Calculation of Peaked Angular Distributions from Legendre Polynomial Expansions and an Application to the Multiple Scattering of Charged Particles*

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A method for evaluating Fourier integrals is presented which relies upon qualitative information about the integral and which combines some of the advantages of both numerical and analytic techniques. A formula is then derived which sums slowly convergent series of Legendre polynomials by making use of a Fourier integral "small angle" approximation. A combination of the two techniques is used to sum the slowly convergent Legendre polynomial series which represents the directional distribution of multiply scattered charged particles. Comparisons with "small angle" calculations by Molière and Snyder-Scott are included.

1. INTRODUCTION

THE directional distribution of charged particles multiply scattered in a thin foil is typical of many physical problems which involve an angular distribution strongly peaked at some particular angle. In this problem as in many others the basic treatment is greatly simplified by making expansions in spherical harmonics; however the series which then represents the desired peaked angular distribution will be very slowly convergent. This slowly convergent series can be summed by making a "small angle approximation," i.e., by transforming the sum into a Fourier integral, which can usually be more easily evaluated than the sum. In the charged particle problem Molière[1] has been able to evaluate such a Fourier integral through the use of a suitable analytic representation and expansion. Snyder and Scott[2] have applied laborious standard numerical integration techniques to the same problem. The first part of this note suggests a method for making Fourier integrations which combines some of the advantages of both numerical and analytical techniques. It has flexibility in that it can be applied in a wide variety of circumstances, being not particularly dependent on a simple analytic form for the integrand. On the other hand it uses analytic expressions in making the integrations, thereby offering the advantages of quick and easy analytic manipulations. We shall illustrate this method by using it to reproduce some of the results of Snyder and Scott (Sec. 3).

The other purpose of this note is to demonstrate a method for summing Legendre polynomial series exactly, but taking advantage of a Fourier integral "approximation." We derive the basic summation formula which enables us thus to dispense with the "small angle approximation" in Sec. 4. We shall illustrate this technique also by applying it to a particular problem (Secs. 5 and 6) and comparing with the "small angle approximation" results of Molière.

Further application of these techniques will be made in a following paper which will contain a full discussion of the problem of multiple scattering of relativistic electrons in a thin foil. A comparison with experiments by Hanson *et al.*[3] will be obtained which utilizes more realistic scattering cross sections than those of Molière and Snyder-Scott.

2. DISCUSSION

In a previous paper,[4] complicated functions are represented by approximate forms which require only qualitative information reinforced by a knowledge of a few well-known parameters of the exact functions. These approximate forms have proved adequate for many purposes. For example, in reference 4 certain integrals were calculated by combining the qualitative information that an unknown factor in the integrand is smooth and single-peaked with a knowledge of a few moments of this unknown factor.

We want to apply this same idea to evaluate the Fourier integrals which appear in the Molière problem. We propose to do this by approximating the integrand with a set of terms which are chosen according to three criteria:

(a) Each term must have a known or easily determined Fourier transform.

(b) Each term must agree with the qualitative information about the integrand or its Fourier transform. Examples of this qualitative information might be smoothness and positiveness, or the known behavior of the integral or integrand for large or small values of the independent variable.

(c) The set of terms must contain constants which are chosen to fit exactly certain numbers characterizing the integrand. These numbers might be integrals, values, or derivatives of the integrand. The choice of parameters which are to be fitted may be made in such a way that the Fourier integral is given particularly accurately for a certain range of values of the independent variable.

What we are proposing is a sort of generalized nu-

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[1] G. Molière, Z. Naturforsch. 3A, 78 (1948). *Note added in proof:* —See also a further development of the Molière technique by H. A. Bethe, Phys. Rev. 90, 000 (1953).

[2] H. S. Snyder and W. T. Scott, Phys. Rev. 76, 220 (1949).

[3] Hanson, Lanzl, Lyman, and Scott, Phys. Rev. 84, 634 (1951).

[4] L. V. Spencer, Phys. Rev. 88, 793 (1952).

merical integration technique. We wish to emphasize that the accuracy of numerical integration techniques increases rapidly with the amount of qualitative information which is utilized.

