the plane $y=0$ and which moves rigidly with velocity $\dot{\xi}(t)$. The current is $i_x=\eta\delta(y)\dot{\xi}$ and the potentials satisfy

$$\nabla^2\varphi-\ddot{\varphi}=-\eta\delta(y),\quad \nabla^2A_x-\partial^2A_x/\partial t^2=-\eta\delta(y)\dot{\xi}.$$

(We use units in which $c=1$.) The discontinuity in H_z across the x axis is equal to the strength of the current sheet $\eta\dot{\xi}$. Using Eq. (5) the elastic interpretation is that there is a discontinuity δw across the x axis for which

$$\delta\dot{w}=\eta\dot{\xi}/\surd\rho \quad\text{or}\quad \partial\delta w/\partial x=-\eta/\surd\rho.$$

The latter quantity does represent a "dislocation density," in the sense that a continuous distribution of infinitesimal dislocations along the x axis with total strength

$$db=-(\partial\delta w/\partial x)dx \quad (23)$$

between the points x and $x+dx$ would give the same elastic field. It is clear that the stresses will be unaltered if, to conform to the Peierls model, the two half-spaces $y\gtrless 0$ are separated by a gap of width a provided the same δw is maintained by the interatomic forces.

By (9a) the stress satisfies

$$\nabla^2p_{zy}-\ddot{p}_{zy}=\mu^{\frac{1}{2}}\{(1-\dot{\xi}^2)\partial\eta/\partial x+\eta d^2\xi/dt^2\}\delta(y). \quad (24)$$

In terms of the Fourier transforms,

$$\bar{p}(\mathbf{k})=\frac{1}{2\pi}\int_{\mathbf{r}}p_{zy}(\mathbf{r})e^{i\mathbf{k}\cdot\mathbf{r}}d\mathbf{r},$$

$$\bar{\eta}(\mathbf{k})=\bar{\eta}(k_x)=\frac{1}{2\pi}\int_{\mathbf{r}}\eta(x)\delta(y)e^{i\mathbf{k}\cdot\mathbf{r}}d\mathbf{r}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\eta(x)e^{ik_xx}dx$$

[9] F. C. Frank, Pittsburgh Symposium on Plastic Deformation, p. 89 (U. S. Office of Naval Research, 1950).
[10] G. Leibfried and H.-D. Dietze, Z. Physik 126, 781 (1949).

(so that $\bar{\eta}$ is the transform of $\eta\delta(y)$, not of η), Eq. (24) becomes the ordinary differential equation,

$$k^2\bar{p}+d^2\bar{p}/dt^2=-\mu^{\frac{1}{2}}\{d^2\xi/dt^2-ik_x(1-\dot{\xi}^2)\}\bar{\eta}(k_x,t),$$

with the solution:

$$\bar{p}=-\mu^{\frac{1}{2}}\int_{-\infty}^{t}\left\{\frac{d^2\xi(\tau)}{d\tau^2}-ik_x[1-\dot{\xi}^2(\tau)]\right\}\times\bar{\eta}(k_x,\tau)\frac{\sin k(t-\tau)}{k}d\tau. \quad (25)$$

It is easy to show that

$$\bar{\eta}(k_x,t)=\bar{\eta}(k_x,0)e^{ik_x\xi(t)},$$

assuming for convenience that $\xi(t)=0$ for $t=0$. According to Parseval's theorem,

$$\int_{\mathbf{r}} f_1(\mathbf{r})f_2(\mathbf{r})d\mathbf{r}=\int_{\mathbf{k}}\bar{f}_1(\mathbf{k})\bar{f}_2{}^*(\mathbf{k})d\mathbf{k},$$

if $\bar{f}_n$ is the Fourier transform of f_n. Applying this to (25) we have

$$\int_{-\infty}^{\infty}p_{zy}\delta\dot{w}dx=\rho^{\frac{1}{2}}\int_{\mathbf{r}}p_{zy}\eta\dot{\xi}\delta(y)d\mathbf{r}=\rho^{\frac{1}{2}}\dot{\xi}(t)\int_{\mathbf{k}}\bar{p}\bar{\eta}^*d\mathbf{k}$$

