The Energy of Nucleon-Nucleon Collisions*

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The problem of determining the energy of the primary particle in a nucleon-nucleon collision at extremely high energies is discussed. A new nomographic method is proposed which is less subject to fluctuations than most estimates used previously. The method allows a redetermination of the energy of the "S" star, which confirms the previous estimate.

A NUMBER of authors have recently analyzed nuclear events of extremely high energy.[1-11] The problem of determining the energy of the primary has been attacked by various methods, some of which will be discussed below. For a complex nucleus interacting with another particle the analysis is complicated by various kinds of phenomena usually referred to as "plural," that is, a succession of multiple or single events inside the same nucleus. As we shall discuss, the symmetry of an extremely high energy nucleon-nucleon collision allows one to draw some conclusions with a fair degree of confidence.

We will denote quantities in the laboratory (L) system of coordinates with unprimed letters and the corresponding quantities in the center-of-mass (C) system with primed letters. The direction of motion of the primary is chosen along the positive X axis in both systems. (See Fig. 1.) If the velocity of the C system as measured in the L system is β, measured in units of the velocity of light c, then applying the well-known formulas of the Lorentz transformation to the X component of momentum of the ith particle, we obtain

$$p_{x_i}'=\gamma(p_{x_i}-\beta E_i/c), \quad \gamma=1/(1-\beta^2)^{\frac{1}{2}}, \tag{1}$$

where p_{x_i} is the X component of the momentum of the ith particle and E_i is the total energy of the ith particle. If we write Eq. (1) for $i=1, \cdots, n$ (where n is the total number of secondary particles, both charged and uncharged), we get the X component of the total momentum, which must be zero in the C system:

$$0=P_x'=\sum_{i=1}^{n} p_{x_i}'=\sum_{i=1}^{n} \gamma\left(p_{x_i}-\beta\frac{E_i}{c}\right)$$

$$=\gamma\left(\sum_{i=1}^{n} p_{x_i}-\beta\sum_{i=1}^{n}\frac{E_i}{c}\right). \tag{2}$$

From (2) we get

$$\beta=c\sum_{i=1}^{n} p_{x_i}/\sum_{i=1}^{n} E_i. \tag{3}$$

Using the value of β found in Eq. (3) one gets the following exact expression for the total energy of the primary:

$$E_p=Mc^2(1+\beta^2)/(1-\beta^2), \tag{4}$$

where M is the nucleon mass.

Three points must be discussed in connection with the use of Eqs. (3) and (4):

1. Both energy and momentum of the particles enter into this formula, but at very high energies these quantities cannot be separately determined with present day techniques.

2. Some of the secondaries may be neutral and hence unobserved, except through subsequent decays or secondary interactions.

3. At extremely high energies, even some of the secondaries are beyond the range of even the best multiple scattering measurements.

We can deal with these problems as follows:

1. In the region beyond the relativistic increase in ionization[12,13] the energy is so much greater than the rest energy that the relation $pc=(E^2-m_0c^2)^{\frac{1}{2}}$ reduces to $pc\simeq E$. Let θ_i be the angle between the X axis and the momentum vector of the particle (in the L system), so $p_{x_i}=p_i\cos\theta_i$. Then, using Eq. (3), where we assume all the particles are in the relativistic region, we have

$$\beta=\sum_{i=1}^{n} E_i\cos\theta_i/\sum_{i=1}^{n} E_i. \tag{5}$$

To within the approximations used in the derivation of the formula, the quantities E_i, the particle energies, can be determined, for example, by multiple scattering measurements.[14,15]

2. At such high energies that spin interactions can be neglected, which may be reached when the primary energy is of the order of several Bev, one can assume that a nucleon-nucleon event has in the C system a

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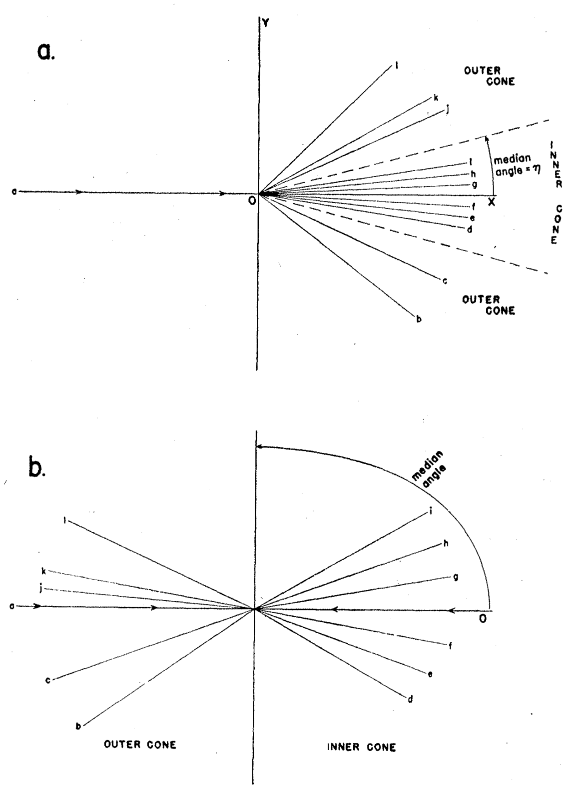