3. THE FOURIER INTEGRAL OF MOLIÈRE AND SNYDER-SCOTT

In order to obtain the (projected) angular distribution $F_1(\omega)$ of originally monodirectional charged particles multiply scattered by a thin foil, Molière and Snyder-Scott evaluate the following Fourier integral [reference 2, Eq. (8, 2)]:

$$F_1(\omega)=(2\pi)^{-1}\int_{-\infty}^{\infty} d\sigma e^{i\omega\sigma}\exp\{-A[1-L_1(\sigma)]\}, \qquad (1)$$

where A is a constant, and we use the notation[5] $L_\nu(\sigma)=\sigma^\nu K_\nu(\sigma)$, K_ν being the Bessel function of the second kind with imaginary argument as given by Watson.[6] The letter ω refers to the total deflection angle divided by a scaling factor which is immaterial for our present purpose.

The available qualitative information about $F_1(\omega)$ is:

(a) $F_1(\omega)$ is a smooth, positive, single-peaked function of ω^2.

(b) $F_1(\omega)$ tends to be Gaussian for small values of ω.

(c) For large values of ω, F_1 can be calculated by expanding $\exp\{-A[1-L_1(\sigma)]\}$, i.e.,[7]

$$F_1(\omega)\rightarrow(2\pi)^{-1}\int_{-\infty}^{\infty} d\sigma e^{i\omega\sigma} A L_1(\sigma)=(A/2)(1+\omega^2)^{-\frac{3}{2}}. \qquad (2)$$

Furthermore, if ω is small F_1 is essentially an integral over the function $\exp\{-A[1-L_1(\sigma)]\}$, each value of σ contributing to the result according to the magnitude of this exponential function. This suggests that for small ω accurate answers can be obtained if an approximate function is fitted to a series of *values* of this exponential function distributed over, e.g., the range $1\geqslant\exp\{-A[1-L_1(\sigma)]\}\gtrsim\frac{1}{10}$.

On the other hand, if ω is large F_1 is essentially determined by the region of small σ, as illustrated, for example, by (c). This suggests that a good approximation for large ω, which calls for an accurate representation of $\exp\{-A[1-L_1(\sigma)]\}$ for small σ, can be obtained by using *derivatives* of $\exp\{-A[1-L_1(\sigma)]\}$ evaluated at some small value of σ.

A. Approximation by Gaussians

These considerations suggest that for small ω we approximate the integrand by Gaussians:

$$\exp\{-A[1-L_1(\sigma)]\}\approx\textstyle\sum_n a_n\exp(-\alpha_n\sigma^2), \qquad (3)$$

where the a_n and α_n are determined so that the approximation agrees with the exact function at values of σ distributed over a range determined by

$$1\geqslant\exp\{-A[1-L_1(\sigma)]\}\geqslant\tfrac{1}{10}.^8$$

Since Gaussians transform to Gaussians, this yields the approximation:

$$F_1(\omega)\approx\frac{1}{2\sqrt{\pi}}\sum_n a_n\alpha_n^{-\frac{1}{2}}\exp[-\omega^2/(4\alpha_n)]. \qquad (4)$$

Sample calculations of this type indicate that the approximation (4) with two or three terms is accurate to within 3 percent for values of ω in the range

$$1\geqslant F_1(\omega)/F_1(0)\gtrsim 0.02.$$

In this simple calculation, no account has been taken of (c). This additional information may be introduced in two ways. We write either

$$\exp\{-A[1-L_1(\sigma)]\}\approx\textstyle\sum_n b_n\exp(-\beta_n\sigma^2)+B_0L_1(B_1\sigma), \qquad (5)$$

or

$$\exp\{-A[1-L_1(\sigma)]\}\approx\{\textstyle\sum_n c_n\exp(-\gamma_n\sigma^2)\}L_1(C\sigma), \qquad (6)$$

where B_0, B_1, and C are assigned values at the outset which will insure the correct asymptotic behavior. The constants b_n, β_n, c_n, and γ_n are chosen in each case so that the approximate function agrees with the exact one at values distributed over the range $1\geqslant\exp\{-A[1-L_1(\sigma)]\}\geqslant\frac{1}{10}$.[8] Of the two forms the more meaningful and more accurate is (6). Each term is the sum (6) refers to a Gaussian diffusion superimposed on a single scattering which very nearly obeys the correct differential scattering law. This form has the drawback that the angular distribution is represented by a folding integral:

$$F_1(\omega)\approx\int_{-\infty}^{\infty}d\omega'\left\{\frac{1}{2\sqrt{\pi}}\sum_n(c_n\gamma_n^{-\frac{1}{2}})\times\exp[-(\omega-\omega')^2/(4\gamma_n)]\right\}\{(C^2/2)(C^2+\omega'^2)^{-\frac{3}{2}}\}. \qquad (7)$$

Fortunately, this is a practicable integration to perform numerically.