$$=\dot{\xi}(t)\int_{-\infty}^{t}d\tau\int_{\mathbf{k}}d\mathbf{k}\left\{\frac{d^2\xi(\tau)}{d\tau^2}-ik_x[1-\dot{\xi}^2(\tau)]\right\}|\bar{\eta}(k_x,0)|^2\times\exp\{ik_x[\xi(\tau)-\xi(t)]\}\frac{\sin k(t-\tau)}{k},$$

which gives the equation of motion,

$$bp_{zy}{}^A=\frac{W(t)}{\dot{\xi}(t)}+\pi\int_{-\infty}^{t}d\tau\int_{-\infty}^{\infty}dk_x\left\{\frac{d^2\xi(\tau)}{d\tau^2}-ik_x[1-\dot{\xi}^2(\tau)]\right\}\times\exp\{ik_x[\xi(\tau)-\xi(t)]\}|\bar{\eta}(k_x,0)|^2J_0\{k_x(t-\tau)\}. \quad (26)$$

We have now to choose an expression for δw. A physically reasonable form is

$$\delta w=\frac{b}{\pi}\tan^{-1}\frac{\frac{1}{2}a}{x-\xi(t)},$$

since it satisfies the Peierls equation approximately for small velocities. Then we have

$$\eta(x,0)=\frac{b\rho^{\frac{1}{2}}}{2\pi}\frac{a}{x^2+\frac{1}{4}a^2},\quad \bar{\eta}(k_x,0)=\frac{b\rho^{\frac{1}{2}}}{2\pi}e^{-\frac{1}{2}a|k_x|}. \quad (27)$$

The expression (26) can then be evaluated by expanding exp{ } in a power series and using the relation

$$\int_0^{\infty}e^{-zv}J_0(\rho v)v^n dv=\frac{n!}{(\rho^2+z^2)^{\frac{1}{2}n+\frac{1}{2}}}P_n\left\{\frac{z}{(\rho^2+z^2)^{\frac{1}{2}}}\right\}.$$

The result with $n=0$ is well known; regarding it as the potential of a point charge in cylindrical coordinates we reach the general case by differentiating n times with respect to z and comparing with the potential of a linear multipole in polar coordinates,

$$\partial^n r^{-1}/\partial z^n=(-1)^n n!r^{-n-1}P_n(\cos\theta) \quad \text{with} \quad r^2=\rho^2+z^2,\ z=r\cos\theta.$$

Because our physical assumptions are invalid for large velocities, it would be pointless to carry the expansion beyond terms of order $\dot{\xi}/c$. In addition the higher terms give a nonlinear relation between the stress and ξ and its derivatives. Because of the dependence on past history we must be content to give the remainder in terms of $V(t)$, the greatest value of $|\dot{\xi}(\tau)|$ for $\tau\leqslant t$. The result is

$$bp_{zy}{}^A=W(t)/\dot{\xi}(t)+\frac{\rho b^2}{4\pi}\int_{-\infty}^{t}\frac{d^2\xi(\tau)/d\tau^2}{\{(t-\tau)^2+a^2/c^2\}^{\frac{1}{2}}}d\tau+\frac{\rho b^2}{8\pi}\int_{-\infty}^{t}\frac{a^2/c^2}{\{(t-\tau)^2+a^2/c^2\}^{\frac{3}{2}}}\frac{d}{d\tau}\frac{\xi(t)-\xi(\tau)}{t-\tau}d\tau+O[V^2(t)/c^2]. \quad (28)$$

In estimating the remainder we use the fact that $[\xi(t)-\xi(\tau)]/(t-\tau)$, the mean velocity between t and τ, must be numerically less than $V(t)$ and that $|P_n(x)|\leqslant 1$.

