The Continuum in Special Relativity. II

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A covariant description is given of the properties, relative to an arbitrary secondary frame, of an element of anisotropic continuum composed of a single chemical species. These properties are measured by an observer in a primary reference frame relative to which the motion of the secondary frame is constant. This is a generalization of an earlier formalism in which the secondary frame coincided with the rest frame of the element. The equation of continuity of mass in the classical sense is formulated, and Gauss' theorem for transformation of volume integrals of space divergences into surface integrals is stated for the secondary reference frame. If the equation of motion relative to the secondary frame is assumed given by the statement that the divergence in the secondary frame of the energy-momentum tensor vanishes, then the equation for the first law of thermodynamics, the equation of continuity of energy, the dynamical equations of motion including the force equation and the kinetic energy equation, and the equation of continuity of momentum result; each equation is formulated for motion of an element of continuum relative to an arbitrary secondary frame as measured by a primary observer. The divergence in the secondary frame of the entropy tensor is assumed to be a time vector whose projection upon the four-vector velocity of the element relative to the primary frame is non-negative; this is shown to imply the second law of thermodynamics relative to the secondary frame, and, in the reversible case, an equation of continuity of entropy. The equation defining the reversible stress in the element of continuum is introduced. The chemical potential function which appears in this equation has the properties of the specific Helmholtz free energy in the thermodynamic consequences of this equation and the properties of the specific Gibbs free energy in the dynamical consequences, a circumstance which obviates the necessity of formulating a Gibbs free energy for anisotropic media. The conditions for reversibility of a process in the continuum are found to be independent of the secondary reference frame chosen for description of the process. Finally, a covariant theory of transport processes is given for a one-component, anisotropic medium. It is shown that in the theory of diffusion and thermal conduction the motion of every element of the continuum must satisfy a special transport condition relative to a secondary frame; in the theory of viscosity this condition need not be met.

A. MATHEMATICAL FORMALISM

In a preceding paper,[1] a formalism was developed for resolving four-dimensional tensors into space and time components by means of the unit space tensor $\bar{\delta}_{\sigma\tau}$ and unit time vector $\hat{u}_\sigma$. We consider the case of two reference frames, one attached momentarily to the physical system whose properties are to be measured, the rest frame, and the other a frame relative to which the rest frame has a four-velocity $u_\sigma = c\hat{u}_\sigma$, where c is the speed of light. We call the latter frame a primary frame. Any four-vector whose components in the rest frame are P_σ^0 will have co nponents in the primary frame,

$$P_\sigma = L_{\sigma\tau} P_\tau^0. \quad (1)$$

$L_{\sigma\tau}$ is the general Lorentz transformation.

The transformation $L_{\sigma\tau}$ may be expressed as

$$L_{\sigma\tau} = A_{\sigma\mu} l_{\mu\nu} A_{\tau\nu}^0. \quad (2)$$

If the velocity in space of the rest frame relative to the primary frame is represented by the three-vector $\mathbf{v}$, then the matrices on the right side of Eq. (2) have the following meanings: $A_{\tau\nu}^0$ is an orthogonal space rotation with Euler angles θ^0, ϕ^0, ψ^0 which transforms a coordinate system in the rest frame whose Z^0 axis is parallel to $\mathbf{v}$ into an arbitrarily oriented coordinate system in the rest frame; $A_{\sigma\mu}$ is a similar rotation with Euler angles θ, ϕ, ψ which transforms a coordinate system in the primary frame whose Z axis is parallel to $\mathbf{v}$ into an arbitrarily oriented coordinate system in the primary frame; $l_{\mu\nu}$ is a simple Lorentz transformation from a coordinate system in the rest frame whose Z^0 axis is parallel to $\mathbf{v}$ to a coordinate system in the primary frame whose Z axis is parallel to $\mathbf{v}$. Accordingly, $L_{\sigma\tau}$ transforms one arbitrarily oriented coordinate system in the rest frame into another arbitrarily oriented coordinate system in the primary frame.

In Paper I, we used the Hermitian matrix $a_{\sigma\tau}$ of Abraham and Becker in place of $L_{\sigma\tau}$. It may be obtained as a special case of $L_{\sigma\tau}$ by letting $\theta=\theta^0$, $\phi=\phi^0$, $\psi=\psi^0$. However, $a_{\sigma\tau}$ does not possess the group property under conditions general enough for our purposes. In fact, if $a_{\sigma\tau}'$ and $a_{\sigma\tau}^*$ represent successive transformations associated with relative velocities $\mathbf{v}'$ and $\mathbf{v}^*$, their product cannot be written as a single transformation $a_{\sigma\tau}$ associated with a velocity $\mathbf{v}$ unless the velocities $\mathbf{v}'$, $\mathbf{v}^*$ and $\mathbf{v}$ are parallel. The generality of $a_{\sigma\tau}$ over the simple Lorentz transformation $l_{\sigma\tau}$ consists only in the fact that whereas the direction of $\mathbf{v}'$, $\mathbf{v}^*$, and $\mathbf{v}$ in the former case is unspecified, in the latter case it must coincide with the common direction of the Z axes of the three reference frames involved. In addition, in either case, as is well known, the magnitude of $\mathbf{v}$ must be obtained from that of $\mathbf{v}'$ and $\mathbf{v}^*$ by the Einstein addition law.

The requirement that the general Lorentz matrix $L_{\sigma\tau}$ possess the group property imposes the following conditions on the velocities: Let β' be the magnitude of the velocity $\mathbf{v}'$ of the first reference frame relative to a second frame, β^* be the magnitude of the velocity $\mathbf{v}^*$ of the second frame relative to a third frame, and β be the magnitude of the resultant velocity $\mathbf{v}$ of the first frame relative to the third frame, all magnitudes measured in

[1] B. Leaf, Phys. Rev. 84, 345 (1951) (referred to in the following as Paper I).

units of the speed of light. Then,

(1) $\beta/(1-\beta^2)^{\frac{1}{2}}=(\beta^{*2}+\beta'^2+2\beta^*\beta'\cos\theta_2 -\beta^{*2}\beta'^2\sin^2\theta_2)^{\frac{1}{2}}/(1-\beta^{*2})^{\frac{1}{2}}(1-\beta'^2)^{\frac{1}{2}}$.

Here θ_2 is the angle between $\mathbf{v}'$ and $\mathbf{v}^*$ as measured by an observer in the second frame. If $\theta_2=0$, this equation reduces to the Einstein addition law. This relation is the relativistic form of the trigonometric law of cosines, to which it reduces for small velocities.

(2) $\sin\theta_2(1-\beta^2)^{\frac{1}{2}}/\beta=\sin\theta_1(1-\beta^{*2})^{\frac{1}{2}}/\beta^2)^{\frac{1}{2}}/\beta^* =-\sin\theta_3(1-\beta'^2)^{\frac{1}{2}}/\beta'$.

Here θ_1 is the angle between $\mathbf{v}$ and $\mathbf{v}'$ as measured by an observer in the first frame; θ_3 is the angle between $\mathbf{v}$ and $\mathbf{v}^*$ as measured by an observer in the third frame. This relation is the relativistic form of the law of sines.

(3) We also have the relationships:

$$\cos\theta_2=\cos\theta_1\cos\theta_3+\sin\theta_1\sin\theta_3/(1-\beta^2)^{\frac{1}{2}},$$
$$\sin\theta_2/(1-\beta^{*2})^{\frac{1}{2}}=-\cos\theta_1\sin\theta_3+\sin\theta_1\cos\theta_3/(1-\beta^2)^{\frac{1}{2}},$$
$$\sin\theta_2/(1-\beta'^2)^{\frac{1}{2}}=\cos\theta_3\sin\theta_1-\sin\theta_3\cos\theta_1/(1-\beta^2)^{\frac{1}{2}},$$

which reduces for small velocities to well-known trigonometric equations for $\theta_1=\theta_2+\theta_3$.

(4) As in the classical case, the vector $\mathbf{v}$ must be coplanar with $\mathbf{v}'$ and $\mathbf{v}^*$.