For a sample calculation of this type we chose $A=C^2=100$. Three Gaussians were used and the c_n, γ_n were determined so that the approximate function agreed with $\exp\{-A[1-L_1(\sigma)]\}$ at $\sigma^2=0$, 0.004, 0.008, 0.012, 0.016, and 0.020.[8] The resulting angular distribution obtained from expression (7) is given in the second column of Table I. It compares quite well with Snyder-Scott results for $F_1(\omega)/F_1(0)\gtrsim 0.005$. For larger values of ω, differences of $\sim$6 percent appear. If the tabulation had been continued to still larger, ω, the

[5] Compare, for example, with Eq. (A.2), G. Molière, Z. Naturforsch. 2A (145), 1947.

[6] G. N. Watson, *Bessel Functions* (MacMillan Company, New York, 1945).

[7] See reference 6, p. 172. Also, see Campbell and Foster, *Fourier Integrals* (D. Van Nostrand Company, Inc., New York, 1948), p. 125.

[8] For an account of the fitting procedure see reference 4, Appendices B and C.

TABLE I. A comparison of Gaussian and inverse power approximations with the Snyder-Scott results for $A=100$.

ω	Snyder-Scott	Gaussian approx.	Inverse power approx.	ω	Snyder-Scott	Gaussian approx.	Inverse power approx.
	$\times 10^{-6}$	$\times 10^{-6}$	$\times 10^{-6}$		$\times 10^{-6}$	$\times 10^{-6}$	$\times 10^{-6}$
0	22850	22740		120	34.38		34.79
10	18740	18670		140	20.76		20.89
20	10780	10780		160	13.52		13.57
30	4904	4905		180	9.132		9.329
40	2041	2051	1833	200	6.691		6.699
50	888.4	887.9	836.7	220	4.973		4.976
60	430.0	428.1	423.7	240	3.798		3.800
70	236.3	233.4	237.1	260	2.968		2.969
80	140.8	137.1	144.4	280	2.364		2.364
90	93.34	89.39	94.23	300	1.913		1.914
100	63.92	60.22	64.92	330	1.430		1.430
110	45.96	43.14	46.71				

discrepancy would have increased slightly and then decreased again as $\omega \rightarrow \infty$.

B. Approximation by Inverse Powers

For large ω, F_1 tends to obey an inverse power law. The first idea which springs to mind, therefore, is to approximate by functions of the type $(K^2+\omega^2)^{-\kappa-\frac{1}{2}}$, where K and κ are constants. This form is not only an inverse power for large ω, but it is also capable of behaving like a Gaussian for small ω. Because of the Fourier relationship,[7]

$$(2\pi)^{-1}\int_{-\infty}^{\infty} d\sigma e^{i\omega\sigma} L_\kappa(K\sigma) = \frac{(2\kappa^2)^\kappa(\kappa-\frac{1}{2})!}{2\sqrt{\pi}(K^2+\omega^2)^{\kappa+\frac{1}{2}}}, \quad (8)$$

we write

$$\exp\{-A[1-L_1(\sigma)]\} \approx \sum_n k_n L_{\kappa n}(K\sigma). \quad (9)$$

Knowledge of the exact asymptotic trend leads us to specify the constants in one term of (9) in such a way that the asymptotic trend will be given correctly, i.e., $\kappa_0=1$, $k_0=AK^{-2}$.

The function $(K^2+\omega^2)^{-\kappa-\frac{1}{2}}$ behaves like a Gaussian for $\omega^2 \ll K^2$ and like an inverse power for $\omega^2 \gg K^2$. Thus $\omega=K$ is a transition region between the two types of behavior. This suggests that K be identified with the abscissa of the point of inflection of the function $\log F_1(\omega)$.[9] This value for K can be determined from the simple Gaussian approximation described in A.