In the electromagnetic analogy, Eq. (22) divided through by $\dot{\xi}$ expresses the fact that the total force on a rigidly moving charge distribution balances a retarding force $-W/\dot{\xi}$. Thus our method is equivalent to that of Lorentz for the electron. We could use the same argument for the dislocation provided we admit that there is a force $(p_{zy}{}^A+p_{zy})db$ on each element of the distribution (23) according to the usual rule. The analogy with Lorentz' method suggests that we might apply to a point dislocation [$a\rightarrow 0$ in Eq. (27)] recent methods which give an equation of motion for a point electron. This is not so, however. First, these methods introduce advanced potentials either explicitly or surreptitiously, and whatever they may signify in electrodynamics we cannot very well allow advanced quantities in our elastic problem, particularly as in two dimensions they would involve integrals over the whole future motion of the dislocation. Secondly, they eliminate the electromagnetic mass, allowing an arbitrary inertial mass to be ascribed to the electron, whereas a dislocation is clearly the analog of a weightless charged rod with purely electromagnetic mass.

Equation (26) is the true equation of motion of a hypothetical "rigid" dislocation. It takes no account of the change of shape of the function $\delta w(x,t)$ which a rigorous solution of (21) would give. For small accelerations the main feature of the change in shape would presumably be a Lorentz contraction appropriate to the instantaneous velocity $\dot{\xi}$. Because of the lack of contraction in our model, there are none of the relativistic effects encountered in Sec. IV. Indeed, nothing untoward happens to Eq. (26) even if the velocity is supersonic. It is easy to show from (26) and (27) that a moving supersonic dislocation would experience a retarding force $bp_{zy}{}^A=(v^2/c^2-1)^{\frac{1}{2}}\mu b^2/2\pi a$ even in the absence of dissipative effects. Physically, this is because it is continually creating a greater disturbed region, or from another point of view, because the leading elements of the dislocation density distribution exert a force on the trailing elements, but not conversely, since each element produces a disturbance only in a wake behind it. In principle a "rigid" dislocation can reach supersonic velocities, but not one which contracts to zero width as the velocity of sound is approached. Admittedly a dislocation which obeys the Peierls law (or some generalization of it) contracts, but the effective width of the source can hardly be much less than one interatomic spacing, so that a supersonic dislocation is a formal possibility.

VI. EXAMPLES AND DISCUSSION

Because nothing is certainly known about dissipative effects, we shall give some examples of dislocation motion on the assumption that W is zero. The possibility of a contribution to W from the nonlinear region $-\frac{1}{2}a<y<\frac{1}{2}a$ of the Peierls model is taken up later.

(a) Constant acceleration f starting from rest at $t=0$. Here $d^2\xi/dt^2=fH(t)$ and for $t>0$

$$bp_{zy}{}^A=\frac{\rho b^2}{4\pi}f\sinh^{-1}\frac{ct}{a}+\frac{\rho b^2}{8\pi}f\frac{t}{t+(t^2+a^2/c^2)^{\frac{1}{2}}}.$$

This gives an effective mass $(\rho b^2/4\pi)\ln(2e^{\frac{1}{2}}ct/a)$ for large t, agreeing roughly with the results of Sec. IV.

(b) Impulsive change of velocity from 0 to v at $t=0$. Here $d^2\xi/dt^2=v\delta(t)$ and $bp_{zy}{}^A$ can be found from the previous result by differentiating and multiplying by v/f. Roughly speaking, we have to apply a constant stress for a time of order a/c followed by a stress falling off as $1/t$ ever afterwards. Since the acceleration is zero, except initially, we cannot calculate a mass directly. However, the time integral of the applied force times the velocity is the work done by the applied stress, and if this is equated to $\frac{1}{2}mv^2$ we find the same effective mass as before.

