index; all time components vanish in the rest frame. Substitution of Eq. (79) into Eq. (58′) gives

$$-S_\sigma\partial_\sigma T+\eta_{\sigma\tau\mu\nu}\Sigma_{\sigma\tau}\Sigma_{\mu\nu}\geq 0, \tag{81}$$

where $\frac{1}{2}\eta_{\sigma\tau\mu\nu}\Sigma_{\sigma\tau}\Sigma_{\mu\nu}$ is the Rayleigh dissipation rate for an anisotropic medium. In the case of a transport process consisting only of heat conduction this viscous dissipation must vanish, since as we have seen, the entire dissipation in this case is $-S_\sigma\partial_\sigma T\geq 0$. Accordingly, in this case $\Sigma_{\mu\nu}$ must vanish, and $\phi_{\sigma\tau}=\phi_{\sigma\tau}{}^r$ everywhere. Because of the symmetry of the stress and rate-of-strain tensors in Eqs. (79) and (81), only 21 independent viscosity coefficients exist in the general anisotropic case. In the isotropic case, in which the excess stress and the rate-of-strain tensors can be diagonalized simultaneously, the number is reduced to 2, according to the usual arguments; here $\phi_{\sigma\tau}{}^r=-p\bar{\delta}_{\sigma\tau}$, where p is the hydrostatic pressure.

PHYSICAL REVIEW VOLUME 90, NUMBER 6 JUNE 15, 1953

The Tamm-Dancoff Formalism and the Symmetric Pseudoscalar Theory of Nuclear Forces

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(Received February 11, 1953)

The general method of deducing the Tamm-Dancoff equal-times formalism, as generalized by Lévy, from the relativistic two-body equation of Bethe-Salpeter and Schwinger is given. Only processes which are finite *ab initio* are considered. The essence of the procedure is the relation between a set of conventional matrix elements of the Tamm-Dancoff formalism and the Feynman diagram which summarizes them; this relationship provides a convenient guide for enumerating all matrix elements of a specified type and precludes the possibility of omission of any members of the set. Rules are also given for writing down any matrix element. The method is then applied to the derivation of the fourth-, sixth-, and eighth-order adiabatic potentials on the symmetrical pseudoscalar-pseudoscalar theory. Some discrepancies with the results of Lévy are noted: In connection with the fourth-order potential these are first, that a more careful treatment of the energy denominators of the leading two-pair terms brings to light contributions that cancel with all other two-pair matrix elements that are of relative order μ/M compared to the leading ones; second, that the one-pair terms do not vanish but yield a repulsive interaction which substantially alters the qualitative picture of the fourth-order potential; third, that for the no-pair terms the result should agree with the previously calculated fourth-order potential for the pseudoscalar-pseudovector theory. The sixth- and eighth-order results are also in disagreement with Lévy. Finally, an analysis of the problem of many-particle forces is given and explicit results obtained for the leading terms of the three- and four-particle forces as well as for certain smaller contributions to the three-particle interaction.

I. INTRODUCTION

In a pair of extremely interesting papers recently published Lévy[1] has derived a three-dimensional equation for the relative motion of two particles with an interaction kernel that, in principle, can be computed to any order in the coupling constant; he has used his formalism for the most thorough examination of the nuclear forces predicted by weak coupling theory so far attempted and from the results has give a plausible account of the low energy properties of the deuteron.

Lévy's approach is a hybrid one. It consists, first of all, in an extension of the Fock space method of Tamm[2] and Dancoff[3] to include higher order processes involving multiple meson exchange and pair creation, with the proviso, however, that all infinite matrix elements associated with "radiative" corrections be omitted. It is then possible to eliminate all amplitudes except that for the two bound nucleons and to obtain an equation for the latter, which is interpreted as the wave function of the two-particle system in momentum space. To incorporate radiative corrections Lévy turns to the relativistic two-body equation[4–6] (henceforth called R.E.). He shows that by an appropriate iteration suggested by the solution for an instantaneous interaction the finite terms of the R.E. can be placed in a one-to-one correspondence with those of the T.D. (Tamm-Dancoff) formalism. It is then possible to carry out all required renormalizations *before* the reduction to equal times for the two particles is effected, and the finite residues can be incorporated into the three-dimensional interaction kernel.

The present work, begun after the author's reading of L1, was motivated by the belief that the demonstration given there of the equivalence between the T.D. formalism and the appropriately reduced R.E., though undoubtedly concerned with a true result, lacked cogency in certain details and completeness. It was felt, moreover, that since the R.E. was required for the

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[1] M. M. Lévy, Phys. Rev. **88**, 72, 725 (1952); hereafter referred to as L1 and L2, respectively.

[2] I. Tamm, J. Phys. U.S.S.R. **9**, 449 (1945).

[3] S. M. Dancoff, Phys. Rev. **78**, 382 (1950).

[4] J. Schwinger, Proc. Natl. Acad. Sci. U. S. **37**, 452, 455 (1951).

[5] E. Salpeter and H. A. Bethe, Phys. Rev. **84**, 1232 (1951), referred to as S.B.

[6] M. Gell-Mann and F. Low, Phys. Rev. **84**, 350 (1951).

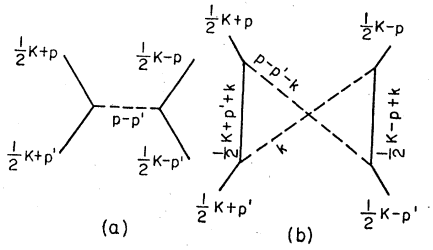

FIG. 1. Feynman diagrams for the interactions representing exchange of one or two quanta.

proper treatment of radiative corrections, it seemed reasonable, for the applications contemplated, to rest entirely within its frame work for the derivation of all results.

This paper is therefore concerned with a re-examination of some of the methods and results of L1 and L2. We shall confine ourselves completely to the parts of the theory that are finite *ab initio*, since we have nothing to add to Lévy's general prescription for treating renormalizations. In Sec. II we define the problem and indicate an apparent difficulty in generating an *equation* for equal times for the two particles using the form of unequal times wave function suggested by the solution for a static potential. There is no obstacle, however, to the derivation of such an equation for the projection of the wave function on the positive energy free-particle subspace. As in L1 and L2 it is this latter quantity that we shall designate as the wave function of the system.

Section III is devoted to a derivation of the complete interaction kernel up to the fourth order in the coupling constant. The main result of this section is the connection between a single covariant interaction term of a given order (as expressed by an appropriate four-dimensional integral and represented by a single Feynman diagram) and the number and kinds of more conventional matrix elements to which this "reduces" in the equal-times formalism. In fact, once this relationship is seen and expressed in complete generality, the equivalence of the result with the T.D. formalism becomes obvious. This relationship is undoubtedly understood by many physicists. From our present point of view, however, it becomes an extremely useful way of classifying matrix elements and of making symmetry relationships clear.

The remainder of the paper is concerned with the application of the formalism to the derivation of static nuclear potentials. In Sec. IV, we compute the leading contributions of the fourth-order adiabatic potential for the symmetric pseudoscalar theory. Our results are in disagreement with those of L2 for the two-pair, one-pair, and no-pair terms.[7] In Sec. V we outline, using our classification procedure, the derivation of the leading terms in the sixth- and eighth-order potentials. Again the results differ in detail from those of L2. The calculation of many-body forces is undertaken in Sec. VI. Explicit expressions are given for the most important terms of the three- and four-particle forces as well as contributions of one order of magnitude smaller to the three-body force. Appendix A contains a procedure for a concise derivation of the three-dimensional formalism from the R.E. and Appendix B points out that the theory contains spin-orbit forces.

The paper concludes in Sec. VII with a qualitative discussion of the status and significance of the Lévy theory. No numerical results are given, however.

[7] See reference 1, L2, Sec. III.

II. DEFINITION AND PRELIMINARY DISCUSSION OF THE PROBLEM

In this and in the following section we shall be dealing exclusively with some form of the equation

$$[\gamma(\tfrac{1}{2}K+p)+M]^{(1)}[\gamma(\tfrac{1}{2}K-p)+M]^{(2)}\psi(p_\mu) = \int d^4p' I(p, p'; K)\psi(p_\mu'). \quad (1)$$

Before defining more closely the contents of Eq. (1) let us settle once and for all the matter of notation. In so far as it is possible, we shall follow the notation of S.B. and of S.[8] The Dirac matrices γ_μ are, however, the skew-symmetric ones defined by Schwinger,[9] and the reader should especially bear in mind that we shall use a γ_5 matrix whose square is minus one. However, when we pass to the α, β matrices, we shall employ the standard Dirac representation. For the necessary purposes of comparison, the notation of L1, 2 will also enter, especially from Sec. IV forward.

Thus, Eq. (1) is the two-body R.E. in momentum space for particles of equal mass M, total four-momentum K, and relative four-momentum p. For the kernel I we confine ourselves to the assumption that

$$I(p, p'; K) = (2\pi i)^{-1}\lambda(\gamma_5\tau_i)^{(1)}(\gamma_5\tau_i)^{(2)}[(p-p')_\lambda{}^2 + \mu^2]^{-1} + (2\pi i)^{-2}\lambda^2\int d^4k[k_\lambda{}^2+\mu^2]^{-1}[(p-p'-k)_\nu{}^2 + \mu^2]^{-1}[\gamma_5\tau_i G(\tfrac{1}{2}K+p'+k)\gamma_5\tau_j]^{(1)} \times[\gamma_5\tau_j G(\tfrac{1}{2}K-p+k)\gamma_5\tau_i]^{(2)}, \quad (2)$$

representing the two Feynman diagrams shown in Fig. 1 for the symmetrical pseudoscalar theory with pseudoscalar coupling;[10] here

$$\lambda = g^2(2\pi)^{-3}, \quad G(p) = (\gamma p + M)^{-1}. \quad (3)$$

We shall not include in our discussion the terms of order λ^2 which are radiative corrections to the λ-term. Our problem is then as follows: We wish to derive from Eqs. (1) and (2) an equation for the three-dimensional function $\phi(\mathbf{p})$ defined by

$$\phi(\mathbf{p}) = \int_{-\infty}^{\infty}\psi(p_\mu)dp_0. \quad (4)$$

[8] E. Salpeter, Phys. Rev. **87**, 328 (1952), referred to as S.

[9] J. Schwinger, Phys. Rev. **82**, 664 (1951).

[10] In order to derive Eq. (2) and all other terms involving radiative corrections that have been omitted, we actually employed the technique of reference 4. See also R. Karplus and A. Klein, Phys. Rev. **87**, 848 (1952).