As discussed early in this section we want to fit the approximate function to the exact function by means of derivatives or combinations of derivatives evaluated at small σ. We make use of the relations:

$$\{S\}L_\nu(\sigma) = \left\{-\frac{1}{\sigma}\frac{d}{d\sigma}\right\}L_\nu(\sigma) = L_{\nu-1}(\sigma), \quad (10)$$

$$\{T\}L_\nu(\sigma) = \{-\tfrac{1}{2}(\sigma^2 S^2 - 1)\}L_\nu(\sigma) = (\nu-1)L_{\nu-1}(\sigma). \quad (11)$$

If we apply to (9) the operators $\{S^3\}$, $\{TS^2+3S^3\}$, $\{T^2S+5TS^2+9S^3\}$, and $\{T^3+6T^2S+19TS^2+27S^3\}$, and if we evaluate the resulting four equations at the same small value of σ, we obtain four nonlinear equations which can be solved simultaneously.[8] This enables us to determine four constants and include two terms in the sum (9) in addition to the term giving the asymptotic trend exactly. The recurrence relations of the L_ν's enable us to calculate the desired derivatives of the left side of (9) quite readily.

Calculations of this nature have been accomplished for the case $A=100$. We chose $K=35$, which is about the value for which $(d^2/d\omega^2)\log F_1(\omega)=0$. The derivatives were all evaluated at $\sigma^2=10^{-4}$. During the solution of the set of nonlinear equations it becomes necessary to evaluate L_ν for nonintegral ν. There are series expansions which can be used for this purpose; however we preferred to obtain the desired values from graphical interpolation using the easily obtained integral[10] and half-integral L_ν's. The final results of the calculation are given in the third column of Table I. They agree well with Snyder-Scott for $\omega \gtrsim 55$. For small ω this approximation is in error by 10–20 percent.[11]

As shown by Table I, there is a considerable overlap between the Gaussian and inverse power approximations. A combination of the two yields an approximation accurate everywhere to better than 3 percent.

It should be borne in mind that the calculations described in this section represent but two of a number of ways in which the general approach described in Sec. 2 may be used to evaluate the Fourier integral (1).

The number of man hours involved in one of these calculations may be of interest. Assuming that procedural details have been ironed out, but allowing rather generously time for checking and for tabulating the final results, we estimate that either of the two types of calculation requires about 10–12 hours.

4. A SUMMATION FORMULA FOR LEGENDRE POLYNOMIALS

The actual summation of a slowly convergent Legendre polynomial series will now be discussed. We will derive a summation formula which converts a Fourier integral approximation into the exact sum. Thus, suppose we want to sum the series

$$G(\vartheta) = \sum_{l=0}^{\infty} g_l P_l(\cos\vartheta). \quad (12)$$

We let $g(l+\frac{1}{2})$ be any continuous function of $(l+\frac{1}{2})$ over the range $-\infty \leqslant (l+\frac{1}{2}) \leqslant \infty$ which has the property that for positive, integral values of l, $g(l+\frac{1}{2})=g_l$. (This does not define $g(l+\frac{1}{2})$ uniquely. Each of the various functions satisfying these criteria is equally good for our purposes.) If $g(l+\frac{1}{2})$ is, e.g., an antisymmetric

[9] On semilog paper a Gaussian becomes monotonically steeper whereas an inverse power becomes monotonically flatter.

[10] Tables of the Bessel Functions $Y_0(x)$, $Y_1(x)$, $K_0(x)$, $K_1(x)$, $0 \leqslant x \leqslant 1$, Natl. Bur. Stand. Applied Mathematical Series 1, February 12, 1948.

[11] In a separate calculation of this same type we have achieved an accuracy everywhere to about 7 percent. This was done by choosing K in such a way that the approximation was correct at $\omega=0$. The value of K determined in this way was slightly higher, i.e., 45 instead of 35.

function of $(l+\frac{1}{2})$ we may define a function $R(r)$ by the integral

$$R(r)=(2\pi)^{-1}\int_{-\infty}^{\infty} d(l+1)\sin[(l+\tfrac{1}{2})r]g(l+\tfrac{1}{2}). \quad (13)$$

We have then the reciprocal relationship,

$$g(l+\tfrac{1}{2})=\int_{-\infty}^{\infty} dr \sin[r(l+\tfrac{1}{2})]R(r). \quad (14)$$