FIG. 1. Nucleon-nucleon collision. a. Laboratory system; b. Center-of-mass system. *a*: incident nucleon; *b-l*: secondaries; *o*: target nucleon.

distinguished axis along the direction of relative motion (X axis) of the primary particles but no other distinguished direction. It follows then that the collision should have axial symmetry about the X axis and also reflective symmetry in the plane perpendicular to this axis at the point of collision. Then the probability that any neutral particle has any given momentum p' is the same as the probability that it has the momentum $-p'$. Thus, the neutral particles will have a total momentum zero in the C system, so that Eq. (2) and hence (3), which was derived from (2), still hold, on the average. This, however, may be subject to fluctuations in individual events.

3. From Eq. (1) and the symmetry argument already referred to, it has been observed that any particle is equally likely to go forward or backward in the C system. Hence, if a particle is at the median angle, i.e., that angle such that half the particles are inside and half outside the cone with that half-angle, its X component of momentum must be close to zero in the C system. (Fig. 1.) Thus, some authors[9] have used the equation

$$\beta=(pc\cos\eta)/E, \tag{6}$$

derived from (1) by setting p_{x_i}' equal to zero. In this equation E, p are the energy and momentum of that particle which is at the median angle η (L system). One may, in the region of very high energies used the approximate relation $pc\simeq E$ and get for β and E_p the following relations:

$$\beta\simeq\cos\eta,\quad E_p=(2-\sin^2\eta)/\sin^2\eta\simeq 2/\eta^2. \tag{7}$$

The principle objection to this form of estimate is based on the fact that it is subject to large fluctuations. In cases where the angular distribution in the C system is bimodal, that is, peaked at angles near zero and 180° (see Fig. 2), which is predicted for collisions of large impact parameter[16] and has been observed,[3] the fluctuations are exceptionally marked. We are proposing another method which is not as subject to these fluctuations.

As mentioned earlier, if we have a nucleon-nucleon collision of such high energy that spin interactions may be neglected, then it can be assumed that in the C system the only spatial restrictions to the general homogeneity of space are at the point at which the collision occurs and in the directions of motions of the primary particles. This means that the system must have rotational symmetry around this axis and reflective symmetry in the $Y'-Z'$ plane. In particular, one concludes that the probability of any particle being emitted at an angle θ' with the X' axis is independent of the orientation with respect to the Y' axis and is equal to the probability of its being emitted at the angle

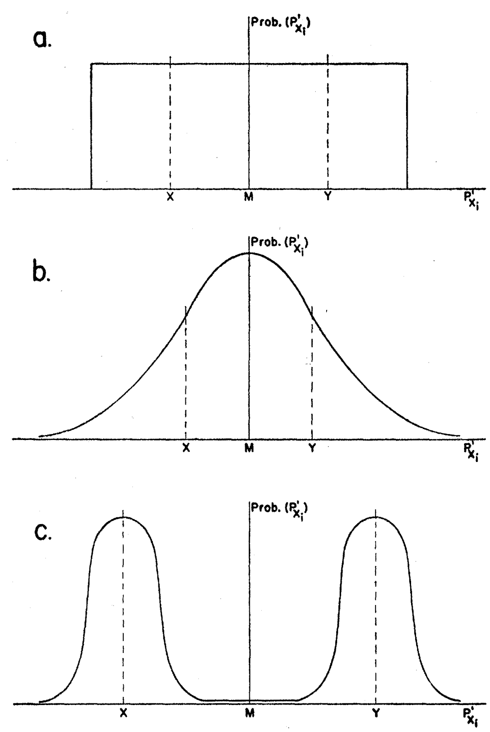

FIG. 2. Typical symmetric distributions. a. Uniform or rectangular distribution; b. Gaussian or normal distribution; c. Extreme bimodal distribution. X: lower quartile; Y: upper quartile; M: median.

[16] E. Fermi, Phys. Rev. **81**, 683 (1951).