This is, in general, not possible, since we do not know the dependence of $\psi(p_\mu)$ on the relative energy variable p_0. Both S. and L1 have shown, however, that if one restricts oneself to the ladder approximation and assumes a static potential

$$[k_i^2-k_0^2+\mu^2]^{-1}\to[k_i^2+\mu^2]^{-1}, \tag{5}$$

then Eq. (1) yields a three-dimensional equation without further approximation [see S., Eq. (13) and sequel]. In terms of the solution $\phi_s(\mathbf{p})$ of this equation, $\psi(p_\mu)$ has the form:[11]

$$\psi_s(p_\mu)=[2\pi i\mathfrak{F}_W(\mathbf{p},p_0)]^{-1}\lambda\Gamma_i^{(1)}\Gamma_i^{(2)}\times\int\frac{d\mathbf{k}}{(k^2+\mu^2)}\phi_s(\mathbf{p}-\mathbf{k})$$
$$=-[2\pi i\mathfrak{F}_W(\mathbf{p},p_0)]^{-1}[W-H_1(\mathbf{p})-H_2(\mathbf{p})]\chi_s(\mathbf{p}) \tag{6}$$

with

$$\mathfrak{F}_W(\mathbf{p},p_0)=F_W^{(1)}(\mathbf{p},p_0)F_W^{(2)}(-\mathbf{p},-p_0)=[\tfrac{1}{2}W+p_0-H_1(\mathbf{p})][\tfrac{1}{2}W-p_0-H_2(\mathbf{p})] \tag{7}$$

and

$$\Gamma_i^{(r)}=(\gamma_0\gamma_5\tau_i)^{(r)}\equiv(\beta\gamma_5\tau_i)^{(r)},\ (r=1,2;\ i=1,2,3). \tag{8}$$

Equation (6) *defines* the function $\chi(\mathbf{p})$, which is then related in this approximation to $\phi(\mathbf{p})$ by the equation[12]

$$\phi(\mathbf{p})=\{\Lambda_+^{(1)}(\mathbf{p})\Lambda_+^{(2)}(\mathbf{p})-\Lambda_-^{(1)}(\mathbf{p})\Lambda_-^{(2)}(\mathbf{p})\}\chi(\mathbf{p}). \tag{9}$$

Up to this point everything in Eq. (6) is well defined, and it can, in fact, be used as the basis of a perturbation theory treatment of a "small" noninstantaneous interaction.[8,10] This is not our purpose, however; we are interested in deriving an improved *equation* for $\phi(\mathbf{p})$. Toward this end it is reasonable to adopt the second form of Eq. (6) as an *ansatz*, the physical content of which is the statement that the particles having propagated up to certain common time in a bound state, the propagation of one or the other of the particles "further" in time is according to free-particle behavior. [See Eq. (28) of S.B. for a nonrelativistic example.] We therefore insert Eq. (6) into the right-hand side (r.h.s.) of the ladder approximation

$$\mathfrak{F}_W(\mathbf{p},p_0)\psi(p_\mu)=(2\pi i)^{-1}\lambda\Gamma_i^{(1)}\Gamma_i^{(2)}\int\frac{d^4k}{k^2+\mu^2}\psi(p_\mu-k_\mu). \tag{10}$$

If we first carry out the integration over k_0 (with the usual hole theory definition of the poles), we obtain first of all an improved form for $\psi(p_\mu)$ which is a generalized version of L1, Eq. (35). We shall not record it here since we shall eventually find useful only a somewhat more limited version of the equation. Now, however, if we divide by $\mathfrak{F}_W(\mathbf{p},p_0)$ and carry out the p_0 integral, we obtain an equation which is precisely equivalent[13] to L1, Eq. (41), if, and only if, we establish the following correspondence between free-particle energy projections:

$$\phi_{++}=\chi_{++},\ \phi_{--}=-\chi_{--},\ \phi_{+-}=\chi_{+-},\ \phi_{-+}=\chi_{-+}. \tag{11}$$

The first original comment of this section is that Eq. (11) is *a priori* an incorrect identification. Though the plus-plus and minus-minus equalities are unobjectionable, since they follow from Eq. (6), the plus-minus and minus-plus equalities cannot be true, since in the adiabatic limit $\phi_{+-}=\phi_{-+}=0$, whereas χ_{+-} and χ_{-+} are finite well-defined functions.[14] The point is that the functions χ_{+-} and χ_{-+} are just not defined by anything which has been said or done up to this juncture. The resolution of the dilemma will be given below. As the final observation of this section we merely note that if one arbitrarily drops all terms of L1, Eq. (41) (or the equivalent equation which we have derived without writing down), which involve χ_{+-} and χ_{-+}, and if one eliminates ϕ_{--} in favor of ϕ_{++} as described in L1, one obtains to order λ^2 the equation

$$(W-2E_p)\phi_{++}(\mathbf{p})=\Lambda_+^{(1)}(\mathbf{p})\Lambda_+^{(2)}(\mathbf{p})\int d\mathbf{p}'[\lambda\Delta_2(\mathbf{p},\mathbf{p}')+\lambda^2\Delta_4(\mathbf{p},\mathbf{p}')]\phi_{++}(\mathbf{p}'), \tag{12}$$

where $\Delta_2(\mathbf{p},\mathbf{p}')$ recorded as Eq. (13) in the next section, is the kernel equivalent to that of L1, Eq. (36), and $\Delta_4(\mathbf{p},\mathbf{p}')$ is that equivalent to L2, Eq. (44). The latter, as we shall see in the next section is a small part of the entire fourth-order kernel or even of the pair effects to this order.

III. DERIVATION OF THE INTERACTION KERNEL TO FOURTH ORDER

The solution of the difficulty raised at the end of the previous section can be obtained by combining an observation with a more diligent job of computation. The observation is that our goal is precisely an equation of the form Eq. (12), whereby $\Delta_4(\mathbf{p},\mathbf{p}')$ we shall understand the complete T.D. interaction in fourth order. In other words, we interpret $\phi_{++}(\mathbf{p})$ as the bound state wave function in momentum space.[15] As we shall see

[11] Note that we are now working in the frame of reference defined by $K_\mu=(0,0,0,W)$; we have also used the relationship $\beta G^{-1}(p)=-F(p)$ so that we are working with the usual $\boldsymbol{\alpha},\beta$ matrices of Dirac. Note also that quantities referring to particle 2 such as $H_2(\mathbf{p}),\Lambda_+^{(2)}(\mathbf{p})$ are functions of $-\mathbf{p}$, though this dependence on $-\mathbf{p}$ is indicated explicitly for $F_W^{(2)}(-\mathbf{p},-\mathbf{p}_0)$.

[12] A nice logical point overlooked by S. is that having defined $\chi(\mathbf{p})$, say by Eq. (6), one can deduce therefrom by Eq. (9) the equation obeyed by the adiabatic $\phi(\mathbf{p})$, but because of the singularity of the operator involved in Eq. (9), one cannot proceed in the reverse direction.

[13] Lévy has already taken spin-matrix elements, whereas we prefer to leave our expressions in operator form until a later stage of the calculation.

[14] There appears to be an awareness of this difficulty in L2. There, the proper use of Eq. (52) can lead to no errors in the results sought, though formally it contains the incorrect statements that $\psi_{+-}(p_\mu)=\psi_{-+}(p_\mu)=0$ in the adiabatic case, whereas only their integrals with respect to p_0 have this property.

[15] See reference 1, L2, Sec. II, for a fuller and more precise statement of this connection.

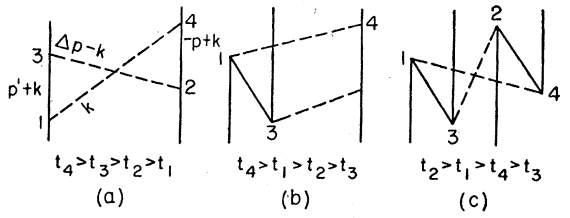

FIG. 2. Diagrams representing three typical conventional matrix elements of the twenty-four such contained in Fig. 1(b). The labeling of (a) accords with the variables used in Eq. (17).

shortly, an equation for this function can be obtained without having to be concerned directly about equations for ϕ_{++}, ϕ_{-+}, and ϕ_{--}. Before showing this, we record the form for $\Delta_2(\mathbf{p}, \mathbf{p}')$,

$$\Delta_2(\mathbf{p}, \mathbf{p}')=\frac{2\Gamma_i^{(1)}\Gamma_i^{(2)}}{2\omega(\mathbf{p}-\mathbf{p}')[W-E_p-E_{p'}-\omega(\mathbf{p}-\mathbf{p}')]}. \quad (13)$$

We have explicitly adjoined the factor of two to the quantity $\omega(\mathbf{p}-\mathbf{p}')$ in the denominator to show their joint origin in the Fourier analysis of the meson field and to point to the fact that Eq. (13) is the sum of the two possible matrix elements of the conventional type that can be associated with the exchange of a single field quantum. Since our task will later deteriorate into the mere enumeration of such matrix elements, it is well to establish good counting habits from the very beginning.

Now it is clear that if we are to derive the complete $\Delta_4(\mathbf{p}, \mathbf{p}')$, we must include both terms of Eq. (2) in Eq. (1). As a starting point we rewrite Eq. (1) as [see Eq. (7) and reference 11]

$$\begin{aligned}
&F_W^{(1)}(\mathbf{p}, p_0)F_W^{(2)}(-\mathbf{p}, -p_0)\psi(p_\mu)\\
&=(2\pi i)^{-1}\lambda\int d^4k\frac{\Gamma_i^{(1)}\Gamma_i^{(2)}}{k^2+\mu^2}\psi(p_\mu-k_\mu)\\
&+(2\pi i)^{-2}\lambda^2\int d^4k d^4p'\frac{1}{(k^2+\mu^2)[(p-p'-k)^2+\mu^2]}\\
&\times[\Gamma_iF_W^{-1}(\mathbf{p}'+\mathbf{k}, p_0+k_0)\Gamma_j]^{(1)}\\
&\times[\Gamma_jF_W^{-1}(-\mathbf{p}+\mathbf{k}, -p_0+k_0)\Gamma_i]^{(2)}\psi(p_\mu'). \quad (14)
\end{aligned}$$

In the explicit λ^2 term we can certainly approximate $\psi(p_\mu')$ by [see Eq. (6)]

$$\begin{aligned}
\psi(p_\mu)&\cong\psi_{++}^{(0)}(p_\mu)\\
&=-[2\pi i\mathfrak{F}_{W++}(\mathbf{p}, p_0)]^{-1}[W-2E_p]\phi_{++}(\mathbf{p})\\
&=-(2\pi i)^{-1}[(\tfrac{1}{2}W+p_0-E_p)^{-1}\\
&\qquad+(\tfrac{1}{2}W-p_0-E_p)^{-1}]\phi_{++}(\mathbf{p}), \quad (15)
\end{aligned}$$

an explicit form that permits us to start grinding out the integrations with respect to p_0', k_0, and p_0 successively. In the first term on the r.h.s. of Eq. (14) we require the next approximation for $\psi(p_\mu)$. This is obtained by dropping the second term on the r.h.s. of Eq. (14), writing $\psi_{++}^{(0)}(p_\mu-k_\mu)$ for $\psi(p_\mu-k_\mu)$ in the first term on the r.h.s. and solving for $\psi(p_\mu)$; thus,[16]

$$\begin{aligned}
\psi^{(1)}(p_\mu)&=[2\pi iF_W^{(1)}(\mathbf{p}, p_0)F_W^{(2)}(-\mathbf{p}, -p_0)]^{-1}\lambda\\
&\times\int d^4k\frac{\Gamma_i^{(1)}\Gamma_i^{(2)}}{k^2+\mu^2}\psi_{++}^{(0)}(p_\mu-k_\mu). \quad (16)
\end{aligned}$$

If we insert Eq. (16) into the first term on the r.h.s. of Eq. (14), take the positive energy projection of both sides, divide through by the p_0 dependent operator on the l.h.s. and finally integrate with respect to p_0, we obtain the equation

$$\begin{aligned}
&(W-2E_p)\phi_{++}(\mathbf{p})=\Lambda_+^{(1)}(\mathbf{p})\Lambda_+^{(2)}(\mathbf{p})(2\pi i)^{-2}\lambda^2\\
&\times\int d\mathbf{k}dp'dp_0dk_0dp_0'[(\tfrac{1}{2}W+p_0-E_p)^{-1}+(\tfrac{1}{2}W\\
&-p_0-E_p)^{-1}][k^2+\mu^2]^{-1}[(p-p'-k)^2+\mu^2]^{-1}\\
&\times\{[\Gamma_iF_W^{-1}(\mathbf{p}-\mathbf{k}, p_0-k_0)\Gamma_j]^{(1)}[\Gamma_iF_W^{-1}(-\mathbf{p}+\mathbf{k},\\
&-p_0+k_0)\Gamma_j]^{(2)}+[\Gamma_iF_W^{-1}(\mathbf{p}'+\mathbf{k}, p_0+k_0)\Gamma_j]^{(1)}\\
&\times[\Gamma_jF_W^{-1}(-\mathbf{p}+\mathbf{k}, -p_0+k_0)\Gamma_i]^{(2)}\}\psi_{++}^{(0)}(p_\mu'). \quad (17)
\end{aligned}$$

Since the second term of the curly bracket is associated with the "crossed-quantum" diagram, Fig. 1(b), it is clear that the first term of the bracket, obtained by iteration based on Eq. (16), is nothing more than the iteration of Fig. 1(a). The dependence of the r.h.s. of Eq. (17) on the variables p_0, k_0, and p_0' is explicit; all poles are well defined, and therefore the integrations can be effected.