If we make use of (14) together with a well-known integral expression for $P_l(\cos\vartheta)$,[12] we may rewrite (12) as follows:

$$G(\vartheta)=\frac{1}{\pi\sqrt{2}}\sum_{l=0}^{\infty}\left\{\int_{-\infty}^{\infty} dr \sin[r(l+\tfrac{1}{2})]R(r)\right\} \times\left\{\int_{\vartheta}^{2\pi-\vartheta} d\phi \sin[\phi(l+\tfrac{1}{2})](\cos\vartheta-\cos\phi)^{-\frac{1}{2}}\right\}.$$

Upon changing the order of the integrations and the summation this becomes

$$G(\vartheta)=\frac{1}{\pi\sqrt{2}}\int_{\vartheta}^{2\pi-\vartheta} d\phi(\cos\vartheta-\cos\phi)^{-\frac{1}{2}}\int_{-\infty}^{\infty} drR(r) \times\sum_{l=0}^{\infty}\sin[r(l+\tfrac{1}{2})]\sin[\phi(l+\tfrac{1}{2})].$$

The sum over l can now be performed:

$$\begin{aligned}G(\vartheta)&=\frac{1}{\sqrt{2}}\int_{\vartheta}^{2\pi-\vartheta} d\phi(\cos\vartheta-\cos\phi)^{-\frac{1}{2}}\int_{-\infty}^{\infty} drR(r) \times\sum_{m=-\infty}^{\infty}(-1)^m\delta[r-(\phi+2\pi m)]\\ &=\sqrt{2}\int_{\vartheta}^{2\pi-\vartheta} d\phi(\cos\vartheta-\cos\phi)^{-\frac{1}{2}}\sum_{m=0}^{\infty}(-1)^m \times R(\phi+2\pi m).\end{aligned} \quad (15)$$

Expression (15) is the formula we wanted to derive.[13] Whenever the Legendre polynomial sequence (12) converges slowly the series in (15) converges rapidly. In the numerical illustration of Sec. 6, only the $m=0$ term is significant.

In the next section we shall present the $g(l+\frac{1}{2})$ for the Molière problem both exactly and in a very useful approximation. Both differ slightly from the form used in the ordinary "small angle approximation," and therefore the Fourier integral which we shall make use of is not quite the usual one.

[12] See, for example, E. T. Whittaker and G. N. Watson, *Modern Analysis* (MacMillan Company, New York, 1946), p. 315.

[13] This formula assumes a $g(l+\frac{1}{2})$ which is antisymmetric in $(l+\frac{1}{2})$. Circumstances may arise in which it is convenient to define a $g(l+\frac{1}{2})$ which is symmetric or which contains both symmetric and antisymmetric parts. For these situations it is advantageous to make use of the formula

$$G(\vartheta)=\sqrt{2}\int_0^{\vartheta} d\phi(\cos\phi-\cos\vartheta)^{-\frac{1}{2}}\sum_{m=0}^{\infty}(-1)^m \times\{R(\phi+2\pi m)-R[(2\pi-\phi)+2\pi m]\},$$

where

$$R(r)=(2\pi)^{-1}\int_{-\infty}^{\infty} d(l+\tfrac{1}{2})\cos[(l+\tfrac{1}{2})r]g(l+\tfrac{1}{2}).$$

TABLE II. A comparison of approximate values for $2\eta C_l$ as given by Eq. (18), with the exact values of Eq. (17). $(2\sqrt{\eta}=0.001765.)$

l	$2\eta C_l$	$1-L_1(\sigma)-\frac{1}{2}\eta L_0(\sigma)-\frac{1}{4}\eta$	$1-L_1(\sigma)$
0	0	$<10^{-12}$	2.977×10^{-6}
1	2.034×10^{-5}	2.039×10^{-5}	2.925×10^{-5}
2	5.636×10^{-5}	5.642×10^{-5}	5.877×10^{-5}
3	10.640×10^{-5}	10.655×10^{-5}	10.877×10^{-5}

5. THE SPHERICAL HARMONIC COEFFICIENTS FOR THE MOLIÈRE PROBLEM

The differential scattering cross section which leads to the Fourier integral of Molière and Snyder-Scott is $\sigma(\vartheta)=2\eta A\{1+2\eta-\cos\vartheta\}^{-2}$, where A and η are constants. The solution of the diffusion equation with this cross section, by expansion into spherical harmonics, yields the following expressions:[14]

$$G(\vartheta)=\sum_l g_lP_l(\cos\vartheta),\quad g_l=(l+\tfrac{1}{2})\exp(-2\eta AC_l),$$
$$C_l=\int_{-1}^{1} d(\cos\vartheta)[P_0-P_l(\cos\vartheta)][1+2\eta-\cos\vartheta]^{-2}. \quad (16)$$