(c) Sinusoidal oscillation: $d^2\xi/dt^2=(d^2\xi/dt^2)_0\cos\omega t$. The solution can be given in terms of Bessel and Struve functions of imaginary argument. For $\omega a/c\ll 1$ it becomes

$$bp_{zy}{}^A=m(d^2\xi/dt^2)_0\cos(\omega t-\alpha),$$

with

$$m=\frac{\rho b^2}{4\pi}\left\{\left(\frac{\pi}{2}\right)^2+\left(\ln\frac{2c}{e^\gamma\omega a}\right)^2\right\}^{\frac{1}{2}},$$

$$\tan\alpha=\tfrac{1}{2}\pi/\ln\left(\frac{2c}{e^\gamma\omega a}\right),\quad e^\gamma=1.78\cdots,$$

in rough agreement with Nabarro.[8]

As it stands, our equation of motion gives the stress required to maintain a given acceleration. Of more interest is the inverse equation giving the acceleration under a given applied field. The exact solution of the last problem, which connects a sinusoidal stress with a sinusoidal acceleration, could be used to invert (28) by expressing the arbitrary applied stress as a Fourier integral. The result is not very helpful and it is more convenient to use Laplace transforms. Consider the case where $p_{zy}{}^A$ (written simply p in what follows) is zero for $t<0$. If we introduce the Laplace transform,

$$\mathfrak{L}\{f\}=\int_0^\infty e^{-st}f(t)dt,$$

and use the result[11]

$$\mathfrak{L}\left\{\int_0^t f_1(t-\tau)f_2(\tau)d\tau\right\}=\mathfrak{L}\{f_1\}\mathfrak{L}\{f_2\},\qquad(29)$$

we find

$$\mathfrak{L}\{d^2\xi/dt^2\}=(8\pi/\rho b)\mathfrak{L}\{p\}[2h(x)+h''(x)-x^{-1}h'(x)+x^{-1}-2x^{-2}]^{-1},\qquad(30)$$

with

$$h(x)=\tfrac{1}{2}\pi\{\mathsf{H}_0(x)-Y_0(x)\},\quad x=sa/c,$$

where Y_0 is the usual second solution of Bessel's equation and H_0 is Struve's function. (Their difference is monotonic though each is oscillatory.) Formally this solves the problem of finding $d^2\xi/dt^2$ in terms of p; we have only to find the function of which $[\ ]^{-1}$ in (30) is the transform and apply (29). However, the necessary Mellin inversion integral is not simple since Y_0 is multiple-valued. But since

$$h(x)\sim x-\ln(\tfrac{1}{2}e^\gamma x),\quad x\ll 1;\quad h(x)\sim x^{-1},\quad x\gg 1,$$

we can find $\dot{\xi}$ for large and small t with the help of a theorem[11] which states that if

$$\mathfrak{L}\{f\}\sim s^{-\beta}L(s^{-1})\quad\text{for}\quad s\to 0(\infty),$$

where

$$L(ux)/L(x)\to 1\quad\text{for}\quad x\to 0(\infty),$$

then

$$\int_0^t f(\tau)d\tau\sim t^\beta L(t)/\Gamma(\beta+1)\quad\text{for}\quad t\to\infty\,(0).$$

(d) A constant stress p_0 is applied at $t=0$. Here $\mathfrak{L}\{p\}=p_0 s^{-1}$, and applying the theorem to (30), we have

$$\dot{\xi}(t)\sim\frac{4\pi}{\rho b}p_0\frac{t}{\ln(ct/e^\gamma a)},\quad t\gg a/c;$$

$$\sim\frac{8\pi a}{3\rho bc}p_0,\qquad t\ll a/c.$$

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[11] G. Doetsch, *Laplace-Transformation* (Julius Springer, Berlin, 1937).

For large t the dislocation gathers speed more slowly than a Newtonian particle; the effective mass is $(\rho b^2/4\pi)\ln(ct/e^\gamma a)$. As well as a free dislocation subjected to a suddenly applied force, we may consider (d) as describing a locked dislocation which breaks free when a gradually increasing stress reaches a certain value and also, though more schematically, a Frank-Read source which passes suddenly from a stable to an unstable state when a critical stress is reached.

(e) An arbitrarily varying stress is applied between $t=0$ and $t=t_1$, so that the total impulse is

$$P=\int_0^{t_1} bp(t)dt.$$

For $s\ll t_1$ $\mathfrak{L}\{p\}=P/b$ and the theorem gives

$$\dot{\xi}(t)\sim\frac{4\pi}{\rho b^2}P\frac{1}{\ln(ct/e^\gamma a)},\quad t\gg a/c,\ t\gg t_1.$$

A dislocation started by an impulse and then allowed to run free gradually loses speed. Again a mass proportional to $\ln(ct/e^\gamma a)$ is appropriate since Newton's law, $d(m\dot{\xi})/dt=0$, is then satisfied during the free motion.