We now claim that the r.h.s. of Eq. (17) contains all of the T.D. $\Delta_4(\mathbf{p}, \mathbf{p}')$ [as well as the iteration of $\Delta_2(\mathbf{p}, \mathbf{p}')$]. We have verified this by actually performing the integrations over the relative energy variables and unscrambling the results. Whether it would have been possible to emerge from the labyrinth of algebraic manipulation involved without knowing the answer beforehand is a moot point. In any case, it is much more instructive to describe the answer than to reproduce the manipulations, since the description leads immediately to the generalization of the result to any order in the coupling constant, whereas actual computation by the means described starting from an equation like (17) becomes prohibitive for any order higher than λ^2. A more concise, and in a sense more physical, procedure for deriving the results described below is given in Appendix A.

Consider first the term in Eq. (17) corresponding to Fig. 1(b). It must contain all the matrix elements of fourth-order perturbation theory which describe the exchange of two mesons in the manner indicated, starting from a bound state of positive energy, ending with such a state, and proceeding *via* free-particle inter-

[16] Equation (16) is the immediate generalization, in our notation, of Eq. (35) of L1, though we have not given the explicit result of the integration with respect to k_0, for reasons of presentation that will become clear below. Note that although $\psi^{(0)}$ contains only a plus-plus part, $\psi^{(1)}$ contains both positive and negative free-particle energies.

mediate states of both positive and negative energy. There are 24 such matrix elements corresponding to the 4! time orderings of the interaction points of the figure. In the terminology of L1, 2 there are six two-pair terms, twelve one-pair terms, and six no-pair terms. Rather than record all twenty-four, we show in Fig. 2 diagrams of the type used by Lévy representing three typical matrix elements. They yield, in our notation, the following contributions to $\Delta_4(\mathbf{p}, \mathbf{p}')$ [in the context of Eq. (12)]:

$$\Delta_4^{(a)}(\mathbf{p}, \mathbf{p}') = \int d\mathbf{k} \frac{[\Gamma_i \Lambda_+(\mathbf{p}'+\mathbf{k})\Gamma_j]^{(1)}[\Gamma_j \Lambda_+(\mathbf{p}-\mathbf{k})\Gamma_i]^{(2)}}{4\omega_k \omega_{p-p'-k}(W-E_p-E_{p-k}-\omega_k)(W-E_{p-k}-E_{p'+k}-\omega_k-\omega_{p-p'-k})(W-E_{p'}-E_{p'+k}-\omega_k)}, \quad (18)$$

$$\Delta_4^{(b)}(\mathbf{p}, \mathbf{p}') = -\int d\mathbf{k} \frac{[\Gamma_i \Lambda_-(\mathbf{p}'+\mathbf{k})\Gamma_j]^{(1)}[\Gamma_j \Lambda_+(\mathbf{p}-\mathbf{k})\Gamma_i]^{(2)}}{4\omega_k \omega_{p-p'-k}(W-E_p-E_{p-k}-\omega_k)(W-E_p-E_{p'}-E_{p-k}-E_{p'+k})(-E_p-E_{p'+k}-\omega_{p-p'-k})}, \quad (19)$$

and

$$\Delta_4^{(c)}(\mathbf{p}, \mathbf{p}') = \int d\mathbf{k} \frac{[\Gamma_i \Lambda_-(\mathbf{p}'+\mathbf{k})\Gamma_j]^{(1)}[\Gamma_j \Lambda_-(\mathbf{p}-\mathbf{k})\Gamma_i]^{(2)}}{4\omega_k \omega_{p-p'-k}(-E_{p'}-E_{p'+k}-\omega_k)(-W-E_{p'+k}-E_{p-k}-\omega_k-\omega_{p-p'-k})(-E_p-E_{p-k}-\omega_k)}. \quad (20)$$

The rules for writing down Eqs. (18)–(20) are almost self-evident from the diagrams: The factors associated with the vertices, sums over intermediate spin states, and Fourier analysis of meson field are indeed so; there is a minus sign for an odd number of pairs, if all factors of the energy denominators are written as $W-E_I$, where E_I is the energy of the intermediate state. The apparently curious form of some of the energy denominators is explained, if we remark that whenever an intermediate state contains two nucleons in the initial or in the final state, their energy[17] is W and not $2E_{p'}$ or $2E_p$. Using these rules one can record all the contributions from Fig. 1(b) as they are actually provided by fairly elaborate calculation.

Turning to the contribution from the "ladder" diagram, the iteration of Fig. 1(a), there are again twenty-four physically distinct matrix elements which can be systematically enumerated. Here, however, a new but anticipated factor enters. Whereas in the previous case, no matrix element contained intermediate states in which only two nucleons are present, here one encounters four such elements, one of which is illustrated in Fig. 3, and with the remaining three obvious variants thereof. In terms of the present formalism, these are "reducible" diagrams, since one obtains their apparent contribution to Δ_4 by iterating Eq. (12) in which Δ_2 is given by Eq. (13). Clearly the correct procedure is to "uniterate" these terms. In this way we have derived the kernel $\lambda\Delta_2+\lambda^2\Delta_4$, the latter consisting of forty-four matrix elements of the conventional type.

Before proceeding, perhaps an additional word is in order justifying the process of "uniteration" beyond the purely formal reason that it leads to a known result. It should be noted that the energy denominator associated with an intermediate state in which only two nucleons are present can become small of the order of the binding energy in the nonrelativistic limit. On the other hand, the energy denominators of intermediate states in which there are mesons are no smaller than the order of μc^2 and those containing pairs of the order of Mc^2 in the nonrelativistic limit. Other things being equal (as is *not* the case in the pseudoscalar theory because of the properties of the γ_5 matrix), the iterated second-order terms would apparently give the largest contribution to the fourth-order potential, a result that would vary inversely with the binding energy. If one dropped all terms of the interaction kernel other than these, one would notice that one solution of the resulting equation could be obtained by "uniterating." This argument is equivalent to that already presented.

At this point, the formal generalization of our procedure to any order can be made. Suppose we were interested in the complete T.D. kernel correct through order $2n$. We could then easily write a generalized version of Eq. (17), which on the r.h.s. would contain n meson propagation functions and in the bracket the factors characteristic of the $n!$ Feynman diagrams (many of them reducible in the sense of S.B.) associated with the exchange of n field quanta and no radiative corrections. If we carried out the integration over all the relative energy variables, we should obtain $(2n)!\times n!$ matrix elements of the conventional type, many of them reducible in the sense defined above. These can all be recognized as arising from anywhere from 1 to $n-1$ iterations of a lower order equation according to the degree of reducibility of the matrix element. The latter is defined as the number of intermediate states with only two nucleons present.

In practice, we shall merely write down any set of matrix elements that may be of interest directly from the diagrams that represent them. This technique together with the underlying rules will be extensively

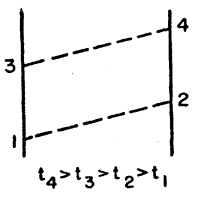

FIG. 3. Diagram representing one of four "reducible" matrix elements in fourth order. Others are $t_4>t_3>t_1>t_2$, $t_3>t_4>t_2>t_1$, $t_3>t_4>t_1>t_2$. All yield precisely the same matrix elements.

[17] This rule is clearly contained in the formalism. It is relevant only when pair production is considered.

used throughout the remainder of this paper. We turn first to a re-examination of the fourth-order potential.

IV. FOURTH-ORDER ADIABATIC NUCLEAR FORCES[18]

As shown in L2, the leading terms of the adiabatic fourth-order potential are of order (in the sense of expansion parameter) $(g^2/4\pi)^2(\mu/2M)^2$ and $(g^2/4\pi)^2(\mu/2M)^3$. To the former order one has strictly a static potential. To the next order, however, one must recognize the appearance of velocity-dependent forces. One can show, however, as in part C of this section (Lévy was undoubtedly aware of this) that such terms need not be considered in connection with the low energy properties of the two-nucleon system. Even omitting these terms we are in disagreement with the results of L2, as demonstrated in parts A and B. We preface the detailed discussion with a simplified formulation of the adiabatic limit.

To define the adiabatic potential one takes matrix elements of both sides of Eq. (12) with respect to positive energy free-particle Dirac spinors[19] and transforms back to coordinate space. The r.h.s. of Eq. (12) then takes the form[20]

$$V\psi(\mathbf{r})=\int d\mathbf{r}'[V_2(\mathbf{r},\mathbf{r}')+V_4(\mathbf{r},\mathbf{r}')]\psi(\mathbf{r}'), \quad (21)$$

where the quantity that now interests us, $V_4(\mathbf{r},\mathbf{r}')$, is given by the expression

$$V_4(\mathbf{r},\mathbf{r}')=\lambda^2(2\pi)^{-3}\int d\mathbf{p}d\mathbf{p}'e^{i\mathbf{p}\cdot\mathbf{r}}e^{-i\mathbf{p}'\cdot\mathbf{r}'} \times(\mathbf{p},-\mathbf{p}|\Delta_4(\mathbf{p},\mathbf{p}')|\mathbf{p}',-\mathbf{p}'), \quad (22)$$

and the matrix elements are between appropriately labelled Dirac spinors for the two particles. We further note that $\Delta_4(\mathbf{p},\mathbf{p}')$ has the form

$$\Delta_4(\mathbf{p},\mathbf{p}')=\int d\mathbf{k}_1 O_4(\mathbf{p},\mathbf{p}',\mathbf{k}_1). \quad (23)$$

To arrive at the adiabatic limit it is convenient to introduce the variable $\mathbf{k}_2$ by setting

$$\mathbf{p}'=\mathbf{p}-\mathbf{k}_1-\mathbf{k}_2. \quad (24)$$

In the limit in which p becomes vanishingly small Eq. (22) reduces to

$$V_4(\mathbf{r},\mathbf{r}')\to\delta(\mathbf{r}-\mathbf{r}')V_4(\mathbf{r}), \quad (25)$$

$$V_4(\mathbf{r})=\lambda^2\int d\mathbf{k}_1 d\mathbf{k}_2\exp[i(\mathbf{k}_1+\mathbf{k}_2)\cdot\mathbf{r}] \times(\mathbf{p},-\mathbf{p}|O_4(\mathbf{p},\mathbf{k}_1,\mathbf{k}_2)| \times\mathbf{p}-\mathbf{k}_1-\mathbf{k}_2,-\mathbf{p}+\mathbf{k}_1+\mathbf{k}_2)|_{p=0}. \quad (26)$$

[18] The results of this section and of the following one were reported by the author in Phys. Rev. **89**, 1158 (1953).

[19] As in reference 1, L1, Sec. III, 1. See also L2, Sec. II. Since we shall be interested only in the adiabatic potential, we need not be as elegant as in the latter reference.

[20] The differences between $\psi(\mathbf{r})$ and the function $\phi^{(0,0)}(\mathbf{r})$ of L2 are immaterial for our purposes.

A. Two-Pair Terms

According to our method of counting there are twelve distinct matrix elements of this type,[21] six having their origin in the ladder approximation (henceforth called M_1 matrix elements) and six in the diagram of Fig. 1(b) (henceforth called M_2 elements). A typical spin matrix element is

$$(\mathbf{p}|\Gamma_i^{(1)}\Lambda_-^{(1)}(\mathbf{p}-\mathbf{k})\Gamma_j^{(1)}|\mathbf{p}-\mathbf{k}_1-\mathbf{k}_2) =\tau_i^{(1)}\tau_j^{(1)}+O((\mu/M)^2), \quad (27)$$

since

$$\gamma_0\gamma_5\Lambda_-\gamma_0\gamma_5=\Lambda_+, \quad (28)$$

and the latter may be replaced by unity to the order indicated.[22] The isotopic spin operator associated with M_1 elements is $(\boldsymbol{\tau}^{(1)}\cdot\boldsymbol{\tau}^{(2)})^2=3-2\boldsymbol{\tau}^{(1)}\cdot\boldsymbol{\tau}^{(2)}$ whereas that associated with M_2 elements is $\tau_i^{(1)}\tau_j^{(1)}\tau_j^{(2)}\tau_i^{(2)}=3+2\boldsymbol{\tau}^{(1)}\cdot\boldsymbol{\tau}^{(2)}$.