In Paper I the group property was not essentially employed so that the results appearing there are not significantly affected by the use of the Abraham and Becker transformation $a_{\sigma\tau}$ in place of $L_{\sigma\tau}$. Only Eqs. (23) and (24) of Paper I depend on the particular transformation matrix used, and must be modified for the general case. In the present paper we shall utilize the group property in an essential way and will, therefore, require $L_{\sigma\tau}$. The transformation $L_{\sigma\tau}$ is orthogonal but not, in general, Hermitian. Its determinant equals unity.

Let us return to the case of two reference frames, the rest frame and the primary frame. Introducing the unit vectors in the rest frame,

$$\bar{\imath}_\sigma{}^0=(1,0,0,0)\quad \bar{\jmath}_\sigma{}^0=(0,1,0,0),$$
$$\bar{k}_\sigma{}^0=(0,0,1,0),\quad \bar{u}_\sigma{}^0=(0,0,0,i),$$

into Eq. (1), we find their values in the primary frame,

$$\bar{\imath}_\sigma=L_{\sigma1},\quad \bar{\jmath}_\sigma=L_{\sigma2},\quad \bar{k}_\sigma=L_{\sigma3},\quad \bar{u}_\sigma=iL_{\sigma4}.$$

Accordingly, $u_\sigma=icL_{\sigma4}$, and we find, on using the values of $L_{\sigma4}$ given by Eq. (2), that

$$u_k=v_k(1-v^2/c^2)^{-\frac{1}{2}} \text{ for } k=1,2,3, \text{ and } u_4=ic(1-v^2/c^2)^{-\frac{1}{2}}. \quad (3)$$

In terms of unit vectors, the Kronecker delta $\delta_{\sigma\tau}$ and the unit space tensor $\bar{\delta}_{\sigma\tau}$ become

$$\delta_{\sigma\tau}=\bar{\delta}_{\sigma\tau}-\bar{u}_\sigma\bar{u}_\tau,\quad \bar{\delta}_{\sigma\tau}=\bar{\imath}_\sigma\bar{\imath}_\tau+\bar{\jmath}_\sigma\bar{\jmath}_\tau+\bar{k}_\sigma\bar{k}_\tau. \quad (4)$$

In Paper I, we defined the space component of a vector P_σ as $\bar{\delta}_{\sigma\tau}P_\tau$ an the time component as $-\bar{u}_\sigma\bar{u}_\tau P_\tau$. Since

$$\bar{\delta}_{\sigma\tau}P_\tau=\bar{\imath}_\sigma P_1{}^0+\bar{\jmath}_\sigma P_2{}^0+\bar{k}_\sigma P_3{}^0=L_{\sigma k}P_k{}^0,$$

we see that the space component of P_σ is the value of $(P_1{}^0, P_2{}^0, P_3{}^0, 0)$ after Lorentz transformation; it is the space component of P_σ relative to the rest frame as measured by a primary observer. Similarly, since

$$-\bar{u}_\sigma\bar{u}_\tau P_\tau=-i\bar{u}_\sigma P_4{}^0=L_{\sigma4}P_4{}^0,$$

the time component of P_σ is the value of $(0,0,0,P_4{}^0)$ after Lorentz transformation; it is the time component of P_σ relative to the rest frame as measured by a primary observer.

Let us consider some examples of space and time components. If x_σ is the position vector of an element of the physical system along its trajectory as measured by a primary observer, then

$$x_\sigma-x_\sigma(0)=L_{\sigma\tau}x_\tau{}^0=\bar{\imath}_\sigma x_1{}^0+\bar{\jmath}_\sigma x_2{}^0+\bar{k}_\sigma x_3{}^0-i\bar{u}_\sigma x_4{}^0, \quad (5)$$

where $x_\sigma(0)$ is the initial position of the origin of the coordinate system in the rest frame as measured by the primary observer, and $x_\sigma{}^0$ is the position vector of the element relative to a coordinate system in the rest frame. The time component of $x_\sigma-x_\sigma(0)$ is $-i\bar{u}_\sigma x_4{}^0 =u_\sigma t^0$; the space component, $\bar{\imath}_\sigma x_1{}^0+\bar{\jmath}_\sigma x_2{}^0+\bar{k}_\sigma x_3{}^0$. It may be observed that the whole trajectory is plotted by the primary observer who uses a single coordinate system in the primary frame, but not by the succession of rest observers each of whom records only one position of the element along the trajectory. The coordinate systems which the rest observers employ need have no relation to each other either in point of origin or in orientation in space. Any vector having components $P_\sigma{}^0$ in such an arbitrary coordinate system of a rest frame will have components P_σ as measured by a primary observer when the appropriate value of $L_{\sigma\tau}$ is used in Eq. (1).

Despite the arbitrary variability of $x_\sigma{}^0$ along the trajectory according to the rest observers' choices of origin and orientation of coordinate systems, the change in $x_\sigma{}^0$ as the physical system moves relatively to any one such coordinate system may be defined. This is the change in position along the tangent at a point on the trajectory as measured by a rest observer at that point using a single coordinate system. Thus

$$dx_k{}^0=0 \text{ for } k=1,2,3;\quad dx_4{}^0=icdt^0.$$

The corresponding change in dx_σ along the tangent, measured by the primary observer, is obtained by differentiating Eq. (5). We find

$$dx_\sigma=u_\sigma dt^0. \quad (6)$$

The "proper" time interval dt^0 is read on a clock fixed in the rest frame. Since $dx_4=icdt$, we see that

$$dt=-i\bar{u}_4dt^0=(1-v^2/c^2)^{-\frac{1}{2}}dt^0. \quad (7)$$

Consequently, $dx_k=u_k(1-v^2/c^2)^{\frac{1}{2}}dt$ for $k=1,2,3$. Comparison with Eq. (3) gives $v_k=dx_k/dt$. The time interval

dt is measured as the difference in readings of two synchronized clocks fixed in the primary frame, one at the initial point, the other at the final point of the displacement dx_k. Accordingly, dt is an interval associated with the trajectory.

The four-vector velocity $u_\sigma = dx_\sigma/dt^0$ is itself a time vector since $\bar{\delta}_{\sigma\tau}u_\tau = 0$. The space component of velocity of an object relative to its own rest frame is zero when measured by any primary observer. The four-vector acceleration du_σ/dt^0 is a space vector, since $u_\sigma du_\sigma/dt^0 = 0$. The time component of acceleration of an object relative to its own rest frame is zero when measured by any primary observer.

The symbol ∂_σ represents the gradient operator, whose time and space components are $-\hat{u}_\sigma\hat{u}_\tau\partial_\tau$ and $\bar{\delta}_{\sigma\tau}\partial_\tau$. The factor $u_\tau\partial_\tau$ in the time component is the operator, $d/dt^0 = u_\tau\partial_\tau$. This operator was designated as $d/d\tau$ in Paper I, but we shall reserve this symbol for a more general operator defined in Eq. (20) of which this is a special case.

In Paper I, any symmetric, second-order tensor $\psi_{\sigma\tau}$ was resolved as follows

$$\psi_{\sigma\tau} = -(1/c^2)(\chi_{\rho\rho}u_\sigma u_\tau + u_\sigma Q_\tau + u_\tau Q_\sigma) + \phi_{\sigma\tau}, \qquad (8)$$

where the combinations of space and time components for each tensor index are

$$\begin{aligned}\phi_{\sigma\tau} &= \bar{\delta}_{\sigma\mu}\bar{\delta}_{\tau\nu}\psi_{\mu\nu},\\ -u_\tau Q_\sigma/c^2 &= \bar{\delta}_{\sigma\mu}(-\hat{u}_\tau\hat{u}_\nu)\psi_{\mu\nu},\\ -u_\sigma Q_\tau/c^2 &= -\hat{u}_\sigma\hat{u}_\mu\bar{\delta}_{\tau\nu}\psi_{\mu\nu},\\ \chi_{\rho\rho}\hat{u}_\sigma\hat{u}_\tau &= (-\hat{u}_\sigma\hat{u}_\mu)(-\hat{u}_\tau\hat{u}_\nu)\psi_{\mu\nu}.\end{aligned}$$

Here, if we identify $\psi_{\sigma\tau}$ as the energy-momentum tensor, we find that Q_σ and $\phi_{\sigma\tau}$ are the components of heat flux and stress relative to the rest frame as measured by a primary observer, whereas $\chi_{\rho\rho}$ is the scalar invariant energy density relative to the rest frame, (i.e., nonkinetic energy) which will appear the same to any primary observer.