$180°-\theta'$. In other words, the probability of a given particle having the X component of momentum, p_x', is independent of the ratio of p_y' to p_z' and is the same as the probability of $-p_x'$. Owing to this fundamental symmetry which is assumed to hold, not only with respect to all the particles but also with respect to the charged particles alone, we can determine the transformation from the L system to the C system by the requirement that we get a distribution in the transformed system which satisfies the above-mentioned symmetry properties.

To put this in a practical form, however, we need to characterize the symmetry property in terms of a parameter lending itself to convenient computation. The best parameter to use, in general, is the mean of the momenta, but as we have discussed above, this is not applicable in our situation, since we cannot determine all of the momenta. However, we can make a theoretical comparison of other estimates with this mean. A common parameter is for instance the median, that point on the momentum distribution (Fig. 2) such that half the secondary particles lie below and half above it. For the normal distribution[17] the efficiency of the median is 64 percent of the efficiency of the mean. For the bimodal distributions [see Fig. 2(c)] it is much less efficient since it is based on an area where the probability of finding a point is low. For this reason we choose to look at the quartiles, that is, the points such that one-fourth of the particles lie above the upper quartile, as shown in Fig. 2. For the normal distribution[17]

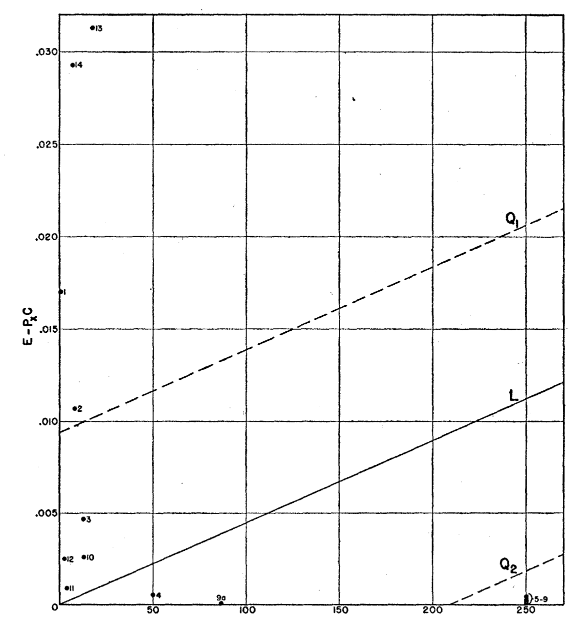

FIG. 3. Analysis of "S" star. Numbering of particle tracks is from right to left in Fig. 1 of Lord, Fainberg, and Schein (reference 3). Tracks 5–9 correspond to 250 Bev or greater; all other energies are measured.

[17] F. Mosteller, Ann. Math. Stat. **17**, 377 (1946).

TABLE I. "S" star energy determinations (in electron volts).

	Median method	Quartile method
Energy of star	5×10^{12}	2×10^{13}
Energy given by random subsets		
A	2×10^{14}	10^{14}
B	4×10^{13}	2.3×10^{13}
C	3.5×10^{12}	2.3×10^{13}
D	5×10^{12}	5×10^{13}
Factor between highest and lowest value of subsets	57	4.3

the quartiles have an efficiency of 81 percent, for other distributions they may be less of an improvement over the median, but for the bimodal distributions, as can be seen in Fig. 2, they represent a far superior estimate of the mean than the median does. It should be noted that there is no fundamental reason to choose just the quartiles as the points to be used. For the normal distribution[17] the points 27 percent from each end of the distribution are the quantiles of maximum efficiency.[18] It is recommended that quantiles as close to this as possible be chosen, consistent with the basic requirement that points near the quantiles must represent particles whose energy can be measured.

A simple nomographic way of representing the derivation of the energy of the primary particle from the momentum distribution has been derived. Since the relation (1) involves only the X component of momentum and energy, we can represent the transformation as a linear transformation of the $p_{xi}-E_i$ plane into itself. While the transformation is not a rotation, it affects the E axis as though it were a rotation. We are interested, as discussed above, in finding a new E axis, such that the secondary particles lie symmetrically with respect to it; this is the interpretation given to the requirement of symmetry which we have imposed. The interpretation given to the median estimate which we have discussed is that it is the line through the origin such that half of the particles lie on either side of it. The interpretation of the quartile estimate is that it is the line L (see Fig. 3) through the origin, such that the two lines Q_1 and Q_2, parallel to L and equidistant from it, divide the distribution into quartiles, i.e., $\frac{1}{4}$ of the particles lie above Q_1 and $\frac{1}{4}$ lie below Q_2. Since the particles generally found in events of this type are all of such high energy that their velocity is near the speed of light, what one does in practice is to plot the $E_i-p_{xi}c$ plane on transformed axes, which allow an easy expansion of scale. If one did not do this, all the points would lie almost on the straight line which represents the equation $E_i=p_{xi}c$. Thus, one plots $E_i-p_{xi}c$ against E_i. The result of this transformation is that the slope of the transformed E' axis which is obtained from the graph is an estimate of $1-\beta$. It is convenient in making the quartile estimate

[18] The efficiency is only 0.2 percent greater than for the quartiles.

to have available a ruler with several lines spaced evenly on each side of a central line.