Following Lüders,[23] we note that it is necessary to treat the energy denominators of the leading terms, involving at most one pair at a given time, with a little extra care. For example, M_1 contributes two denominators of the form (Fig. 1a$_1$ of L2)

$$(-E_p-E_{p-k_1}-\omega_1)(W-E_p-E_{p-k_1-k_2}-\omega_1-\omega_2) \times(-E_{p-k_1-k_2}-E_{p-k_1}-\omega_2)\cong-(2M)^2(\omega_1+\omega_2) \times\left\{1+\frac{\omega_1+\omega_2}{2M}+\frac{1}{(\omega_1+\omega_2)}\left[\frac{p^2+(p-k_1-k_2)^2}{2M}+\epsilon\right]\right\}, \quad (29)$$

where ϵ is the binding energy. Equation (29) is essentially an expansion in powers of (μ/M), and though objectionable from a strict mathematical point of view, it certainly has equal validity with the other approximations carried out in the nonrelativistic domain, for example, the manner of evaluation of spin matrix elements.

It is found that the elements of M_1 and M_2 contribute equally to the final result (the $\boldsymbol{\tau}^{(1)}\cdot\boldsymbol{\tau}^{(2)}$ dependence canceling). With a notation paralleling that of L2, we then find

$$V_4^{(a)}(\mathbf{r},\mathbf{r}')=-\frac{\lambda^2}{(2\pi)^3}\frac{3}{(2M)^2}\int d\mathbf{p}d\mathbf{k}_1d\mathbf{k}_2 \times e^{i\mathbf{p}\cdot(\mathbf{r}-\mathbf{r}')}e^{i(\mathbf{k}_1+\mathbf{k}_2)\cdot\mathbf{r}'}\frac{1}{\omega_1\omega_2(\omega_1+\omega_2)} \times\left\{1-\frac{\omega_1+\omega_2}{2M}-\frac{[p^2+(p-k_1-k_2)^2]/2M+\epsilon}{\omega_1+\omega_2}\right\}. \quad (30)$$

[21] See Fig. 1 of L2 for typical diagrams.

[22] It is shown in Appendix B that the terms neglected in Eq. (27) give rise to spin-orbit coupling.

[23] G. Lüders, Institute for Theoretical Physics, Copenhagen, Denmark (unpublished manuscript). However, Lüders does not include the binding energy terms in Eq. (29), which play an important role in the considerations of part C below.

In Eq. (30) the first term in the face brackets is the $V_4^{(a)}$ of L2; the second term, which results from expansion of denominators associated with intermediate states which contain a pair, precisely cancels the $V_4^{(b)}$ of L2; the third term is velocity dependent. The latter type of term always results from the expansion of an energy denominator in which there are no pairs and one or more mesons. It will be shown quite generally in part C that such terms need not be included in the treatment of the low energy properties of the two-nucleon system. We therefore assert that, except for corrections of relative order $(\mu/M)^2$, the adiabatic potential contributed by the two-pair terms has the form

$$V_4(r)=-3\left(\frac{g^2}{4\pi}\right)^2\left(\frac{\mu}{2M}\right)^2\frac{1}{\mu r^2}\frac{2}{\pi}K_1(2\mu r). \quad (31)$$

B. One-Pair Terms

Of the twenty-four one-pair matrix elements twelve are too small by a factor μ/M. The remaining twelve belong half to M_1 and half to M_2. Three of the diagrams for M_1 are shown in Fig. 4. The remaining contributions to M_1 result from interchange of the two particles and those of M_2 by uncrossing the meson lines. Beneath each diagram is indicated its energy denominator in the adiabatic limit. The first point to be emphasized is that each set of three matrix elements yields precisely the same set of energy denominators. The second fact to be noted is that the spin matrix element for one particle is unity, whereas for the other it has the typical form:[24]

$$[\mathbf{p}|\Lambda_-^{(r)}(\mathbf{p}-\mathbf{k}_1)|\mathbf{p}-\mathbf{k}_1-\mathbf{k}_2]$$
$$=-(2M)^{-2}\boldsymbol{\sigma}^{(r)}\cdot\mathbf{k}_1\boldsymbol{\sigma}^{(r)}\cdot\mathbf{k}_2+O((\mu/M)^4). \quad (32)$$

All the spin matrix elements have this form with the interchange of k_1 and k_2 the only alternative that arises. The remainder of each matrix element is symmetrical in the two momenta, so that this is not a significant change. In short, we claim that the M_1 and M_2 elements yield precisely the same contributions, except for the isotopic spin operators and when added together the $\boldsymbol{\tau}^{(1)}\cdot\boldsymbol{\tau}^{(2)}$ dependence cancels, but the remainder adds. In all we find that

$$V_4'(\mathbf{r})=-\frac{3\lambda^2}{(2M)^3}\int d\mathbf{k}_1 d\mathbf{k}_2\frac{e^{i(\mathbf{k}_1+\mathbf{k}_2)\cdot\mathbf{r}}}{4\omega_1\omega_2}4\mathbf{k}_1\cdot\mathbf{k}_2$$
$$\times\left[\frac{1}{\omega_1(\omega_1+\omega_2)}+\frac{1}{\omega_2(\omega_1+\omega_2)}+\frac{1}{\omega_1\omega_2}\right]$$
$$=-\frac{6\lambda^2}{(2M)^3}\left(\int\frac{\mathbf{k}d\mathbf{k}e^{i\mathbf{k}\cdot\mathbf{r}}}{\omega^2}\right)^2$$
$$=6(g^2/4\pi)^2(\mu/2M)^3(\mu r^2)^{-1}$$
$$\times[1+(\mu r)^{-1}]^2e^{-2\mu r}. \quad (33)$$

[24] The value of the first term of the matrix element is obtained without the adiabatic approximation.

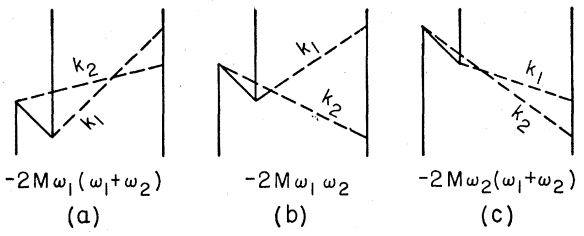

FIG. 4. Diagrams representing three of the six one-pair matrix elements belonging to M_1.

The net result is thus a repulsive potential, which at distances $r\lesssim\mu^{-1}$ is appreciably more significant than the second-order central potential. The neutral theory[25] yields a third of Eq. (33).

The fourth-order potential which we propose is the sums of Eqs. (31) and (33). It is hoped to report separately on the low energy properties of the two-nucleon system predicted by this potential used in conjunction with the well-known second-order result (and the hard-core model of L2). Further qualitative discussion of the model is given in Sec. VII.

C. Velocity Dependent Forces

We turn here to the proof of the assertion made in part *A* that a certain class of velocity dependent forces need not be included in a first treatment of the low energy properties of the two-nucleon system. The procedure to be employed is merely an adaptation to the present case of the proof given in L1 of the well-known result that there is a cancellation in the neutral scalar theory between the fourth-order potential and velocity dependent corrections to the second-order potential. For the sake of concreteness the discussion will be confined to the terms of actual interest, though it will be seen that the method of attack is widely applicable.

Consider the Schrödinger equation in momentum space with interaction terms consisting of the second-order terms plus the leading two-pair fourth-order terms,

$$\left(-\frac{p^2}{M}-\epsilon\right)\phi(\mathbf{p})=\lambda\int\frac{d\mathbf{k}}{\omega}\frac{\boldsymbol{\sigma}^{(1)}\cdot\mathbf{k}\boldsymbol{\sigma}^{(2)}\cdot\mathbf{k}}{(2M)^2}$$
$$\times\frac{\boldsymbol{\tau}^{(1)}\cdot\boldsymbol{\tau}^{(2)}}{W-E_p-E_{p-k}-\omega}\phi(\mathbf{p}-\mathbf{k})+\frac{\lambda^2}{(2M)^2}\int\frac{d\mathbf{k}_1 d\mathbf{k}_2}{4\omega_1\omega_2}$$
$$\times\left[\frac{2(3-2\boldsymbol{\tau}^{(1)}\cdot\boldsymbol{\tau}^{(2)})}{W-E_p-E_{p-k_1-k_2}-\omega_1-\omega_2}\right.$$
$$\left.+\frac{2(3+2\boldsymbol{\tau}^{(1)}\cdot\boldsymbol{\tau}^{(2)})}{W-E_p-E_{p-k_1-k_2}-\omega_1-\omega_2}\right]$$
$$\times\phi(\mathbf{p}-\mathbf{k}_1-\mathbf{k}_2). \quad (34)$$

[25] See L2, reference 14, where the result should be multiplied by a factor of two. It is difficult to see how the symmetric and neutral theory can possibly differ other than in the isotopic spin operator dependence.

In Eq. (34), the adiabatic limit has been selectively taken in the fourth-order terms, namely, in the denominators corresponding to intermediate states with pairs. These denominators have been adequately treated in part A. If we now expand the remaining type of denominator about its adiabatic limit, the resulting equation can be written as

$$\left(-\frac{p^2}{M}-\epsilon\right)\phi(\mathbf{p})=-\lambda\tau^{(1)}\cdot\tau^{(2)}\int d\mathbf{k}\frac{\sigma^{(1)}\cdot\mathbf{k}\sigma^{(2)}\cdot\mathbf{k}}{(2M)^2}$$
$$\times\left\{\frac{1}{\omega^2}+\frac{1}{\omega^3}\left[\frac{(\mathbf{p}-\mathbf{k})^2-p^2}{2M}-\frac{(\mathbf{p}-\mathbf{k})^2}{M}-\epsilon\right]\right\}\phi(\mathbf{p}-\mathbf{k})$$
$$-\frac{3\lambda^2}{(2M)^2}\int\frac{d\mathbf{k}_1d\mathbf{k}_2}{\omega_1\omega_2}\left\{\frac{1}{\omega_1+\omega_2}+\frac{1}{(\omega_1+\omega_2)^2}\right.$$
$$\left.\times\left[\frac{(\mathbf{p}-\mathbf{k}_1-\mathbf{k}_2)^2-p^2}{2M}-\frac{(\mathbf{p}-\mathbf{k}_1-\mathbf{k}_2)^2}{M}-\epsilon\right]\right\}$$
$$\times\phi(\mathbf{p}-\mathbf{k}_1-\mathbf{k}_2). \quad (35)$$

The deliberate rearrangement of the velocity-dependent terms assumes significance, if we take the point of view that these terms are to be treated as a small perturbation to the problem defined by the static second plus fourth order potential. First of all then, we invoke the argument given in L1, Sec. 4.32, based on variational principle Eq. (49) of L1, to show that the terms proportional to the difference of the kinetic energies of initial and final momentum states vanish. We then iterate the remaining velocity-dependent terms by making use of the Schrödinger equation of the unperturbed problem. In this way the velocity dependent corrections to the second-order potential give rise to fourth- and sixth-order *static* potentials, whereas velocity-dependent corrections to the fourth-order potential give rise to sixth- and eighth-order *static* potentials. These potentials which we denote by $\delta V_4(r)$, $\delta V_6(r)$ and $\delta V_8(r)$, respectively, have the form (after Fourier transformation)

$$\delta V_4(r)=\frac{\lambda^2}{(2M)^4}(\tau^{(1)}\cdot\tau^{(2)})^2\int\frac{d\mathbf{k}_1d\mathbf{k}_2}{\omega_1^3\omega_2^2}\exp[i(\mathbf{k}_1+\mathbf{k}_2)\cdot\mathbf{r}]$$
$$\times\sigma^{(1)}\cdot\mathbf{k}_1\sigma^{(1)}\cdot\mathbf{k}_2\sigma^{(2)}\cdot\mathbf{k}_1\sigma^{(2)}\cdot\mathbf{k}_2, \quad (36)$$

$$\delta V_6(r)=\frac{3\lambda^2}{(2M)^4}\tau^{(1)}\cdot\tau^{(2)}\int\frac{d\mathbf{k}_1d\mathbf{k}_2d\mathbf{k}_3}{\omega_1\omega_2\omega_3}$$
$$\times\exp[i(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)\cdot\mathbf{r}]\sigma^{(1)}\cdot\mathbf{k}_1\sigma^{(2)}\cdot\mathbf{k}_1$$
$$\times\left[\frac{1}{\omega_1^2(\omega_2+\omega_3)}+\frac{1}{\omega_1(\omega_2+\omega_3)^2}\right], \quad (37)$$

$$\delta V_8(r)=\frac{9\lambda^4}{(2M)^4}$$
$$\times\int\frac{d\mathbf{k}_1\cdots d\mathbf{k}_4\exp[i(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4)\cdot\mathbf{r}]}{\omega_1\omega_2\omega_3\omega_4(\omega_1+\omega_2)^2(\omega_3+\omega_4)}. \quad (38)$$

Anticipating some of the results to be derived in the sections which follow, we note here that when we derive the no-pair terms of the fourth-order potential and the leading terms of the sixth- and eighth-order potentials, we shall find contributions which respectively cancel Eqs. (36)–(38). This phenomenon, in addition to the entire development of this section, points to the fact, previously remarked,[26] that in the low energy limit it is, in general, inconsistent to consider velocity-dependent corrections to the potential except in conjunction with static potentials of higher order.