In all the resolutions of vectors and tensors into space and time components which we have performed up to this point, we have in every case obtained the components relative to the rest frame as measured by the primary observer. But in many physical experiments the primary observer does not measure the properties of an element of continuum relative to the rest frame of the element, but relative to some other arbitrary frame. We call such a frame a secondary frame. Let the secondary frame have a three-vector velocity $\mathbf{v}^*$ relative to the primary frame as measured by a primary observer; and let the rest frame have a three-vector velocity $\mathbf{v}'$ relative to the secondary frame as measured by a secondary observer. Then, subject to the conditions stated previously on the magnitude and direction of the resultant velocity $\mathbf{v}$ of the rest frame relative to the primary frame, we may write the group property of the Lorentz transformation $L_{\sigma\tau}$ of Eq. (2) as

$$L_{\sigma\tau} = L_{\sigma\rho}^* L_{\rho\tau}'. \qquad (9)$$

The transformation $P_\sigma' = L_{\sigma\tau}'P_\tau^0$ gives the values P_σ' of measurements made upon the rest frame by a secondary observer and $P_\sigma = L_{\sigma\tau}^*P_\tau'$ gives the values P_σ of measurements made upon the rest frame by a primary observer. Also

$$P_\sigma^* = L_{\sigma\tau}^*P_\tau^0 \qquad (10)$$

gives the values P_σ^* of measurements made by the primary observer upon physical systems at rest in the secondary frame. Introducing the unit vectors in the secondary frame, $\bar{\imath}_\sigma^0$, $\bar{\jmath}_\sigma^0$, $\bar{k}_\sigma^0$, $\hat{u}_\sigma^0$, into Eq. (10) we find their values in the primary frame, $\bar{\imath}_\sigma^* = L_{\sigma1}^*$, $\bar{\jmath}_\sigma^* = L_{\sigma2}^*$, $\bar{k}_\sigma^* = L_{\sigma3}^*$, $\hat{u}_\sigma^* = iL_{\sigma4}^*$. The four-vector velocities u_σ' and u_σ^* are related to the corresponding three-vector velocities $\mathbf{v}'$ and $\mathbf{v}^*$ by equations similar to Eq. (3). We may now write the Kronecker delta $\delta_{\sigma\tau}$ and a unit relative space tensor $\bar{\delta}_{\sigma\tau}^*$ as

$$\delta_{\sigma\tau} = \bar{\delta}_{\sigma\tau}^* - \hat{u}_\sigma^*\hat{u}_\tau^*, \quad \bar{\delta}_{\sigma\tau}^* = \bar{\imath}_\sigma^*\bar{\imath}_\tau^* + \bar{\jmath}_\sigma^*\bar{\jmath}_\tau^* + \bar{k}_\sigma^*\bar{k}_\tau^*, \qquad (11)$$

and define the relative space component of P_σ as $\bar{\delta}_{\sigma\tau}^*P_\tau$ and the relative time component as $-\hat{u}_\sigma^*\hat{u}_\tau^*P_\tau$. We readily find that

$$\begin{aligned}\bar{\delta}_{\sigma\tau}^*P_\tau &= \bar{\imath}_\sigma^*P_1' + \bar{\jmath}_\sigma^*P_2' + \bar{k}_\sigma^*P_3' = L_{\sigma k}^*P_k',\\ -\hat{u}_\sigma^*\hat{u}_\tau^*P_\tau &= -i\hat{u}_\sigma^*P_4' = L_{\sigma4}^*P_4'.\end{aligned}$$

Thus $\bar{\delta}_{\sigma\tau}^*P_\tau$ is the value of the space component, $(P_1', P_2', P_3', 0)$, relative to the secondary frame as measured by a primary observer, and $-\hat{u}_\sigma^*\hat{u}_\tau^*P_\tau$ is the time component, $(0, 0, 0, P_4')$, relative to the secondary frame as measured by the primary observer. If the secondary frame and the momentary rest frame are at rest relative to each other, then we revert to the case considered previously, with $\bar{\delta}_{\sigma\tau}^*P_\tau = \bar{\delta}_{\sigma\tau}P_\tau$ and $-\hat{u}_\sigma^*\hat{u}_\tau^*P_\tau = -\hat{u}_\sigma\hat{u}_\tau P_\tau$. On the other hand, if the secondary frame is at rest relative to the primary frame, then the primary observer is measuring the properties of the physical system relative to his own coordinate system in the primary frame. In this case, $L_{\sigma\tau}^*$ becomes a mere space rotation in the primary frame, so that

$$\bar{\delta}_{\sigma\tau}^*P_\tau = (P_1, P_2, P_3, 0), \quad -\hat{u}_\sigma^*\hat{u}_\tau^*P_\tau = (0, 0, 0, P_4).$$

For example, let $P_\sigma = u_0$, the velocity vector. Then the relative space component, $\bar{\delta}_{\sigma\tau}^*u_\tau$, equals zero when the secondary frame coincides with the rest frame, but it equals $(u_1, u_2, u_3, 0)$ when the secondary frame coincides with the primary. At the same time, the relative time component, $-\hat{u}_\sigma^*\hat{u}_\tau^*u_\tau$, equals u_σ in the first case, and $(0, 0, 0, u_4)$ in the latter.

We now define the four-vector v_σ,

$$v_\sigma = u_\sigma/(-\hat{u}_\tau\hat{u}_\tau^*). \qquad (12)$$

The invariant quantity $-\hat{u}_\tau\hat{u}_\tau^*$ will appear frequently in the following and will be represented by k'. We find

$$k' = -i\hat{u}_4' = L_{44}' = (1-\beta'^2)^{-\frac{1}{2}}.$$

The relative space component of v_σ is $\bar{\delta}_{\sigma\tau}^*v_\tau = L_{\sigma k}^*v_k'$, which when measured by a secondary observer is equal to $(v_1', v_2', v_3', 0)$ in which the three space components constitute the three-vector velocity $\mathbf{v}'$ of the rest frame relative to the secondary frame. The relative time component of v_σ is $-\hat{u}_\sigma^*\hat{u}_\tau^*v_\tau = u_\sigma^*$, which when measured measured by a secondary observer is equal to $(0,0,0,ic)$. The vector v_σ differs quantitatively from u_σ only by the factor $1/k'$. But a fundamental qualitative difference appears between them. In the case of u_σ it will be noted that u_σ' is the value measured by a secondary observer of the vector $u_\sigma^0 = (0, 0, 0, ic)$ associated with the rest frame. But in the case of v_σ there is no property of the rest frame which when measured by the secondary observer yields $v_\sigma' = (v_1', v_2', v_3', ic)$. When the primary observer, therefore, measures $v_\sigma = L_{\sigma\tau}^* v_\tau'$ he is measuring a property of the secondary frame. This is particularly evident in the expression for the relative time component of v_σ, $-\hat{u}_\sigma^*\hat{u}_\tau^*v_\tau = u_\sigma^*$.

We conclude this section with a result which we shall find very useful. If A_σ is a space vector and B_σ a time vector, then

$$(\bar{\delta}_{\sigma\rho}^*A_\rho)(\bar{\delta}_{\sigma\tau}^*B_\tau)+(-\hat{u}_\sigma^*\hat{u}_\rho^*A_\rho)(-\hat{u}_\sigma^*\hat{u}_\tau^*B_\tau)=0,$$

so that

$$A_\rho\hat{u}_\rho^* = \bar{\delta}_{\sigma\rho}^*A_\rho B_\sigma/B_\tau\hat{u}_\tau^*,$$

or

$$A_\rho u_\rho^* = -\bar{\delta}_{\sigma\rho}^*A_\rho v_\sigma. \qquad (13)$$

B. EQUATION OF CONTINUITY AND GAUSS' THEOREM

The secondary frames used in physical experiments have certain special properties which we now formulate. While the velocity u_σ of the rest frame relative to the primary may vary in space because of the differing velocities of the various elements of the physical system relative to the primary, and may vary in time because of the acceleration of these elements, this is not the case for a secondary frame. The velocity u_σ^* of the secondary frame relative to the primary must have the same value throughout space; that is, the secondary frame is a rigid reference frame. A necessary condition for the uniformity in space of u_σ^* is $\bar{\delta}_{\rho\tau}^*\partial_\tau^*u_\sigma^*=0$. We shall require also that the secondary frame be unaccelerated, so that $\hat{u}_\tau^*\partial_\tau^*u_\sigma^*=0$. These two conditions imply that

$$\partial_\rho^*u_\sigma^*=0 \quad\text{and}\quad \partial_\sigma^*\bar{\delta}_{\sigma\tau}^*=0. \qquad (14)$$

The gradient operator ∂_σ^* indicates partial differentiation with respect to x_σ^*.