Like the example of Sec. IV the results (a), (b), (d), and (e) fit Frank's picture of a mass $(\rho b^2/4\pi)\ln(r_1/a)$, with r_1 of the order of the radius of the disturbed region surrounding the starting point. In all these cases ξ is a monotonic function of t, and presumably the elastic field in the disturbed region, logarithmically speaking, does not differ much from that round a uniformly moving dislocation. For a more complicated motion (e.g., oscillation) this is no longer true.

It can be shown that the results we have obtained are largely independent of the exact form of the density distribution η provided that, like the particular form (27), it is a bell-shaped curve with a width of order a and enclosing an area $\rho^{\frac{1}{2}}b$.

Even if we assume that there are no dissipative effects, W will not be zero if there is any variation in the energy stored in the gap at the slip plane. The calculation of this energy is difficult because of a certain ambiguity in extending the Peierls approximation to dynamic problems. If we take as our model two semi-infinite elastic solids separated by a gap a, the contribution to W is zero since the weightless interatomic forces have no kinetic energy and their total potential energy is independent of time, depending only on the shape of the prescribed function δw. On the other hand, if we apply the Peierls recipe of replacing the crystal lattice by a continuum only outside the gap, we are left with a number of half-atoms adhering to the faces of the gap, and their kinetic energy changes at a rate

$$2\times\tfrac{1}{2}\rho a\frac{\partial}{\partial t}\int_{-\infty}^{\infty}(\tfrac{1}{2}\delta\dot{w})^2dx=\frac{\rho b^2}{2\pi}\frac{d\xi}{dt}\frac{d^2\xi}{dt^2},$$

if we use the $\delta\dot{w}$ corresponding to (27). The W term in (28) becomes $\rho b^2(d^2\xi/dt^2)/2\pi$, an addition of "inertial" mass to the "electromagnetic" mass represented by the other terms. The modification to our examples is small.

If this or some similar account of the effect of the inertia of the matter in the slip plane is accepted, the results of Sec. IV and the generalized Peierls equation (21) need modification. In particular this spoils the elegant result that an antiplane elastic field satisfying the Peierls condition, or some generalization of it, continues to do so when moving uniformly with an appropriate Lorentz contraction.

It may be as well to contrast our calculation of the effect of the inertia of the material in the slip plane with Nabarro's.[8] In his problem the contribution to $\dot{w}$ at the slip plane from the applied field (a sound wave) is not zero, and the half-atoms above and below the slip plane move in the same direction. In our problem it is assumed that the external stress is applied in such a way that $\dot{w}$ is an odd function of y, and so the half-atoms move in opposite sense above and below the slip plane. The corresponding term in Nabarro's calculation is of an order higher than he retains.

We should also make a slight correction to allow for the fact that strictly not only the dislocation but also the applied elastic field makes a contribution to $\delta\dot{w}$ in (22). If we assume that this contribution is approximately $a\partial^2w^A/\partial y\partial t$, as it would be if there were no nonlinearity at the slip plane, it can be shown that a term $ab\dot{p}_{zy}{}^A/2c$ must be added to the left-hand side of (28). This will be partly offset by a contribution of the same order to the term in W; the exact value depends on the details of the nonlinear behavior in the slip plane. In any case the correction will be negligible if $d(\ln p_{zy}{}^A)/dt\ll c/a\sim 10^{13}$.

In conclusion it should be emphasized that the foregoing is only a first step towards solving the much harder problem of the motion of a dislocation loop. It is not clear how much light the present calculations throw on the general problem. The curious behavior expressed by the integral equation (28) is, of course, due to the fact that a given element of the dislocation is acted on not only by the applied stress, but also by delayed disturbances from other parts of the dislocation line. The same effect will presumably be important in determining how a dislocation loop spreads under the influence of an applied stress.

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