D. No-Pair Terms

Here only two matrix elements are contributed by the ladder diagram (M_1), but there is the full complement of six of type M_2. (Four M_1 elements were of the reducible type.) The result we shall record differs from that given in L2, Eq. (20), in two respects. First, we omit the first term of the latter equation since it cancels, as promised, against $\delta V_4(r)$, Eq. (36). Secondly, we obtain an additional spin-independent, but charge-dependent term not found there. As the adiabatic no-pair term, we therefore take (some rearrangement is involved)

$$V_4''(r)=-\frac{\lambda^2}{(2M)^4}\int\frac{d\mathbf{k}_1d\mathbf{k}_2\exp[i(\mathbf{k}_1+\mathbf{k}_2)\cdot\mathbf{r}]}{\omega_1^2\omega_2(\omega_1+\omega_2)}$$
$$\times[2\tau^{(1)}\cdot\tau^{(2)}(\mathbf{k}_1\cdot\mathbf{k}_2)^2+3\sigma^{(1)}\cdot(\mathbf{k}_1\times\mathbf{k}_2)$$
$$\times\sigma^{(2)}\cdot(\mathbf{k}_1\times\mathbf{k}_2)]\left[\frac{2}{\omega_1}+\frac{1}{\omega_2}\right]. \quad (39)$$

We note parenthetically that if we multiply the equation above by $(2M/\mu)^4$, the resulting expression is just the *leading* term of the fourth-order potential of the pseudoscalar theory with pseudovector coupling. That the no-pair terms give the leading contribution in this case is a consequence, first of all, of the fact that for derivative coupling the spin matrix elements are of the same order of magnitude for no-pair, one-pair, and two-pair interactions. One then chooses for maximum contribution the terms with minimum energy denominators, i.e., the no-pair terms. The equivalence to the pseudoscalar result then follows from the observation that it is the spatial part of the coupling that is largest for the no-pair terms.

After a fairly laborious calculation we find for the potential of Eq. (39),

$$V_4''(\mathbf{r})=\mu(g^2/4\pi)^2(\mu/2M)^4[\tau^{(1)}\cdot\tau^{(2)}U_\tau(\mu r)$$
$$+\sigma^{(1)}\cdot\sigma^{(2)}U_\sigma(\mu r)+S_{12}U_T(\mu r)], \quad (40)$$

where

$$U_\tau(x)=(8/\pi)[-A_1+2A_2-4A_3+2A_4],$$
$$U_\sigma(x)=(8/\pi)[-2A_2+2A_3-A_4], \quad (41)$$
$$U_T(x)=(4/\pi)[2A_2-5A_3+4A_4],$$

[26] Y. Nambu, Progr. Theoret. Phys. **5**, 614 (1950).

and

$$A_1=K_0^{\mathrm{IV}}(2x)/x, \quad A_2=K_0'''(2x)/x^2, \quad A_3=K_0''(2x)/x^3, \quad A_4=K_0'(2x)/x^4, \tag{42}$$

the superscripts indicating derivative. Equations (40) to (42) are in agreement with the work of previous authors who have considered the $ps-pv$ theory of nuclear forces.[27]

For the pseudoscalar coupling case we remark finally that, if we wish to include in a consistent fashion potentials of the order of magnitude of those considered in this section, we must also take into account terms arising from (a) velocity dependence in the spin-matrix elements of the two-pair diagrams; part of these are shown in Appendix B to lead to spin-orbit coupling; (b) corrections to the pair denominators of two-pair and one-pair matrix elements already included in lowest approximation; (c) one-pair matrix elements previously neglected.

V. SIXTH- AND EIGHTH-ORDER ADIABATIC POTENTIALS

A. Sixth-Order Results

The prescription given in Sec. III can be directly applied to the computation of any desired portion of the sixth-order potential. All matrix elements are contained in the six Feynman diagrams, shown in Fig. 5. There are $3!\times 6!=4320$ matrix elements in all; of these 8 from Fig. 5(a) merely contain Δ_2 three times, and 176 from Figs. 5(a), (b), (c) are combinations of Δ_2 and Δ_4 in either order. Only the remaining 4136 matrix elements are *bona fide* members of the sixth-order interaction kernel. Fortunately, only a small fraction of these (128 in all) contribute to the leading term of the adiabatic potential.

To see this we first look for contributions with the largest possible spin matrix elements. These are of the general form[28]

$$(\mathbf{p}_r|\Gamma^{(r)}\Lambda_\pm^{(r)}(\mathbf{p}_r-\epsilon_r\mathbf{k}_i)\Gamma^{(r)}\Lambda_\mp(\mathbf{p}_r-\epsilon_r\mathbf{k}_i-\epsilon_r\mathbf{k}_j)\times\Gamma^{(r)}|\mathbf{p}_r-\epsilon_r(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3))=(\mathbf{p}_r|\Lambda_\mp^{(r)}(\mathbf{p}_r-\epsilon_r\mathbf{k}_i)\times\Gamma^{(r)}\Lambda_\pm(\mathbf{p}_r-\epsilon_r\mathbf{k}_i-\epsilon_r\mathbf{k}_j)|\mathbf{p}_r-\epsilon_r(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)), \tag{43}$$

admitting the restriction that in sixth order there can be at most one pair associated with each nucleon line, but insisting on that pair for maximum value of the matrix element. We shall designate the two matrix elements involved in Eq. (43) by $(+-)_{i+j}$ and $(-+)_i$. Then with $\Gamma^{(r)}=(\gamma_0\gamma_5)^{(r)}$ these have the general values

$$(+-)_{i+j}=(2M)^{-1}i\boldsymbol{\sigma}^{(r)}\cdot\epsilon_r\mathbf{k}_l, \quad (l\neq j\neq i), \tag{44}$$

$$(-+)_i=(2M)^{-1}i\boldsymbol{\sigma}^{(r)}\cdot\epsilon_r\mathbf{k}_i. \tag{45}$$

According to Fig. 5, for $r=1$, $i=1$ and $l=3$, whereas for $r=2$ the values which these indices assume varies from diagram to diagram but are the same for all matrix elements arising from a given diagram. Thus for each of the graphs a–f there are four possible spin elements consistent with our option of choosing a $(+-)$ or a $(-+)$ matrix element independently for each particle.

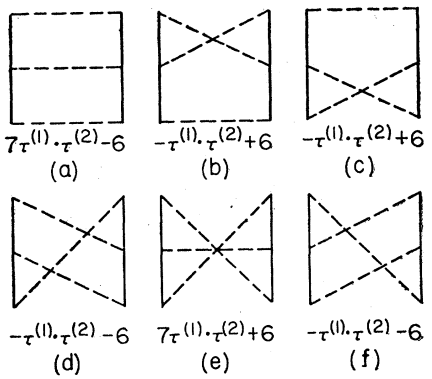

FIG. 5. Feynman diagrams for sixth-order interaction kernel. Beneath each figure stands the associated isotopic operator for matrix elements derived therefrom.

With this much settled and with the isotopic operators already designated in Fig. 5, one can begin the job of enumerating allowed matrix elements as soon as one recognizes that minimum energy denominators are associated with those processes in which a virtual pair created at a given point is annihilated at the next interaction point. (The restriction to one pair at a time is not sufficiently stringent here.) A typical diagram having its origin in our Fig. 3(f) is Fig. 3(a) of L2. A job of straightforward enumeration shows that twenty-four matrix elements of the specified type arise from each of Figs. 5(d), (e), (f), twenty (irreducible) elements each from (b), (c), and sixteen from (a), 128 in all. Counting up is simplified by the fact that (d) and (f) yield precisely the same set of matrix elements, as do (b) and (c). Further, for each Feynman diagram the matrix elements classify into sets associated with the different spin matrix elements, and when one takes advantage of symmetry properties in the meson momenta, there turns out to be in all only four distinct sets, comprising at most six matrix elements each. These then combine further into two sets. Of these one precisely cancels, as promised, the quantity $\delta V_6(r)$ of Eq. (37). The remainder which we take as the sixth-order adiabatic potential has the extremely simple appearance[29]

$$V_6(r)=-\left(\frac{g^2}{4\pi}\right)^3\left(\frac{1}{2M}\right)^4\boldsymbol{\tau}^{(1)}\cdot\boldsymbol{\tau}^{(2)}(2\pi^2)^{-3}\int\frac{d\mathbf{k}_1d\mathbf{k}_2d\mathbf{k}_3}{\omega_1^2\omega_2^2\omega_3^2}\times\exp[i(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)\cdot\mathbf{r}]4\boldsymbol{\sigma}^{(1)}\cdot\mathbf{k}_1\boldsymbol{\sigma}^{(2)}\cdot\mathbf{k}_3$$
$$=(4/3)(g^2/4\pi)^3(\mu/2M)^4\boldsymbol{\tau}^{(1)}\cdot\boldsymbol{\tau}^{(2)}(\mu^2r^3)^{-1}\times[1+(\mu r)^{-1}]^2[\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{\sigma}^{(2)}+S_{12}]e^{-3\mu r}. \tag{46}$$

[27] K. Nishijima, Progr. Theoret. Phys. **6**, 815, 911 (1951). Taketani, Machida, and Onuma, Progr. Theoret. Phys. **6**, 638 (1951).

[28] Our notation parallels L1; thus $\mathbf{r}=1, 2$; $\mathbf{p}_1=-\mathbf{p}_2=\mathbf{p}$; $\epsilon_1=-\epsilon_2=1$. The isotopic dependence has been removed from the matrix element; $i, j, l=1, 2, 3$ each. One seeks the largest value of the spin matrix elements first because in the adiabatic region, these differ from one another by even powers of μ/M, whereas for energy denominators one progresses one power at a time.

[29] We obtain the result of L2, Eq. (23), if we include only those matrix elements which he explicitly stipulates, namely, those with two intermediate states in which there are two mesons present and one intermediate state with three mesons. There are many other possibilities, however.

B. Eighth-Order Results

We proceed as before: As the basis of our analysis, we use the twenty-four eighth-order Feynman diagrams involving the exchange of four mesons. The associated isotopic operators are easily computed. The leading terms of the adiabatic potential are the four-pair terms with no more than one pair present in any intermediate state (see Fig. 3(b) of L2). The spin matrix elements are unity to lowest order, as for the fourth-order potential. The conventional matrix elements associated with each Feynman diagram are readily enumerated—twenty of the diagrams yield six irreducible elements each, and the remaining four diagrams just two each. In the adiabatic limit, all these diagrams yield, aside from the charge dependence, just three distinct sets of matrix elements. Again it is found that a considerable part of the result cancels $\delta V_8(r)$ Eq. (38). The net result is then[30]

$$V_8(r)=-3\left(\frac{g^2}{2M}\right)^4(2\pi)^{-12}\int d\mathbf{k}_1\cdots d\mathbf{k}_4 \times\frac{\exp[i(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4)\cdot\mathbf{r}]}{\omega_1\omega_2\omega_3\omega_4} \times\left[\frac{3}{(\omega_1+\omega_2)(\omega_1+\omega_3)(\omega_2+\omega_4)} +\frac{2}{(\omega_1+\omega_2)(\omega_1+\omega_3)(\omega_1+\omega_4)}\right]. \quad (47)$$

After extensive rearrangement it is found that the integral in Eq. (47) can be performed by methods analogous to those given in the Appendix of L2. It has the value

$$V_8(r)=-6\left(\frac{g^2}{4\pi}\right)^4\left(\frac{\mu}{2M}\right)^4\frac{1}{\mu^3r^4}\frac{2}{\pi}K_1(4\mu r). \quad (48)$$

As pointed out in L2, the effect of $V_6(r)$ and especially of $V_8(r)$ on the low energy properties of the two-nucleon system will provide a test of "convergence" of the asymptotic series for the nuclear potential.