A formalism was presented in Paper I according to which the basic equations of dynamics and thermodynamics for an element of continuum appear, respectively, as space and time components of the same tensor relationships. However, the space and time components were in every case the values relative to the rest frame of the element. We shall now reformulate these equations in order to obtain the laws of dynamics and thermodynamics for an element relative to a secondary frame.

Consider first the equation of continuity of mass. This equation was stated in Paper I as $\partial_\sigma\rho u_\sigma=0$, where ρ is the scalar invariant density of rest mass per unit volume in the rest frame. We now consider instead, the equation,

$$\partial_\sigma^*\rho u_\sigma=0. \qquad (15)$$

We shall see that this equation describes conservation of rest mass in motion relative to the secondary frame as measured by any primary observer. It includes the earlier formulation as a special case when the secondary frame coincides with the rest frame.

Resolving the vectors u_σ and ∂_σ^* in Eq. (15) into relative space and time components and using Eq. (14), we find that

$$-\hat{u}_\sigma^*\hat{u}_\mu^*\partial_\mu^*(-\hat{u}_\sigma^*\hat{u}_\nu^*\rho u_\nu) = -\bar{\delta}_{\sigma\mu}^*\partial_\mu^*(\bar{\delta}_{\sigma\nu}^*\rho u_\nu),$$

or

$$\partial\rho k'/\partial\tau^* = -\bar{\delta}_{\sigma\mu}^*\partial_\mu^*\rho u_\sigma, \qquad (16)$$

where we have written

$$\partial/\partial\tau^* = u_\sigma^*\partial_\sigma^*. \qquad (17)$$

Comparison with the operator $u_\sigma\partial_\sigma = d/dt^0$ shows that $\partial/\partial\tau^*$ gives the rate of change with respect to time as read on a clock fixed in the secondary frame. Evidently, $u_\sigma\partial_\sigma$ and $u_\sigma^*\partial_\sigma^*$ are the same invariant operator according to the primary observer, which implies that the rate of flow of time is the same in every reference frame when measured by a clock at rest in that frame.

From Eq. (15) we also obtain

$$u_\sigma\partial_\sigma^*\rho = -\rho\partial_\sigma^*u_\sigma. \qquad (18)$$

The left side of this equation can be written as

$$u_\sigma\partial_\sigma^*\rho = k'(\partial\rho/\partial\tau^* + \bar{\delta}_{\sigma\mu}^*v_\mu\partial_\sigma^*\rho).$$

It is evident that the operator,

$$d/d\tau^* = \partial/\partial\tau^* + \bar{\delta}_{\sigma\mu}^*v_\mu\partial_\sigma^* = v_\sigma\partial_\sigma^*, \qquad (19)$$

is the usual hydrodynamic "mobile" time differentiation operator which follows the motion of the physical element relative to the secondary frame. We now define the operator,

$$d/d\tau = u_\sigma\partial_\sigma^* = k'd/d\tau^*. \qquad (20)$$

This definition supersedes that given in Paper I where $d/d\tau$ was identified with $d/dt^0 = u_\tau\partial_\tau$, and includes it as a special case when the secondary frame coincides with the momentary rest frame.

It is important to understand the four operators, $d/d\tau^*$, $d/d\tau$, d/dt', and d/dt^0. We shall see that the two latter ones are special cases of the two former. Writing

$$v_\sigma\partial_\sigma^* = v_\sigma'\partial_\sigma^0 = \lim_{\Delta t'=0} \Delta x_\sigma'\partial_\sigma^0/\Delta t',$$

$$u_\sigma\partial_\sigma^* = u_\sigma'\partial_\sigma^0 = \lim_{\Delta t^0=0} \Delta x_\sigma'\partial_\sigma^0/\Delta t^0,$$

we find that $d/d\tau^*$ expresses rate of change relative to the secondary frame along the trajectory per unit of time interval $\Delta t'$; and $d/d\tau$, rate of change relative to the secondary frame along the trajectory per unit of time interval Δt^0. The proper time interval Δt^0 is read on a clock fixed in the rest frame, whereas $\Delta t' = k'\Delta t^0$ is the difference in readings of two synchronized clocks fixed in the secondary frame, one having the same position as the rest clock at the beginning at Δt^0; the other, the same position as the rest clock at the end of Δt^0. We may write similarly

$$d/dt' = \lim_{\Delta t'=0} \Delta x_\sigma^0 \partial_\sigma^0/\Delta t'; \quad d/dt^0 = \lim_{\Delta t^0=0} \Delta x_\sigma^0 \partial_\sigma^0/\Delta t^0.$$

Accordingly, d/dt' gives rate of change relative to the rest frame along the trajectory per unit of $\Delta t'$; and d/dt^0, rate of change relative to the rest frame along the trajectory per unit of Δt^0. Obviously $d/d\tau^*$ and $d/d\tau$ reduce to d/dt' and d/dt^0, respectively, in the special case that the secondary frame coincides with the rest frame.

The rate of dilatation of an element of continuum as it moves relatively to the secondary frame is $-d\ln\rho k'/d\tau^*$ which, according to Eqs. (16) and (19), equals $\bar{\delta}_{\sigma\tau}{}^*\partial_\sigma{}^*v_\tau$. The quantity $\rho k'$ is the density of rest mass per unit volume fixed on the secondary frame. The factor k' appears because of the Lorentz contraction of volume in motion relative to the observer. If $\rho k'$ is constant, then the dilatation rate vanishes, and $\bar{\delta}_{\sigma\tau}{}^*\partial_\sigma{}^*v_\tau = 0$. However, this does not describe the case of an incompressible material. For this case, ρ is constant which, according to Eq. (18), requires $\partial_\sigma{}^*u_\sigma = 0$; dilatation occurs at the rate $-d\ln k'/d\tau^*$.

Volume integrals of space divergences of the form $\bar{\delta}_{\sigma\tau}{}^*\partial_\sigma{}^*P_\tau$ may be transformed by Gauss' theorem into integrals over the surface enclosing the volume. In covariant form, Gauss' theorem is

$$\int_{V^*} (\bar{\delta}_{\sigma\tau}{}^*\partial_\sigma{}^*P_\tau)(-i\hat{u}_4{}^*dV^*) = \int_{\Sigma^*} P_\sigma dn_\sigma{}^*, \qquad (21)$$

where $dn_\sigma{}^* = \bar{\delta}_{\sigma\tau}{}^*dn_\tau{}^*$ is a space vector having the direction of the normal drawn outwardly to the boundary surface Σ^* and magnitude equal to an element of area of Σ^*. The invariance of $-i\hat{u}_4{}^*dV^*$ follows according to the argument given in Paper I, Eq. (72); it represents the differential volume of a region at rest in the secondary frame. The conditions of Eq. (14) upon the secondary frame are essential for this formulation; for this reason, the theorem is not applicable to an integral of a space divergence $\bar{\delta}_{\sigma\tau}\partial_\sigma P_\tau$ in the rest frame.

Forming the volume integral of Eq. (16) over a region fixed in the secondary frame and applying Gauss' theorem, we have

$$\partial m/\partial\tau^* = -\int \rho k' v_\sigma dn_\sigma{}^*, \qquad (22)$$

where $m = \int \rho k'(-i\hat{u}_4{}^*)dV^*$. Equation (22) is the statement of the classical law of conservation of rest mass contained in a region V^* fixed in the secondary frame. A similar integration shows that the integral over the region V^* of the dilatation rate is equal to the surface integral $\int v_\sigma dn_\sigma{}^*$.