In Fig. 3 we have plotted the points obtained from the "S" star, which has been carefully measured in this laboratory.[3] The results of applying both the median and quartile methods to this star to determine its energy are given in Table I. In addition to this, we used the following procedure to test the sensitivity of the methods to fluctuations. We took a random subset of ten particles and applied each method to this subset. This was repeated four times, using the same subset for measurements by both methods. The range of energies given by the method was used as a measure of the deviations to be expected. The table indicates also the values obtained for these measurements and the factor between the highest and lowest measurement. It can easily be seen that for stars of this type, with a pronounced separation into two cones, the method of the quartiles is much superior to the method of the medians since the fluctuations are much smaller.

On the basis of this analysis, one can conclude that the energy of the S star is $2^{+1.45}_{-0.85}\times10^{13}$ ev. The estimated errors include only statistical fluctuation errors, estimated as indicated above, and not experimental errors. This estimate compares well with the estimate in the original paper[3] of 3×10^{13} ev based in part on the interpretation of the two cones contained in Fermi's theory.[16]

The method was also applied to the star of Hopper, Biswas, and Darby.[8] This star contained only six charged particles of which only four had measurable energies, so that there are inherent large limitations on the accuracy of any energy estimate. However, the method gives an estimate of 2×10^{12} ev, in agreement with the conclusions of the authors.

In general, one can conclude that the method contained in this paper of determining the energy of stars which result from extremely high energy nucleon-nucleon collisions by the use of the quartiles of the momentum distribution is a useful computational technique.

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Extension of Heisenberg's Model of Turbulence to Critical Reynolds Numbers

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By considering a few extra physical complications introduced by the presence of boundaries, small velocities, etc., it is possible to extend Heisenberg's methods to the calculation of critical Reynolds numbers. Calculations are carried out for Couette, plane Poiseuille, and Blasius flow. The results are found to be within about 20 percent of the observed values.

USING the dimensional relations on which Heisenberg's theory of isotropic homogeneous turbulence is based,[1] it is possible to extend the treatment of turbulence to the anisotropic, inhomogeneous field. To demonstrate this extension let us consider the problem of computing critical Reynolds numbers. Let us consider a steady state flow $\mathbf{v}_0$. We shall investigate the instability of an infinitesimal perturbation $\mathbf{v}_1$. To describe the perturbation we introduce the parameter k which is an inverse measure of the scale of $\mathbf{v}_1$, though it is not to be interpreted outright as a wave number arising from a Fourier transformation. (Later on we shall specify $2\pi/k$ to refer to the linear size of an eddy.) We represent the energy density by the function $F(k)$. Again let it be stated that $F(t)$ is not to be interpreted directly as the Fourier transform of the correlation function. We define $F(k)$ for the time being only by analogy, based on dimensional arguments, to its Fourier counterpart: thus we let $2F(k)k^3$ be the increase in $\langle(\text{curl}\mathbf{v})^2\rangle_{\text{Av}}$ resulting from the perturbation $\mathbf{v}_1$, i.e.,

$$2F(k)k^3=2\langle\text{curl}\mathbf{v}_0\cdot\text{curl}\mathbf{v}_1\rangle_{\text{Av}}+\langle(\text{curl}\mathbf{v}_1)^2\rangle_{\text{Av}}. \quad (1)$$

The angular brackets denote a time average or an average over the members of an ensemble. We have chosen to define $F(k)$ in terms of the curl of the velocity rather than the velocity itself in order that $F(k)$ be independent of the translation of the particular observer involved.

After the manner of Heisenberg we construct the eddy viscosity by dimensional arguments as

$$\nu_e=\beta\langle|\text{curl}\mathbf{v}_1|\rangle_{\text{Av}}/k^2, \quad (2)$$

where β is a constant. Now, for fully developed turbulence there is little or no correlation between $\text{curl}\mathbf{v}_0$ and $\text{curl}\mathbf{v}_1$. Thus, one obtains

$$2F(k)k^3=\langle|\text{curl}\mathbf{v}_1|^2\rangle_{\text{Av}}, \quad (3)$$

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