VI. MANY-BODY FORCES

A. General Analysis of Leading Terms.

The analysis of many-body forces[31,32] could be carried out from first principles by deriving a many-particle relativistic equation and reducing this to a three-dimensional formalism. Such an undertaking, because of its length, is out of place here and will be reserved for a possible later publication. We begin our efforts, therefore, with the results of such an analysis, which for n particles is an equation of the form

$$(W-E_{p1}-E_{p2}-\cdots-E_{pn})\delta(\mathbf{p}_1+\mathbf{p}_2+\cdots+\mathbf{p}_n) \times\phi(\mathbf{p}_1\cdots\mathbf{p}_n)=\delta(\mathbf{p}_1+\cdots+\mathbf{p}_n)\Lambda_+^{(1)}(\mathbf{p}_1)\cdots\Lambda_+^{(n)}(\mathbf{p}_n) \times\int d\mathbf{p}_1'\cdots d\mathbf{p}_n'\Delta(\mathbf{p}_1\cdots\mathbf{p}_n;\mathbf{p}_1'\cdots\mathbf{p}_n') \times\delta(\mathbf{p}_1'+\cdots+\mathbf{p}_n')\phi(\mathbf{p}_1'\cdots\mathbf{p}_n'). \quad (49)$$

The kernel $\Delta(\mathbf{p}_1\cdots\mathbf{p}_n;\mathbf{p}_1'\cdots\mathbf{p}_n')$ contains interactions between $2, 3, \cdots n$ particles whose form is determined completely by the rules of perturbation theory as formulated in Sec. III of this paper. In particular, the adiabatic potentials can be defined in a manner precisely analogous to the definition given in Eq. (26).

The leading terms in the n-body force arise from Feynman diagrams involving n mesons in which two meson lines emanate from each nucleon line and in which there are n pairs but at most one pair at a time. Diagrams for typical matrix elements of the three- and four-particle forces are illustrated in Fig. 6. The expansion parameter is then $(g^2/4\pi)^n(\mu/2M)^n$. There are also n-particle forces involving $n-1$ mesons, but these can give rise to only $n-2$ pairs (see below for the three-particle case), and therefore the expansion parameter is $(g^2/4\pi)^{n-1}(\mu/2M)^n$, smaller roughly by an order of magnitude. By restricting our attention to the dominant n-pair terms a number of general statements can be made about the nature of the resulting potential in the adiabatic limit:

(1) All spin matrix elements are unity to lowest order. The resulting forces are spin independent.

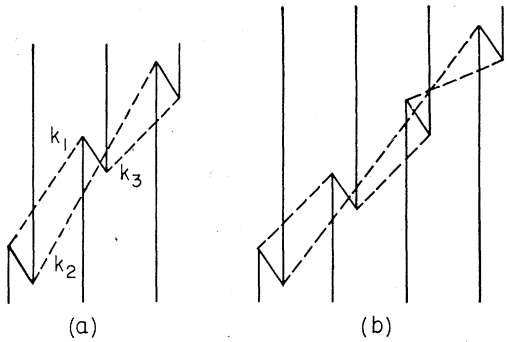

FIG. 6. Diagrams for typical matrix elements of leading contributions to three- and four-body forces.

[30] See Eq. (24) of L2. We obtain the result quoted there if we omit the considerable class of matrix elements which have at most two mesons in all intermediate states in which there are no pairs.

[31] The analytic form of the n-particle force was first given by G. Wentzel, Helv. Phys. Acta **XV**, 111 (1942), using equivalent pair theory methods. For the saturation problem see G. Wentzel, Helv. Phys. Acta **XXV**, 569 (1952).

[32] The major results of this section have been derived independently by S. D. Drell (private communication) using a method based on the canonical transformation of F. J. Dyson, Phys. Rev. **73**, 929 (1948). See also S. D. Drell and E. M. Henley, Phys. Rev. **88**, 1053 (1952). The problem of saturation of nuclear forces is under investigation by Drell and Huang. We wish to take this opportunity to thank Dr. Drell for an illuminating discussion of his work.

(2) The potential will consist of a sum of terms of the form $V(\{|\mathbf{r}_{i_1}-\mathbf{r}_{i_2}|+|\mathbf{r}_{i_2}-\mathbf{r}_{i_3}|+\cdots+|\mathbf{r}_{i_n}-\mathbf{r}_{i_1}|\})$, where $i_1, i_2, \cdots i_n$ is some permutation of $1, 2, \cdots n$. Aside from isotopic spin dependence to be dealt with in a moment, the number of distinct terms of this form is the number of ways of forming n-vector differences from n-distinct vectors, each vector being utilized twice. This number is $(n-1)!/2$. For there are $\frac{1}{2}n(n-1)$ choices for the pair $|\mathbf{r}_{i_1}-\mathbf{r}_{i_2}|$, then $n-2$ choices for $\mathbf{r}_{i_3}, \cdots$, one choice for $\mathbf{r}_{i_n}$. But then any of the n-coordinate differences can stand first in the sum and we must therefore divide by n. The number of Feynman diagrams contributing to one such ordering is 2^n, for having fixed on which particles are directly coupled by meson lines, we have still the possibility of interchanging the role of the two mesons which emerge from a given particle. And we can do this for n particles. The same result is obtained by dividing $(n-1)!/2$ into $2^{n-1}(n-1)!$, the total number of Feynman diagrams of the type under consideration.

(3) The contributions from each of the 2^n diagrams belonging to a given connection scheme differ only in isotopic spin dependence. The sum of the 2^n isotopic spin operators has the form

$$\{\tau_{\alpha_1}{}^{i_1}, \tau_{\alpha_n}{}^{i_1}\}\{\tau_{\alpha_1}{}^{i_2}, \tau_{\alpha_2}{}^{i_2}\}\{\tau_{\alpha_2}{}^{i_3}, \tau_{\alpha_3}{}^{i_3}\}\cdots\{\tau_{\alpha_{n-1}}{}^{i_n}, \tau_{\alpha_n}{}^{i_n}\}$$
$$=2^n\delta_{\alpha_n\alpha_1}\delta_{\alpha_1\alpha_2}\cdots\delta_{\alpha_{n-1}\alpha_n}=3\times 2^n. \quad (50)$$

The factor 2^n will cancel a similar factor in the denominator of the potential having its origin in the Fourier analysis of the meson field.

(4) One need calculate the analytic form of the potential for only a single connection scheme for the others will differ only by a relabelling of particle coordinates.

(5) We fix therefore on a standard connection scheme defined by $|\mathbf{r}_1-\mathbf{r}_2|+|\mathbf{r}_2-\mathbf{r}_3|+\cdots|\mathbf{r}_n-\mathbf{r}_1|$ and need consider only a single Feynman diagram associated with this connection. There will be just $n!$ time-ordered n-pair diagrams to compute corresponding to the permutation in time of the n pairs. Only half of these at most are distinct because of symmetry with respect to the inversion of the (vertical) time axis.

(6) The sign of the n-body force is $(-1)^{n-1}$ as follows from the fact that the matrix elements contain $2n-1$ negative energy denominators, but that there is an additional minus sign for an odd number of pairs.

Summarizing statements (1)–(6), we have reduced the problem to the evaluation of the quantity

$$V(\{|\mathbf{r}_1-\mathbf{r}_2|+\cdots+|\mathbf{r}_n-\mathbf{r}_1|\})$$
$$=(-1)^{n-1}6\left(\frac{g^2}{(2\pi)^3}\right)^n\left(\frac{1}{2M}\right)^n\int\frac{d\mathbf{k}_1\cdots d\mathbf{k}_n}{\omega_1\cdots\omega_n}$$
$$\times\exp[i\{\mathbf{k}_1\cdot(\mathbf{r}_1-\mathbf{r}_2)+\cdots k_n\cdot(\mathbf{r}_n-\mathbf{r}_1)\}]$$
$$\times[\quad], \quad (51)$$

where the square bracket is a sum of $n!/2$ terms, each comprising the adiabatic limit of the reciprocal of a product of energy denominators for the intermediate states without pairs

B. Three- and Four-Body Forces

We apply the general considerations given in the preceding section to the cases $n=3$ and $n=4$. In the former instance there are only three terms in the unspecified square bracket of Eq. (51). These are

$$[\quad]=\frac{1}{(\omega_1+\omega_2)(\omega_1+\omega_3)}+\frac{1}{(\omega_1+\omega_2)(\omega_2+\omega_3)}$$
$$+\frac{1}{(\omega_1+\omega_3)(\omega_2+\omega_3)}$$
$$=\frac{2\omega_1}{(\omega_2+\omega_3)(k_1^2-k_3^2)}+\frac{2\omega_3}{(\omega_1+\omega_2)(k_3^2-k_1^2)}, \quad (52)$$

the last form following from the observation that

$$\frac{1}{(\omega_1+\omega_2)(\omega_2+\omega_3)}=\frac{1}{(\omega_1-\omega_3)}\left[\frac{1}{\omega_2+\omega_3}-\frac{1}{\omega_1+\omega_2}\right]. \quad (53)$$

Using the last form of Eq. (52), the required integrations in Eq. (51) can be carried by methods indicated in the Appendix of L2. We find for the three-particle force the expression

$$V_{3p}=12\left(\frac{g^2}{4\pi}\right)^3\left(\frac{\mu}{2M}\right)^3\frac{2}{\pi}$$
$$\times\frac{K_1[\mu(|\mathbf{r}_1-\mathbf{r}_2|+|\mathbf{r}_2-\mathbf{r}_3|+|\mathbf{r}_3-\mathbf{r}_1|)]}{\mu^2|\mathbf{r}_1-\mathbf{r}_2||\mathbf{r}_2-\mathbf{r}_3||\mathbf{r}_3-\mathbf{r}_1|}. \quad (54)$$

For the four-particle force the bracket of Eq. (51) consists of twelve terms, which the reader can easily derive for himself from a set of diagrams. By extensive combination and rearrangement these can be brought into the useful form

$$[\quad]=\frac{4\omega_3\omega_4}{(\omega_1+\omega_2)(k_1^2-k_3^2)(k_2^2-k_4^2)}$$
$$+\frac{4\omega_1\omega_2}{(\omega_3+\omega_4)(k_1^2-k_3^2)(k_2^2-k_4^2)}$$
$$+\frac{4\omega_1\omega_4}{(\omega_2+\omega_3)(k_1^2-k_3^2)(k_4^2-k_2^2)}$$
$$+\frac{4\omega_2\omega_3}{(\omega_1+\omega_4)(k_3^2-k_1^2)(k_2^2-k_4^2)}. \quad (55)$$

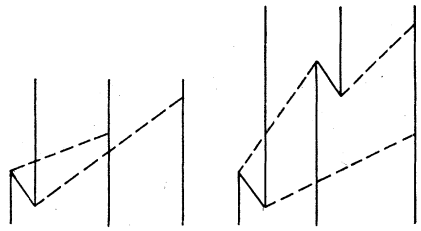

FIG. 7. Diagrams for typical matrix elements of the three-particle force: (a) One-pair term involving interchange of two mesons. (b) Two-pair terms analogous to one-pair terms of two-particle fourth-order potential.