The symbol ρ can also be interpreted as the quantity of electrical charge per unit volume of the rest frame, in which case Eq. (15) represents the equation of continuity of charge. In a molecular theory the equation of continuity expresses conservation of number of particles rather than of rest mass or charge; conservation of the latter results from the fixed quantity of rest mass or charge per particle. It is obvious that Eq. (15) applies only when a single chemical species is considered and chemical reaction excluded, since in the course of chemical transformation the number of particles of a species is not conserved.

C. EQUATIONS OF MOTION

The basic dynamical and thermodynamic hypothesis of Paper I, $\partial_\sigma\psi_{\sigma\tau} = 0$, is replaced here by the more general assumption,

$$\partial_\sigma{}^*\psi_{\sigma\tau} = 0. \qquad (23)$$

The tensor $\psi_{\sigma\tau}$ is the energy-momentum tensor of Eq. (8) so that, accordingly,

$$\partial_\sigma{}^*[(\chi_{\rho\rho}u_\sigma + Q_\sigma)u_\tau/c^2] = \partial_\sigma{}^*(\phi_{\sigma\tau} - u_\sigma Q_\tau/c^2). \qquad (24)$$

Four equations can be obtained from Eq. (23) by equating to zero the time components and the space components of $\partial_\sigma{}^*\psi_{\sigma\tau}$ relative to the rest frame and the secondary frame. We consider these in turn.

Multiplying Eq. (24) by $-u_\tau$, we obtain the time component equation relative to the rest frame,

$$\partial_\sigma{}^*(\chi_{\rho\rho}u_\sigma + Q_\sigma) = \phi_{\sigma\tau}\partial_\sigma{}^*u_\tau - (Q_\tau/c^2)(du_\tau/d\tau). \qquad (25)$$

Resolving the gradient operator into space and time components, and using Eq. (13), we find

$$\begin{aligned}(\partial/\partial\tau^*)(\chi_{\rho\rho}k' + \bar{\delta}_{\sigma\tau}{}^*Q_\sigma v_\tau/c^2) + \bar{\delta}_{\sigma\tau}{}^*\partial_\sigma{}^*(\chi_{\rho\rho}u_\tau + Q_\tau) \\ = (1/c^2)(\bar{\delta}_{\sigma\rho}{}^*v_\rho\phi_{\sigma\tau} - k'Q_\tau)(\partial u_\tau/\partial\tau^*) \\ + (\phi_{\sigma\tau} - k'Q_\tau v_\sigma/c^2)\bar{\delta}_{\sigma\rho}{}^*\partial_\rho{}^*u_\tau. \qquad (25')\end{aligned}$$

Neglecting terms which vanish in the classical limit, $c \to \infty$, we recognize the first law of thermodynamics relative to a secondary frame. On the left side appear the rate of increase of energy density $\chi_{\rho\rho}k'$ which is the rest energy per unit volume in the secondary frame, and the space divergence of the flux of heat and convected rest energy, $Q_\tau + \chi_{\rho\rho}u_\tau$. On integration over a volume fixed in the secondary frame, the integral of the space divergence term can be transformed by Gauss' theorem into the surface integral, $\int(Q_\sigma + \chi_{\rho\rho}k'v_\sigma)dn_\sigma{}^*$. On the right side appears the rate at which thermodynamic work is performed on unit volume of the secondary frame.

Multiplying Eq. (24) by $-u_\tau{}^*$, we obtain the time component equation relative to the secondary frame,

$$\partial_\sigma{}^*[k'(\chi_{\rho\rho}u_\sigma + Q_\sigma)] = \partial_\sigma{}^*[(\phi_{\sigma\tau} - u_\sigma Q_\tau/c^2)\bar{\delta}_{\tau\mu}{}^*v_\mu], \qquad (26)$$

which may be written as

$$(\partial/\partial\tau^*)[k'(\chi_{\rho\rho}k'+\bar{\delta}_{\sigma\tau}{}^*Q_\sigma v_\tau/c^2)]+\bar{\delta}_{\sigma\tau}{}^*\partial_\sigma{}^*[k'(\chi_{\rho\rho}u_\tau+Q_\tau)]$$
$$=(1/c^2)(\partial/\partial\tau^*)[(\phi_{\sigma\tau}v_\rho\bar{\delta}_{\sigma\rho}{}^*-k'Q_\tau)\bar{\delta}_{\tau\mu}{}^*v_\mu]$$
$$+\bar{\delta}_{\sigma\mu}{}^*\partial_\sigma{}^*[(\phi_{\mu\tau}-u_\mu Q_\tau/c^2)\bar{\delta}_{\tau\rho}{}^*v_\rho]. \quad (26')$$

Equation (26) has the form of a continuity equation for energy relative to a secondary frame. In the classical limit, there appear on the left side of Eq. (26′) the rate of increase of energy density, $\chi_{\rho\rho}k'/(1-\beta'^2)^{\frac{1}{2}}$ which includes the kinetic energy density relative to the secondary frame, and the space divergence of heat and convected energy flux, also augmented by the factor $k'=(1-\beta'^2)^{-\frac{1}{2}}$ as compared to the flux in the corresponding term of Eq. (25′). On the right side appears the rate at which total work, both thermodynamic and dynamical, is performed on unit volume. Equation (26) is equivalent to

$$\partial_\sigma{}^*(\psi_{\sigma\tau}u_\tau{}^*)=\partial_\sigma{}^0(ic\psi_{\sigma4}')=0. \quad (26'')$$

Thus in Eq. (26′) the operator $\partial/\partial\tau^*$ differentiates $\psi_{44}'=-\psi_{\sigma\tau}\hat{u}_\sigma{}^*\hat{u}_\tau{}^*$.

$$\psi_{44}'=\chi_{\rho\rho}k'^2+2\bar{\delta}_{\sigma\tau}{}^*Q_\sigma u_\tau/c^2-\phi_{\sigma\tau}\bar{\delta}_{\sigma\rho}{}^*v_\rho\bar{\delta}_{\tau\mu}{}^*v_\mu/c^2.$$

It is because of Eq. (26″) that it is customary to interpret ψ_{44}' as the total energy density and $ic\psi_{m4}'$ as the three-vector of total energy flux relative to the secondary frame.[2]

Multiplying Eq. (24) by $\bar{\delta}_{\alpha\tau}$, we obtain the space component equation relative to the rest frame,

$$(1/c^2)(\chi_{\rho\rho}du_\alpha/d\tau+Q_\sigma\partial_\sigma{}^*u_\alpha)=K_\alpha, \quad (27)$$

where K_α is a Minkowski force density,

$$K_\alpha=\bar{\delta}_{\alpha\tau}\partial_\sigma{}^*(\phi_{\sigma\tau}-u_\sigma Q_\tau/c^2). \quad (28)$$

Obviously, K_α is a space vector, so that $K_\alpha u_\alpha=0$ and, according to Eq. (13), $-u_\alpha{}^*K_\alpha=\bar{\delta}_{\alpha\beta}{}^*K_\alpha v_\beta$. Equation (27) is the basic equation of dynamics relative to a secondary frame. It may be resolved in turn into two equations. Multiplication by $\bar{\delta}_{\alpha\beta}{}^*$ gives the relative space component equation,

$$(1/c^2)(\chi_{\rho\rho}du_\alpha\bar{\delta}_{\alpha\beta}{}^*/d\tau+Q_\sigma\partial_\sigma{}^*u_\alpha\bar{\delta}_{\alpha\beta}{}^*)=\bar{\delta}_{\alpha\beta}{}^*K_\alpha, \quad (29)$$

which is the force equation; multiplication by $-u_\alpha{}^*$ gives the relative time component equation,

$$\chi_{\rho\rho}dk'/d\tau+Q_\sigma\partial_\sigma{}^*k'=\bar{\delta}_{\alpha\beta}{}^*K_\alpha v_\beta, \quad (30)$$

which is the kinetic energy equation. Thus, the product of the space component of force relative to the secondary frame, $\bar{\delta}_{\alpha\beta}{}^*K_\alpha$, by the relative space component of velocity, $\bar{\delta}_{\rho\beta}{}^*v_\rho$, gives the rate of performance of dynamical work, $\bar{\delta}_{\alpha\beta}{}^*K_\alpha v_\beta$, which according to Eq. (30) equals the rate of increase of kinetic energy. We may readily show that

$$\bar{\delta}_{\alpha\beta}{}^*K_\alpha v_\beta=\partial_\sigma{}^*[(\phi_{\sigma\tau}-u_\sigma Q_\tau/c^2)\bar{\delta}_{\tau\mu}{}^*v_\mu]$$
$$-k'(\phi_{\sigma\tau}-u_\sigma Q_\tau/c^2)\partial_\sigma{}^*u_\tau,$$

so that, according to Eqs. (25) and (26), the rate of performance of dynamical work per unit volume in the secondary frame is equal to the excess of the rate of performance of total work over that of thermodynamic work per unit volume.