We may evaluate rather easily the integral for C_l:[15]

$$\begin{aligned}C_l&=-\frac{\partial}{\partial(2\eta)}\int_{-1}^{1} d(\cos\vartheta) \times[P_0-P_l(\cos\vartheta)][1+2\eta-\cos\vartheta]^{-1}\\ &=-\frac{\partial}{\partial(2\eta)}\{2Q_0(1+2\eta)-2Q_l(1+2\eta)\} \quad (17)\\ &=\frac{1}{2\eta(1+\eta)}\{1+l[(1+2\eta)Q_l(1+2\eta)-Q_{l-1}(1+2\eta)]\}.\end{aligned}$$

A very accurate approximation (for small η) which we shall use instead of (17) is

$$C_l\approx(2\eta)^{-1}\{1-L_1(\sigma)-\tfrac{1}{2}\eta L_0(\sigma)-\tfrac{1}{4}\eta\}, \quad (18)$$

where $\sigma=2\eta^{\frac{1}{2}}(l+\frac{1}{2})$. Table II compares (18) and (17) and with the "small angle approximation," i.e., $C_l\approx(2\eta)^{-1}[1-L_1(\sigma)]$, for the first four values of l. The form (18) was suggested by Lewis' Eq. (15),[14] but it is asymptotically correct as $l\rightarrow\infty$.

[14] H. W. Lewis, Phys. Rev. 78, 527 (1950).

[15] See, e.g., reference 12, p. 320. I am indebted to Dr. C. H. Blanchard for the expression of C_l in terms of Q_l's.

TABLE III. The Legendre polynomial sum as compared with Molière's "small angle" results for $A=118.4$, $2\sqrt{\eta}=0.001765$.[a]

ϑ	Molière	Polynomial sum	ϑ	Molière	Polynomial sum
(degrees)	$\times 10^{-6}$	$\times 10^{-6}$	(degrees)	$\times 10^{-6}$	$\times 10^{-6}$
0	2850	2870	8.411	8.24	8.43
0.561	2710	2730	11.214	2.15	2.17
1.121	2320	2320	12.500	1.32	1.35
1.682	1800	1780	15.000	0.589	0.601
2.243	1270	1260	20.000	0.173	0.175
3.364	511	511	23.750	0.0843	0.0864
4.486	179	182	27.500	0.0461	0.0479
5.607	67.7	65.2	30.000	0.0322	0.0338
6.728	27.1	26.1			

[a] The tabulated quantity is actually 4η times the polynomial sum (16).

6. AN ILLUSTRATION OF THE SPHERICAL HARMONIC SUMMATION

We now want to make an application of the summation formula (15) by performing the summation (16) for specific values of A and η, corresponding to a 15.7-Mev gold foil scattering experiment of Hanson *et al.*,[3] i.e., $A=118.4$, $2\sqrt{\eta}=0.001765$.

Our first step is the evaluation of the integral

$$\begin{aligned} R(2\eta^{\frac{1}{2}}\omega) &= (2\pi)^{-1}(2\eta^{\frac{1}{2}})^{-2}\int_{-\infty}^{\infty} d\sigma\,\sin(\sigma\omega)\sigma \\ &\quad\times\exp\{-A[1-L_1(\sigma)-\tfrac{1}{2}\eta L_0(\sigma)-\tfrac{1}{4}\eta]\} \\ &= -(4\eta)^{-1}\frac{\partial}{\partial\omega}(2\pi)^{-1}\int_{-\infty}^{\infty} d\sigma\,\cos(\sigma\omega) \\ &\quad\times\exp\{-A[1-L_1(\sigma)-\tfrac{1}{2}\eta L_0(\sigma)-\tfrac{1}{4}\eta]\}, \end{aligned} \tag{19}$$

where $\sigma=2\eta^{\frac{1}{2}}(l+\frac{1}{2})$. This is very nearly the same calculation as that discussed in Sec. 3, since the last two terms in the exponent are very small. We make very nearly the same approximations, i.e.,

$$\exp\{-A[1-L_1(\sigma)-\tfrac{1}{2}\eta L_0(\sigma)-\tfrac{1}{4}\eta]\} \approx \{\sum_{n=1}^{3} c_n'\exp(-\gamma_n'\sigma^2)\}L_1(A^{\frac{1}{2}}\sigma) \quad (\text{small } \omega)$$