The integrations can now be carried out as before, and the total result for the four-particle potential given as

$$V_{4p}=-24\left(\frac{g^2}{4\pi}\right)^4\left(\frac{\mu}{2M}\right)^4\frac{2}{\pi}$$
$$\times\left\{\frac{K_1[\mu(|\mathbf{r}_1-\mathbf{r}_2|+|\mathbf{r}_2-\mathbf{r}_3|+|\mathbf{r}_3-\mathbf{r}_4|+|\mathbf{r}_4-\mathbf{r}_1|)]}{\mu^3|\mathbf{r}_1-\mathbf{r}_2||\mathbf{r}_2-\mathbf{r}_3||\mathbf{r}_3-\mathbf{r}_4||\mathbf{r}_4-\mathbf{r}_1|}\right.$$
$$+\frac{K_1[\mu(|\mathbf{r}_1-\mathbf{r}_3|+|\mathbf{r}_3-\mathbf{r}_2|+|\mathbf{r}_2-\mathbf{r}_4|+|\mathbf{r}_4-\mathbf{r}_1|)]}{\mu^3|\mathbf{r}_1-\mathbf{r}_3||\mathbf{r}_3-\mathbf{r}_2||\mathbf{r}_2-\mathbf{r}_4||\mathbf{r}_4-\mathbf{r}_1|}$$
$$\left.+\frac{K_1[\mu(|\mathbf{r}_1-\mathbf{r}_2|+|\mathbf{r}_2-\mathbf{r}_4|+|\mathbf{r}_4-\mathbf{r}_3|+|\mathbf{r}_3-\mathbf{r}_1|)]}{\mu^3|\mathbf{r}_1-\mathbf{r}_2||\mathbf{r}_2-\mathbf{r}_4||\mathbf{r}_4-\mathbf{r}_3||\mathbf{r}_3-\mathbf{r}_1|}\right\}. \quad (56)$$

We have not tried to push our diligence beyond this point, although it is conceivable that by sufficient searching one could obtain by these methods the general coefficient of the n-body force. The case of five particles, at any rate, is certainly amenable to direct calculation by these techniques.

C. Other Three-Body Forces

We consider briefly two types of three-body force which are roughly an order of magnitude smaller than the leading term. Of course these have their analogues in forces between any number of particles.

We turn first to the three-particle force which involves the total interchange of only two mesons, two mesons emanating from one of the particles, say, and each terminating on one of the others. There are six Feynman diagrams in all: for the particle which is the source of the two mesons we may interchange the role of the latter. The leading terms are one-pair terms, the pair subsisting only between neighboring interaction points as illustrated in Fig. 7(a). Six matrix elements of the specified type arise from each Feynman diagram. The calculation is straightforward and yields the result

$$V_{3p}'=2\left(\frac{g^2}{4\pi}\right)^2\left(\frac{\mu}{2M}\right)^3$$
$$\times\left[\boldsymbol{\tau}^{(2)}\cdot\boldsymbol{\tau}^{(3)}\frac{\boldsymbol{\sigma}^{(2)}\cdot(\mathbf{r}_1-\mathbf{r}_2)}{|\mathbf{r}_1-\mathbf{r}_2|}\frac{\boldsymbol{\sigma}^{(3)}\cdot(\mathbf{r}_1-\mathbf{r}_3)}{|\mathbf{r}_1-\mathbf{r}_3|}\right.$$
$$\times\left(1+\frac{1}{\mu|\mathbf{r}_1-\mathbf{r}_2|}\right)\left(1+\frac{1}{\mu|\mathbf{r}_1-\mathbf{r}_3|}\right)$$
$$\times\frac{\exp[-\mu(|\mathbf{r}_1-\mathbf{r}_2|+|\mathbf{r}_1-\mathbf{r}_3|)]}{\mu|\mathbf{r}_1-\mathbf{r}_2||\mathbf{r}_1-\mathbf{r}_3|}$$
$$\left.+1\leftrightarrow2+1\leftrightarrow3\right]. \quad (57)$$

We consider finally corrections of relative order μ/M to the leading three-pair terms computed in part B. The terms which contribute are as follows: corrections arise from the expansion of energy denominators of the leading three-pair terms. These are first the velocity-dependent terms supplied by the energy denominators for intermediate states without pairs. In accordance with previous results (see Sec. IV, C) these will not be given further consideration here since they must be regarded in connection with still higher order three-body forces. There are, however, the terms provided by pair denominators. In exact analogy to the situation which obtains in the case of fourth-order two-particle forces, these are canceled by the leading contribution from the three-pair terms which involve two pairs at a time.

There remains the leading two-pair terms. One easily enumerates twelve matrix elements for each Feynman diagram. A typical contribution is illustrated in Fig. 7(b). Calculation shows the result to be spin and charge independent and of the form

$$V_{3p}''=-12\left(\frac{g^2}{4\pi}\right)^3\left(\frac{\mu}{2M}\right)^4$$
$$\times\frac{\exp[-\mu(|\mathbf{r}_1-\mathbf{r}_2|+|\mathbf{r}_1-\mathbf{r}_3|+|\mathbf{r}_2-\mathbf{r}_3|)]}{\mu^2|\mathbf{r}_1-\mathbf{r}_2||\mathbf{r}_1-\mathbf{r}_3||\mathbf{r}_2-\mathbf{r}_3|}$$
$$\times\left[\frac{(\mathbf{r}_1-\mathbf{r}_2)\cdot(\mathbf{r}_1-\mathbf{r}_3)}{|\mathbf{r}_1-\mathbf{r}_2||\mathbf{r}_1-\mathbf{r}_3|}\left(1+\frac{1}{\mu|\mathbf{r}_1-\mathbf{r}_2|}\right)\right.$$
$$\left.\times\left(1+\frac{1}{\mu|\mathbf{r}_1-\mathbf{r}_3|}\right)+1\leftrightarrow2+1\leftrightarrow3\right]. \quad (58)$$

VII. CONCLUDING REMARKS

We have confined ourselves in this paper so far to the purely formal task of deriving static potentials with no radiative corrections. It is our opinion that Lévy's treatment of such corrections, aside possibly from ques-

tions of detail is qualitatively accurate. Quantitatively, one can say little more than that there is no *a priori* justification for equating the coupling constants associated with the second- and fourth-order potentials.[33] The relative values of the constants can only be adjusted by experiment. The result of Lévy that one obtains good agreement with experience by equating the two constants in no sense constitutes a proof of uniqueness. It is, in fact, almost self-evident that if one can obtain agreement by the use of two parameters only (coupling constant and cut-off radius), one should, within the present accuracy of experiment, be able to obtain as good or better agreement by the addition of an extra coupling parameter. In short, one must expect a range of acceptable relative values of the coupling constants.

Now, however, we must consider the alteration of the picture by the results of this paper. In particular, for equal values of the coupling constant the efficacy of the central force has been severely suppressed relative to the tensor force. Assuming a three-parameter scheme, there is again every reason to expect that there will be an arc of parameter space which gives agreement with experiment. Whether this arc passes through the plane determined by equating the values of the two coupling constants is a question that only detailed calculation can answer.

Another problem which requires further clarification is the exact nature of the role of the repulsive core. This problem has two facets. First, there is the field-theoretical task of establishing that the pseudoscalar theory actually predicts a hard core. The present methods of field theory are certainly not adequate to yield a definitive answer to this question and as Lévy himself is the first to admit, his arguments are only of the plausibility variety. Second, however, assuming the hard core as a useful *ad hoc* hypothesis, the question then arises as to the sensitivity of the results to the detailed form of the asymptotic potential. One is inclined to suspect that the answer, at least within the framework of a three-parameter theory, is that they are not very sensitive. One can quote in support of this contention the numerical results of Taketani *et al.*[34] for the low energy neutron-proton system using the second plus fourth potential of the $ps-pv$ theory and a cut-off. They find reasonable agreement with experiment (using only two parameters) for a theory in which for triplet even states the central force is repulsive and the tensor force much more strongly attractive than the tensor force of the second-order potential. There obviously remains then much work to be done before one can speak of a successful meson-theoretical model of nuclear forces. *Note added in proof:*—Numerical calculations show that the potential obtained in the paper does not agree with experiment. Further work, to be published, indicates moreover, that the perturbation theory doesn't even converge.

[33] We wish to make clear that there is no claim of originality for the observations of this section. Several of the ideas were brought to the author's attention during a series of discussions at the Cambridge meeting of the American Physical Society. The observation of the nonuniqueness of the numerical results of Lévy was variously attributed to R. Jastrow and J. Blatt.

[34] Taketani, Machida, and Onuma, Progr. Theoret. Phys. **7**, 45 (1952). It has also been brought to the author's attention that R. Jastrow has shown that using the Lévy potential, one can reduce V_4 by a factor of 4 and still obtain a decent fit with the $n-p$ data with a choice of $g^2/4\pi=15$.

APPENDIX A

The procedure described in Sec. III for reducing the R.E. to a three-dimensional equation, following closely the work of previous authors, depends on carrying out a sequence of integrations over energy variables. However, the interpretation of the resulting matrix elements is indissolubly tied to their representation by time-ordered Feynman diagrams. It makes more physical sense, therefore, and also renders the derivation more concise, if we reserve for last the sequence of time integrations. By appropriately breaking up the latter into a sum of terms defined by integrations over mutually exclusive subregions of time, we divide a given Feynman diagram automatically into the time-ordered regions corresponding to three-dimensional matrix elements.

We illustrate first by deriving the g^2 approximation in this manner, starting from the equation (for relative motion in center-of-mass system)

$$[\tfrac{1}{2}W+p_0-H_1(\mathbf{p})][\tfrac{1}{2}W-p_0-H_2(\mathbf{p})]\psi(x) = -ig^2\Gamma_i^{(1)}\Gamma_i^{(2)}\Delta(x)\psi_{++}(x). \quad (A.1)$$

Here p, p_0 will refer to momentum operators or momenta according to context, $x_\mu=x_{1\mu}-x_{2\mu}$; $\Delta(x)$, the meson propagation function, will be represented by its three-dimensional Fourier transform

$$\Delta(x)=\frac{i}{(2\pi)^3}\int\frac{\exp[i\mathbf{k}\cdot\mathbf{r}-i\omega|x_0|]}{2\omega}d\mathbf{k}. \quad (A.2)$$

Finally by $\psi_{++}(x)$ we mean the wave function which is the coordinate space transform of the wave function of Eq. (15) of the text,

$$\psi_{++}(x)=\frac{1}{(2\pi)^2}\int d^4p e^{ipx}(-2\pi i)^{-1}[(\tfrac{1}{2}W+p_0-E_p)^{-1} + (\tfrac{1}{2}W-p_0-E_p)^{-1}]\phi_{++}(\mathbf{p})=\frac{1}{(2\pi)^2}\int d\mathbf{p} \times\exp[i\mathbf{p}\cdot\mathbf{r}-i(E_p-\tfrac{1}{2}W)|x_0|]\phi_{++}(\mathbf{p}). \quad (A.3)$$

We insert Eqs. (A.2) and (A.3) into the r.h.s. of (A.1). For the l.h.s., we introduce the Fourier transform of $\psi(x)$,

$$\psi(x)=\frac{1}{(2\pi)^2}\int d^4p e^{ipx}\phi(\mathbf{p}), \quad (A.4)$$

project with respect to positive energies, and then integrate both sides with respect to p_0 in the manner al-

ready explained several times in the text. We thus obtain the equation

$$(W-2E_p)\phi_{++}(\mathbf{p})=-ig^2\frac{1}{(2\pi)^4}\frac{i}{(2\pi)^3}$$
$$\times\Lambda_+^{(1)}(\mathbf{p})\Lambda_+^{(2)}(\mathbf{p})\Gamma_i^{(1)}\Gamma_i^{(2)}\int d\mathbf{r}dx_0d\mathbf{k}dp'dp_0$$
$$\times\frac{1}{2\omega}\left[\frac{1}{\frac{1}{2}W+p_0-E_p}+\frac{1}{\frac{1}{2}W-p_0-E_p}\right]e^{ip_0x_0}e^{i\mathbf{r}\cdot[\mathbf{k}-(\mathbf{p}-\mathbf{p}')]}$$
$$\times\exp[-i|x_0|(\omega+E_{p'}-\tfrac{1}{2}W)]\phi_{++}(p')$$
$$=-i\frac{g^2}{(2\pi)^3}\Lambda_+^{(1)}(\mathbf{p})\Lambda_+^{(2)}(\mathbf{p})\Gamma_i^{(1)}\Gamma_i^{(2)}\int dx_0d\mathbf{k}\frac{1}{2\omega}$$
$$\times\exp[-i|x_0|(\omega+E_p+E_{p-k}-W)]\phi_{++}(\mathbf{p}-\mathbf{k}). \quad (A.5)$$

The final step in the derivation is to break up the integral over x_0 into two regions defined by $x_0>0$ and $x_0<0$, and to use the relation

$$\int_0^\infty e^{-i\lambda x}dx=\int_{-\infty}^0 e^{i\lambda x}dx=\pi\delta(\lambda)-iP(1/\lambda), \quad (A.6)$$

noting that in application to Eq. (A.5) (and in all subsequent applications of the appendix) the delta-function does not contribute since its argument can never vanish. We finally obtain therefore, the equation

$$(W-2E_p)\phi_{++}(\mathbf{p})=\lambda\Lambda_+^{(1)}(\mathbf{p})\Lambda_+^{(2)}(\mathbf{p})\Gamma_i^{(1)}\Gamma_i^{(2)}$$
$$\times\int d\mathbf{k}\frac{2}{2\omega(W-E_p-E_{p-k}-\omega)}\phi_{++}(\mathbf{p}-\mathbf{k}), \quad (A.7)$$

which is just the Tamm-Dancoff approximation.