Multiplying Eq. (24) by $\bar{\delta}_{\alpha\tau}{}^*$, we obtain the space component equation relative to the secondary frame

$$\partial_\sigma{}^*[(\chi_{\rho\rho}u_\sigma+Q_\sigma)u_\tau\bar{\delta}_{\alpha\tau}{}^*/c^2]$$
$$=\partial_\sigma{}^*[(\phi_{\sigma\tau}-u_\sigma Q_\tau/c^2)\bar{\delta}_{\alpha\tau}{}^*]. \quad (31)$$

This is the equation of continuity of momentum relative to the secondary frame. It is equivalent to

$$\partial_\sigma{}^*(\psi_{\sigma\tau}\bar{\delta}_{\alpha\tau}{}^*)=\partial_\sigma{}^0\psi_{\sigma m}'=0, \quad m=1,2,3. \quad (31')$$

Here we have the basis for the usual interpretation of $ic\psi_{4m}'$ as the total momentum density three-vector and ψ_{km}' as the total stress three-tensor relative to the secondary frame.

Just as Eqs. (25) and (26) yielded Eqs. (25′) and (26′), respectively, when the gradient operator was resolved into its space and time components, similarly each of Eqs. (27) to (31), inclusive, may be written in more detail. These equations will not be given here. However, an important case arises, the condition of the steady state, the covariant requirement for which is that the application of operator $\partial/\partial\tau^*$ to any quantity characterizing the continuum produce a derivative whose value is zero. Since the operator $\partial/\partial\tau^*$, as we have seen, is equal to d/dt^0, the steady-state condition does not depend on the choice of secondary frame to which the motion of the physical medium is referred. For example, the kinetic energy equation, Eq. (30), gives for the steady state

$$\chi_{\rho\rho}k'\bar{\delta}_{\sigma\tau}{}^*v_\sigma\partial_\tau{}^*k'+\bar{\delta}_{\sigma\tau}{}^*Q_\sigma\partial_\tau{}^*k'=\bar{\delta}_{\sigma\tau}{}^*K_\sigma v_\tau,$$

where the terms in the expression for K_σ, Eq. (28), which contain $\partial/\partial\tau^*$ also vanish.

The equations of this section all stem from Eq. (23), which we have seen implies, among other things, the expression for the first law of thermodynamics. They will appear more familiar as a description of the continuum after being modified by the statement of the second law and the equation for the stress in the system. They will be employed in the following discussion only in considering the case of a continuum composed of a single chemical species in the absence of chemical reaction.

D. ENTROPY TENSOR AND SECOND LAW OF THERMODYNAMICS

As in Paper I we write the entropy tensor,

$$S_{\sigma\tau}=-(1/c^2)(S_{\rho\rho}u_\sigma u_\tau+S_\sigma u_\tau+S_\tau u_\sigma), \quad (32)$$

where $S_{\rho\rho}$ is the scalar entropy density in the rest frame and S_σ is the entropy flux vector given by the heat flux vector divided by the scalar temperature T.

$$S_\sigma=Q_\sigma/T. \quad (33)$$

[2] See, for example, R. Tolman, *Relativity, Thermodynamics and Cosmology* (Oxford University Press, London, 1934), p. 73.

It was shown in Paper I that $S_{\rho\rho}$ and T have the properties, respectively, of the Clausius entropy and Kelvin temperature. We shall now require that the vector $\partial_\sigma^* S_{\sigma\tau}$ be a time vector. Accordingly,

$$\bar{\delta}_{\sigma\tau}\partial_\sigma^* S_{\sigma\tau}=0. \quad (34)$$

Furthermore, we require

$$u_\tau \partial_\sigma^* S_{\sigma\tau} \geq 0. \quad (35)$$

Equation (35) is a generalization of Eq. (55) of Paper I, but Eq. (34) is a new condition, of a dynamical rather than thermodynamic nature, which completes the specification of $\partial_\sigma^* S_{\sigma\tau}$.

Equation (35) is the statement of the second law of thermodynamics relative to a secondary frame. It may be written

$$\partial_\sigma^*(S_{\rho\rho}u_\sigma + S_\sigma) + (S_\tau/c^2)(du_\tau/d_\tau) \geq 0, \quad (35')$$

or, in more detail,

$$(\partial/\partial\tau^*)(S_{\rho\rho}k' + S_\sigma v_\tau \bar{\delta}_{\sigma\tau}^*/c^2) + \bar{\delta}_{\sigma\tau}^* \partial_\sigma^*(S_{\rho\rho}u_\tau + S_\tau) + (S_\tau/c^2)(du_\tau/d\tau) \geq 0. \quad (35'')$$

Apart from terms which vanish in the classical limit, $c \to \infty$, these equations state that the local entropy production rate per unit volume in the secondary frame is non-negative. A process occurring in the continuum for which the equality in Eq. (35) holds at every point will be called a reversible process relative to the secondary frame.

Equation (34) may be written

$$S_{\rho\rho}du_\alpha/d\tau + S_\sigma \partial_\sigma^* u_\alpha = -\bar{\delta}_{\alpha\tau}\partial_\sigma^*(S_\tau u_\sigma). \quad (34')$$

From this equation two component equations may be derived. Multiplying by $-u_\alpha^*$ gives the time component equation,

$$S_{\rho\rho}dk'/d\tau + S_\sigma \partial_\sigma^* k' = -\bar{\delta}_{\alpha\beta}^* v_\beta \bar{\delta}_{\alpha\tau}\partial_\sigma^*(S_\tau u_\sigma), \quad (36)$$

or multiplying by $\bar{\delta}_{\alpha\beta}^*$ gives the space component equation,

$$S_{\rho\rho}du_\alpha \bar{\delta}_{\alpha\beta}^*/d\tau + S_\sigma \partial_\sigma^*(u_\alpha \bar{\delta}_{\alpha\beta}^*) = -\bar{\delta}_{\alpha\beta}^* \bar{\delta}_{\alpha\tau}\partial_\sigma^*(S_\tau u_\sigma). \quad (37)$$

These equations describe a sort of entropy dynamics, a consequence of Eq. (34).

If $\partial_\sigma^* S_{\sigma\tau}$ is multiplied by u_τ^*, we find that

$$\partial_\sigma^*(S_{\sigma\tau}u_\tau^*) = k' u_\tau \partial_\sigma^* S_{\sigma\tau} \geq 0, \quad (38)$$

since k' is non-negative. This may also be written as

$$\partial_\sigma^*[k'(S_{\rho\rho}u_\sigma + S_\sigma + \bar{\delta}_{\alpha\beta}^* S_\alpha v_\beta v_\sigma/c^2)] \geq 0. \quad (38')$$

In a reversible process for which the equality holds, we find a continuity equation for entropy relative to a secondary frame. This result cannot be obtained from the second law, Eq. (35), alone without the use of Eq. (34). Equation (38) is equivalent to

$$\partial_\sigma^0(icS_{\sigma 4}') \geq 0, \quad (38'')$$

which permits interpretation of S_{44}' as the total entropy density and S_{m4}' as the three-vector of total entropy flux relative to the secondary frame.