$$\exp\{-A[1-L_1(\sigma)-\tfrac{1}{2}\eta L_0(\sigma)-\tfrac{1}{4}\eta]\} \approx \sum_{n=1,2} k_n'L_{\kappa_n'}(K'\sigma)+AK'^{-2}L_1(K'\sigma) + \tfrac{1}{2}A\eta L_0(K'\sigma) \quad (\text{large } \omega),$$

where the c_n', ω_n' in the "small ω" approximation are assigned values which fit the approximation to the exact expression at six values $\sigma_j^2=\eta_i+0.003j$. In the "large ω" approximation the four constants k_n', κ_n' are given values which fit the approximation to the exact expression for the four combinations of derivatives mentioned in Sec. 3, part B, evaluated again at $\sigma^2=10^{-4}$. The constant K' is again given the value 35.

Since we shall make numerical integrations which involve the $R(2\sqrt{\eta}\omega)$ we tabulate the functions

$$\begin{aligned} R(2\eta^{\frac{1}{2}}\omega) &\approx (4\eta)^{-1}\int_{-\infty}^{\infty} d\omega'\left\{\frac{1}{2\sqrt{\pi}}\sum_{n=1}^{3}(c_n'\gamma_n'^{-\frac{1}{2}}) \right. \\ &\quad\left.\times[2(\omega-\omega')/(4\gamma_n')]\exp[-(\omega-\omega')^2/(4\gamma_n')]\right\} \\ &\quad\times\{(A/2)(A+\omega'^2)^{-\frac{3}{2}}\}, \end{aligned}$$

$$\begin{aligned} R(2\eta^{\frac{1}{2}}\omega) &\approx (4\eta)^{-1}(2\sqrt{\pi})^{-1}(2\omega) \\ &\quad\{\sum_{n=1,2}[k_n'(2K'^2)^{\kappa_n'}(\kappa_n'+\tfrac{1}{2})!](K'^2+\omega^2)^{-\kappa_n'-3/2} \\ &\quad+\tfrac{3}{4}A(K'^2+\omega^2)^{-5/2}+\tfrac{1}{8}\eta A(K'^2+\omega^2)^{-3/2}\}. \end{aligned}$$

The two approximations overlap within a few percent in the region $\omega\approx 80$, as expected. Because we have taken the derivative, the two do not join quite as well as in Table I. Since we shall be integrating again this is not serious.

Finally, we evaluate the integral

$$G(\vartheta)=\sqrt{2}\int_{\vartheta}^{2\pi-\vartheta} d(2\eta^{\frac{1}{2}}\omega)[\cos\vartheta-\cos(2\eta^{\frac{1}{2}}\omega)]^{-\frac{1}{2}} \times\sum_{m=0}^{\infty}(-1)^m R[(2\eta^{\frac{1}{2}}\omega)+2\pi m].$$

Only the first term in the sum is significant and we neglect the others. This integration can be made very simply by writing

$$G(\vartheta)=2\sqrt{2}\int_{0}^{(\cos\vartheta+1)^{\frac{1}{2}}} d[\cos\vartheta-\cos(2\eta^{\frac{1}{2}}\omega)]^{\frac{1}{2}} \times\{[\sin(2\eta^{\frac{1}{2}}\omega)]^{-1}[R(2\eta^{\frac{1}{2}}\omega)+R(2\pi-2\eta^{\frac{1}{2}}\omega)]\}.$$

We plot the quantity in curly brackets against $[1-\cos(2\eta^{\frac{1}{2}}\omega)]$. For each value of $\cos\vartheta$ we then read quadratically spaced values from the graph and sum them according to a standard numerical integration formula.

Table III gives $G(\vartheta)$ calculated in this manner as compared with Molière's results obtained by Fourier inversion. The two calculations agree quite well for small angles. The oscillating discrepancy of 3 or 4 percent in the region 5° to 11° seems to be due largely to the neglect of the f^2 term in Molière's approximation. At large angles where single scattering is predominant, the exact sum is larger than the "small angle approximation" by about a factor $\sigma(\text{exact})/\sigma(\text{small angles}) \approx\frac{1}{4}\vartheta^4/(1-\cos\vartheta)^2$, which amounts to 1.047 at 30°.