We turn next to the more complicated case of the diagram of Fig. 1(b). We add to the r.h.s. of Eq. (A.1) the term

$$(-ig^2)^2\int d^4x'd^4X'\exp[iW(X_0-X_0')]\Delta(X-X'$$
$$-\tfrac{1}{2}(x+x'))\Delta(X-X'+\tfrac{1}{2}(x+x'))[\Gamma_iG(X-X'$$
$$+\tfrac{1}{2}(x-x'))\tau_j\gamma_5]^{(1)}[\Gamma_jG(X-X'$$
$$-\tfrac{1}{2}(x-x'))\tau_i\gamma_5]^{(2)}\psi_{++}(x'). \quad (A.8)$$

In Eq. (A.8), in addition to c.m. coordinates for the points x_1, x_2, we have introduced similar coordinates for x_3, x_4. To transform Eq. (A.8), we require, in addition to Eqs. (A.2) and (A.3), the representation

$$G(x)=\frac{i}{(2\pi)^3}\int d\mathbf{p}\exp[i\mathbf{p}\cdot\mathbf{r}-iE_p|x_0|]$$
$$\times\frac{[H(\mathbf{p})+E_p\,\mathrm{sgn}x_0]\gamma_0}{2E_p}, \quad (A.9)$$

where $\mathrm{sgn}x_0$ means the sign of x_0. Incorporating Eqs. (A.2), (A.3), and (A.9) into (A.8) and carrying out the sequence of operations described prior to Eq. (A.5), the additional term in (A.5) contributed by (A.8) has the form

$$(-ig^2)^2\left(\frac{i}{(2\pi)^3}\right)^4\frac{1}{(2\pi)^4}\Lambda_+^{(1)}(\mathbf{p})\Lambda_+^{(2)}(\mathbf{p})$$
$$\times\int d^4xd^4x'd^4X'dp_0d\mathbf{k}_1d\mathbf{k}_2d\mathbf{p}_1d\mathbf{p}_2d\mathbf{p}'$$
$$\times\frac{1}{4\omega_1\omega_2}e^{-ipx}\left[\frac{1}{\frac{1}{2}W+p_0-E_p}+\frac{1}{\frac{1}{2}W-p_0-E_p}\right]$$
$$\times\exp[iW(X_0-X_0')]\exp[i\mathbf{k}_1\cdot(\mathbf{R}-\mathbf{R}'$$
$$-\tfrac{1}{2}(\mathbf{r}+\mathbf{r}'))-i\omega_1|X_0-X_0'-\tfrac{1}{2}(x_0+x_0')|]$$
$$\times\exp[i\mathbf{k}_2\cdot(\mathbf{R}-\mathbf{R}'+\tfrac{1}{2}(\mathbf{r}+\mathbf{r}'))-i\omega_2|X_0-X_0'$$
$$+\tfrac{1}{2}|x_0+x_0'|]\exp[i\mathbf{p}_1\cdot(\mathbf{R}-\mathbf{R}'+\tfrac{1}{2}(\mathbf{r}-\mathbf{r}'))$$
$$-iE_{p1}|X_0-X_0'+\tfrac{1}{2}(x_0-x_0')|]\exp[i\mathbf{p}_2\cdot(\mathbf{R}-\mathbf{R}'$$
$$-\tfrac{1}{2}(\mathbf{r}-\mathbf{r}'))-iE_{p2}|X_0-X_0'-\tfrac{1}{2}(x_0-x_0')|]$$
$$\times[\Gamma_i(H(\mathbf{p}_1)+E_{p1}\,\mathrm{sgn}(X_0-X_0'+\tfrac{1}{2}(x_0-x_0'))\Gamma_j]^{(1)}$$
$$\times\frac{1}{2E_{p1}}[\Gamma_j(H(\mathbf{p}_2)+E_{p2}\,\mathrm{sgn}(X_0-X_0'$$
$$-\tfrac{1}{2}(x_0-x_0'))\Gamma_i]^{(2)}\frac{1}{2E_{p2}}\exp[i\mathbf{p}'\cdot\mathbf{r}'-i|x_0'|$$
$$\times(E_{p'}-\tfrac{1}{2}W)]\phi_{++}(\mathbf{p}'). \quad (A.10)$$

As the next stage in the development, we record the result of performing the integrations with respect to p_0, $\mathbf{r}$, $\mathbf{r}'$, $\mathbf{R}'$, $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}'$. It is also useful to introduce the variable $x_0''=x_{20}-x_{30}=X_0-X_0'-\frac{1}{2}(x_0+x_0')$. Equation (A.10) now takes the somewhat simplified form

$$\lambda^2i\int\frac{d\mathbf{k}_1d\mathbf{k}_2}{4\omega_1\omega_2}\int_{-\infty}^\infty dx_0dx_0'dx_0''\exp\{iW[x_0''+\tfrac{1}{2}(x_0+x_0')]$$
$$-i(E_p-\tfrac{1}{2}W)|x_0|-i\omega_1|x_0''|-i\omega_2|x_0''+x_0+x_0'|$$
$$-iE_{p1}|x_0''+x_0|-iE_{p2}|x_0''+x_0'|-i(E_{p'}-\tfrac{1}{2}W)|x_0'|\}$$
$$\times\left\{\Gamma_i\frac{[H(\mathbf{p}_1)+E_{p1}\,\mathrm{sgn}(x_0''+x_0)]}{2E_{p1}}\Gamma_j\right\}^{(1)}$$
$$\times\left\{\Gamma_j\frac{[H(\mathbf{p}_2)+E_{p2}\,\mathrm{sgn}(x_0''+x_0')]}{2E_{p2}}\Gamma_i\right\}^{(2)}$$
$$\times\phi_{++}(\mathbf{p}'), \quad (A.11)$$

where it is understood that

$$\mathbf{p}_1=\mathbf{p}-\mathbf{k}_2, \quad p_2=-\mathbf{p}+\mathbf{k}_1, \quad \mathbf{p}'=\mathbf{p}-\mathbf{k}_1-\mathbf{k}_2. \quad (A.12)$$

The final step in the derivation of the twenty-four matrix elements contained in Eq. (A.11) consists in

dividing the regions of integration with respect to x_0, x_0', x_0'' into appropriate subregions, each corresponding to a single three-dimensional matrix element. Contenting ourselves here with a sample indication, one easily verifies that the no-pair terms, six in all, are given by the six integrals

$$\Big[\int_0^\infty\int_0^\infty\int_0^\infty+\int_0^\infty\int_0^\infty\int_{-x_0''}^0+\int_0^\infty\int_{-x_0''}^0\int_0^\infty$$
$$+\int_0^\infty\int_{-x_0''}^0\int_{-x_0''}^{-x_0-x_0''}+\int_0^\infty\int_{-x_0''}^0\int_{-x_0-x_0''}^0$$
$$+\int_{-\infty}^0\int_{-x_0''}^\infty\int_{-x_0''}^\infty\Big]dx_0''dx_0dx_0'=(t_1>t_2>t_3>t_4)$$
$$+(t_1>t_2>t_4>t_3)+(t_2>t_1>t_3>t_4)+(t_2>t_4>t_1>t_3)$$
$$+(t_2>t_1>t_4>t_3)+(t_1>t_3>t_2>t_4). \quad \text{(A.13)}$$

The l.h.s. of Eq. (A.13) gives the ranges of integration variables corresponding to the matrix elements defined by the time orderings on the r.h.s. It is easily verified that the removal of the absolute value signs for the combinations of time variables found in Eq. (A.11) is well defined for each of the matrix elements in (A.13). The remaining matrix elements may be similarly derived. The same technique applies also to the ladder diagram.

The procedure described above appears to be the most concise means for extracting some small subset of desired matrix elements from higher order diagrams where the derivation of all matrix elements contained therein would be prohibitively laborious. In most of the applications in this paper, however, we have merely recorded the known results of such a derivation.

APPENDIX B

We wish to point out here that the existence of velocity-dependent corrections to the two-pair matrix elements of the same order of magnitude as the no-pair terms and in particular that one obtains from these terms a spin-orbit coupling. This follows from a more detailed consideration of the spin matrix element

$$(p_r|\Lambda_+^{(r)}(\mathbf{p}_r-\epsilon_r\mathbf{k}')|\mathbf{p}_r')=\frac{1}{2E_{p-k'}}\left[\frac{(E_p+M)(E_{p'}+M)}{2E_p2E_{p'}}\right]^{\frac{1}{2}}$$
$$\times\left[E_{p-k'}+M+\frac{\boldsymbol{\sigma}^{(r)}\cdot\mathbf{p}}{E_p+M}(E_{p-k}-M)\frac{\boldsymbol{\sigma}^{(r)}\cdot\mathbf{p}_r'}{E_{p'}+M}\right.$$
$$\left.+\boldsymbol{\sigma}^{(r)}\cdot(\mathbf{p}_r-\epsilon_r\mathbf{k}')\frac{(\boldsymbol{\sigma}^{(r)}\cdot\mathbf{p}_r')}{E_{p'}+M}+\frac{\boldsymbol{\sigma}^{(r)}\cdot\mathbf{p}_r}{E_p+M}\boldsymbol{\sigma}^{(r)}\cdot(\mathbf{p}_r-\epsilon_r\mathbf{k}')\right]$$
$$\rightarrow 1+(2M)^{-2}[\boldsymbol{\sigma}^{(r)}\cdot(\mathbf{p}_r-\epsilon_r\mathbf{k}')\boldsymbol{\sigma}^{(r)}\cdot p_r'+\boldsymbol{\sigma}^{(r)}\cdot p_r\boldsymbol{\sigma}^{(r)}$$
$$\cdot(\mathbf{p}_r-\epsilon_r\mathbf{k}')]\rightarrow 1+(2M)^{-2}i\boldsymbol{\sigma}^{(r)}\cdot(\mathbf{k}_1+\mathbf{k}_2)\times\mathbf{p}. \quad \text{(B.1)}$$

The successive simplifications indicated by the arrows consist at the first stage of discarding all terms beyond the leading one which do not contribute a spin-orbit coupling and at the second stage more of the same accompanied by substitution $p_r'=p_r-\epsilon_r(k_1+k_2)$. The result (B.1) is seen to depend on the particular particle involved only through its spin. Consequently, when we multiply the spin matrix elements for the two particles, the cross-term leads to an interaction that depends on the total spin of the system and is, in fact, of the spin-orbit type. If one inserts the latter into the leading two-pair terms, one finds easily an interaction,

$$V_4^{\text{S.O.}}=-6(g^2/4\pi)^2(\mu/2M)^4(\mu^3r)^{-1}(\partial/\partial r)$$
$$\times[K_1(2\mu r)/r^2]\mathbf{L}\cdot\mathbf{S}, \quad \text{(B.2)}$$

with

$$\mathbf{L}=\mathbf{r}\times\mathbf{p}, \qquad \mathbf{S}=\tfrac{1}{2}(\boldsymbol{\sigma}^{(1)}+\boldsymbol{\sigma}^{(2)}). \quad \text{(B.3)}$$

Since $K_1(x)$ is a monotonically decreasing function of its argument, the quantity multiplying $\mathbf{L}\cdot\mathbf{S}$ in Eq. (B.2) is positive. This is the sign required for the Case and Pais[35] interpretation of the $p-p$ scattering data at high energies. It is therefore of the wrong sign for application to the shell model of heavy nuclei. In the latter instance, however, there are many other mechanisms that can produce the desired result.

[35] K. M. Case and A. Pais, Phys. Rev. **80**, 203 (1950).