Similarly, if $\partial_\sigma^* S_{\sigma\tau}$ is multiplied by $\bar{\delta}_{\tau\alpha}^*$, we have

$$\partial_\sigma^*(S_{\sigma\tau}\bar{\delta}_{\tau\alpha}^*) = -(u_\tau \bar{\delta}_{\tau\alpha}^*/c^2)(u_\rho \partial_\sigma^* S_{\sigma\rho}), \quad (39)$$

which is equivalent to

$$\partial_\sigma^0 S_{\sigma m}' = -(u_m'/c^2)(u_\rho \partial_\sigma^* S_{\sigma\rho}) \text{ for } m=1, 2, 3. \quad (39')$$

The right side vanishes in a reversible process, so that an additional continuity equation appears in this case,

$$\partial_\sigma^*[(S_{\rho\rho}u_\sigma u_\tau + S_\sigma u_\tau + S_\tau u_\sigma)\bar{\delta}_{\tau\alpha}^*] = 0. \quad (40)$$

These equations which have been obtained from Eqs. (34) and (35) may be combined with those of the preceding section obtained from Eq. (23). We write, using Eq. (33),

$$\psi_{\sigma\tau} = -(1/c^2)(\chi_{\rho\rho} - TS_{\rho\rho})u_\sigma u_\tau + TS_{\sigma\tau} + \phi_{\sigma\tau}, \quad (41)$$

so that

$$-(1/c^2)\partial_\sigma^*[(\chi_{\rho\rho} - TS_{\rho\rho})u_\tau u_\sigma] = -T\partial_\sigma^* S_{\sigma\tau} - S_{\sigma\tau}\partial_\sigma^* T - \partial_\sigma^* \phi_{\sigma\tau}. \quad (42)$$

Letting

$$\rho A = \chi_{\rho\rho} - TS_{\rho\rho} - \rho c^2, \quad (43)$$

where A is the specific Helmholtz free energy, as in Paper I, and using the equation of continuity, Eq. (15), we have

$$-(\rho/c^2)d(A+c^2)u_\tau/d\tau = -T\partial_\sigma^* S_{\sigma\tau} - S_{\sigma\tau}\partial_\sigma^* T - \partial_\sigma^* \phi_{\sigma\tau}. \quad (44)$$

Equation (44) is an alternative formulation of Eq. (24) and yields equivalent thermodynamic and dynamical consequences when the time component and space component equations relative to the rest frame and the secondary frame are obtained. We shall list these in turn, each combined with the appropriate component equation derived from Eqs. (34) and (35).

Multiplying Eq. (44) by u_τ and using Eq. (35) gives

$$\rho dA/d\tau \leq -S_{\rho\rho}dT/d\tau - S_\sigma \partial_\sigma^* T + \phi_{\sigma\tau}\partial_\sigma^* u_\tau. \quad (45)$$

Multiplying Eq. (44) by u_τ^* and using Eq. (38) gives

$$\rho d(A+c^2)k'/d\tau \leq -k'(S_{\rho\rho}u_\sigma + S_\sigma)\partial_\sigma^* T - (S_\tau v_\sigma \bar{\delta}_{\sigma\tau}^*/c^2)(dT/d\tau) + \partial_\sigma^*(\phi_{\alpha\tau}v_\mu \bar{\delta}_{\tau\mu}^*). \quad (46)$$

Multiplying Eq. (44) by $-\bar{\delta}_{\alpha\tau}$ and using Eq. (34) gives

$$\rho(1+A/c^2)du_\alpha/d\tau = -(S_\alpha/c^2)(dT/d\tau) + \bar{\delta}_{\alpha\tau}\partial_\sigma^* \phi_{\sigma\tau}. \quad (47)$$

Equation (47), when multiplied by $-u_\alpha^*$, gives the kinetic energy equation,

$$\rho(A+c^2)dk'd\tau = -(S_\alpha v_\beta \bar{\delta}_{\alpha\beta}^*/c^2)(dT/d\tau) + v_\beta \bar{\delta}_{\alpha\beta}^* \bar{\delta}_{\alpha\tau}\partial_\sigma^* \phi_{\sigma\tau}, \quad (48)$$

or, when multiplied by $\bar{\delta}_{\alpha\beta}^*$, gives the force equation,

$$\rho(1+A/c^2)du_\alpha \bar{\delta}_{\alpha\beta}^*/d\tau = -S_\alpha(\bar{\delta}_{\alpha\beta}^*/c^2)(dT/d\tau) + \bar{\delta}_{\alpha\beta}^* \bar{\delta}_{\alpha\tau}\partial_\sigma^* \phi_{\sigma\tau}. \quad (49)$$

Finally, multiplying Eq. (44) by $-\bar{\delta}_{\alpha\tau}^*$ and using Eq. (39) gives

$$(\rho/c^2)d(A+c^2)u_\tau\bar{\delta}_{\alpha\tau}^*/d\tau$$
$$=\partial_\sigma^*(\phi_{\sigma\tau}\bar{\delta}_{\tau\alpha}^*)-(\bar{\delta}_{\alpha\tau}^*/c^2)[(S_{\rho\rho}u_\tau+S_\tau)(dT/d\tau)$$
$$+u_\tau S_\sigma\partial_\sigma^*T]-T(u_\tau\bar{\delta}_{\tau\alpha}^*/c^2)(u_\rho\partial_\sigma^*S_{\sigma\rho}), \quad (50)$$

in which the last term on the right side vanishes in a reversible process. These equations describe the thermodynamics and dynamics of an element of continuum composed of a single, chemically inert species, and conform to the requirements of the first and second laws of thermodynamics.

E. THE STRESS IN THE SYSTEM AND REVERSIBILITY

Let us now introduce the equations defining the reversible stress in the system. This is the stress determined by the properties of the element of continuum at a point in the physical system. It was denoted by $\phi_{\sigma\tau}'$ and defined by Eq. (38) or Eq. (46) in Paper I. We will denote it here by $\phi_{\sigma\tau}^r$ and replace Eq. (46) of Paper I by a more general form,

$$\partial_\sigma^*[(-1/c^2)\rho(A+c^2)u_\tau u_\sigma+\phi_{\sigma\tau}^r]$$
$$=(\rho u_\sigma/c^2)[\partial_\tau^*(A+c^2)u_\sigma$$
$$-\partial_\sigma^*(A+c^2)u_\tau]-S_{\rho\rho}\partial_\tau^*T. \quad (51)$$

With the help of the equation of continuity Eq. (15), we may reduce this equation to

$$-\rho\partial_\tau^*A-S_{\rho\rho}\partial_\tau^*T=\partial_\sigma^*\phi_{\sigma\tau}^r. \quad (51')$$

Consequences of this equation are obtained in the usual manner. Multiplying Eq. (51′) by $-u_\tau$ gives

$$\rho dA/d\tau+S_{\rho\rho}dT/d\tau=\phi_{\sigma\tau}^r\partial_\sigma^*u_\tau. \quad (52)$$

Multiplying Eq. (51′) by $-u_\tau^*$ gives

$$\rho\partial A/\partial\tau^*+S_{\rho\rho}\partial T/\partial\tau^*=\partial_\sigma^*(\phi_{\sigma\tau}^r v_\mu\bar{\delta}_{\tau\mu}^*). \quad (53)$$

Multiplying Eq. (51′) by $\bar{\delta}_{\alpha\tau}$ gives

$$-\rho\bar{\delta}_{\alpha\tau}\partial_\tau^*A-S_{\rho\rho}\bar{\delta}_{\alpha\tau}\partial_\tau^*T=\bar{\delta}_{\alpha\tau}\partial_\sigma^*\phi_{\sigma\tau}^r. \quad (54)$$

Multiplying Eq. (51′) by $\bar{\delta}_{\alpha\tau}^*$ gives

$$-\rho\bar{\delta}_{\alpha\tau}^*\partial_\tau^*A-S_{\rho\rho}\bar{\delta}_{\alpha\tau}^*\partial_\tau^*T=\partial_\sigma^*(\phi_{\sigma\tau}^r\bar{\delta}_{\alpha\tau}^*). \quad (55)$$

In Eq. (52), $dA/d\tau$ expresses the rate of change of specific Helmholz free energy, a point function of the variables specifying the properties of the element of continuum. In the special case that these properties are isotropic, $\phi_{\sigma\tau}^r$ reduces to the hydrostatic pressure, $-p\bar{\delta}_{\sigma\tau}$. We find with the aid of Eq. (18) in this case the usual thermodynamic formula,

$$\rho dA/d\tau=-S_{\rho\rho}dT/d\tau-\rho p d(1/\rho)/d\tau, \quad (52')$$

expressing A as a function of T and specific volume, $1/\rho$.

Equation (54) takes a particularly interesting form in the isotropic case,

$$\rho\bar{\delta}_{\alpha\tau}\partial_\tau^*A=-S_{\rho\rho}\bar{\delta}_{\alpha\tau}\partial_\tau^*T-\bar{\delta}_{\alpha\tau}\partial_\tau^*p-(p/c^2)(du_\alpha/d\tau). \quad (54')$$

But in the classical limit, $c\rightarrow\infty$, the right side is just the expression which would be expected to equal $\rho\bar{\delta}_{\alpha\tau}\partial_\tau^*F$, where F is the specific Gibbs free energy, rather than $\rho\bar{\delta}_{\alpha\tau}\partial_\tau^*A$, where A is the specific Helmholtz free energy. We see, therefore, that ∂_τ^*A in Eq. (51′) has the property that in its time component or thermodynamic aspect, A plays the role of the Helmholtz free energy, but in its space component or dynamical aspect, A has the nature of the Gibbs free energy. It will be seen in the following that the Helmholtz free energy will appear in our dynamical equations whereas in the corresponding classical equations for the isotropic case the Gibbs free energy would appear. This is particularly fortunate since it enables us to formulate equations for anisotropic media which cannot be written classically, because of the inability of classical theory to define an adequate Gibbs free energy for anisotropic media; we use the Helmholz function. In the following we shall refer to A as simply the chemical potential.

Similar remarks, of course, apply to the behavior of $\chi_{\rho\rho}/\rho$ in the equation,

$$-\rho\partial_\tau^*(\chi_{\rho\rho}/\rho)=-T\rho\partial_\tau^*(S_{\rho\rho}/\rho)+\partial_\sigma^*\phi_{\sigma\tau}^r. \quad (56)$$

In the time component of $\partial_\tau^*(\chi_{\rho\rho}/\rho)$ this quantity appears as the specific thermodynamic energy, whereas in the space component it appears as the specific enthalpy. Classical theory is likewise unable to define an enthalpy function for anisotropic media.

Since the change in A in a physical process depends only on the end points of the process and not on the path, therefore, in the reversible case described by the equality in Eq. (45), the expression for $\rho dA/d\tau$ in Eqs. (45) and (52) must be identical. This is the case if we take as conditions for reversibility,

$$S_\sigma=0 \quad \text{and} \quad \phi_{\sigma\tau}=\phi_{\sigma\tau}^r. \quad (57)$$

Following Paper I, Eq. (67), we could have chosen the condition $\bar{\delta}_{\sigma\tau}\partial_\tau^*T=0$ instead of $S_\sigma=0$, offering Fourier's law of heat conduction as evidence that the two conditions are equivalent. However, Fourier's law, as we shall see in the discussion of transport processes which concludes this paper, does not possess the degree of generality which has been assumed in our formulation to this point. It seems preferable to take $S_\sigma=0$ as the condition of reversibility with $\bar{\delta}_{\sigma\tau}\partial_\tau^*T=0$ as equivalent to it when the motion of the medium conforms to the requirements of the theory of transport processes. It is important to note that the conditions stated in Eq. (57) are independent of the choice of secondary reference frame to which the motion of the continuum is referred.

Equations (45) and (52) give, in the general case,

$$-S_\sigma\partial_\sigma^*T+(\phi_{\sigma\tau}-\phi_{\sigma\tau}^r)\partial_\sigma^*u_\tau\geq 0, \quad (58)$$

which is the dissipation equation corresponding to Eq. (57) of Paper I, while Eq. (45) itself corresponds to Eq. (58) of Paper I.

In the remainder of this section we shall consider the dynamical consequences of Eq. (51′), appearing in the space component equation, Eq. (54). If we combine the dynamical equation, Eq. (47), with Eq. (54), we find

$$\rho(1+A/c^2)du_\alpha/d\tau+(S_\alpha/c^2)(dT/d\tau) \\ =-S_{\rho\rho}\bar{\delta}_{\alpha\tau}\partial_\tau{}^*T-\rho\bar{\delta}_{\alpha\tau}\partial_\tau{}^*A-\bar{\delta}_{\alpha\tau}\partial_\sigma{}^*(\phi_{\sigma\tau}-\phi_{\sigma\tau}{}^r), \quad (59)$$

corresponding to Eq. (78) of Paper I, the difference arising from our present assumption of Eq. (34). On multiplying by $\bar{\delta}_{\alpha\beta}{}^*$ we obtain the force equation,

$$\rho(1+A/c^2)du_\alpha\bar{\delta}_{\alpha\beta}{}^*/d\tau+(S_\alpha\bar{\delta}_{\alpha\beta}{}^*/c^2)(dT/d\tau) \\ =-S_{\rho\rho}\bar{\delta}_{\alpha\beta}{}^*\bar{\delta}_{\alpha\tau}\partial_\tau{}^*T-\rho\bar{\delta}_{\alpha\beta}{}^*\bar{\delta}_{\alpha\tau}\partial_\tau{}^*A \\ +\bar{\delta}_{\alpha\beta}{}^*\bar{\delta}_{\alpha\tau}\partial_\sigma{}^*(\phi_{\sigma\tau}-\phi_{\sigma\tau}{}^r). \quad (60)$$

On the right-hand side, the following force densities may be recognized:

(1) $$\bar{\delta}_{\alpha\beta}{}^*\bar{\delta}_{\alpha\tau}\partial_\sigma{}^*(\phi_{\sigma\tau}-\phi_{\sigma\tau}{}^r) \\ =(1/c^2)(\partial/\partial\tau^*)[(\phi_{\sigma\alpha}-\phi_{\sigma\alpha}{}^r)\bar{\delta}_{\alpha\beta}{}^*\bar{\delta}_{\sigma\rho}{}^*v_\rho] \\ +\bar{\delta}_{\sigma\rho}{}^*\partial_\sigma{}^*[(\phi_{\rho\alpha}-\phi_{\rho\alpha}{}^r)\bar{\delta}_{\alpha\beta}{}^*] \\ -(u_\alpha\bar{\delta}_{\alpha\beta}{}^*/c^2)(\phi_{\sigma\tau}-\phi_{\sigma\tau}{}^r)\partial_\sigma{}^*u_\tau.$$

In the classical limit, $c\rightarrow\infty$, only the space divergence term remains on the right. On integration over a volume fixed in the secondary frame, the integral of this space divergence can be converted into a surface integral, $\int(\phi_{\sigma\alpha}-\phi_{\sigma\alpha}{}^r)\bar{\delta}_{\alpha\beta}{}^*dn_\sigma{}^*$, proportional to the area of the boundary surface. This force density corresponds to the classical surface force density of hydrodynamics.

(2) $-\rho\bar{\delta}_{\beta\alpha}{}^*\bar{\delta}_{\alpha\tau}\partial_\tau{}^*A$ is a body force density arising from the gradient of chemical potential. As we know, other forces such as electromagnetic forces and gravitational forces are included among the possible body forces of classical hydrodynamics. The absence of such forces in Eq. (60) indicates the limitation of our present treatment to a single-component chemical system without external body forces. Equation (59) may be written in the form

$$\rho du_\alpha/d\tau+(S_\alpha/c^2)(dT/d\tau) \\ =(\rho u_\tau/c)(\partial_\alpha{}^*A\hat{u}_\tau-\partial_\tau{}^*A\hat{u}_\alpha) \\ -S_{\rho\rho}\bar{\delta}_{\alpha\tau}\partial_\tau{}^*T+\bar{\delta}_{\alpha\tau}\partial_\sigma{}^*(\phi_{\sigma\tau}-\phi_{\sigma\tau}{}^r). \quad (59')$$

The term containing the skew-symmetric tensor, $\partial_\alpha{}^*A\hat{u}_\tau-\partial_\tau{}^*A\hat{u}_\alpha$, suggests the electromagnetic Lorentz force, $(\rho u_\tau/c)(\partial_\alpha{}^*A_\tau-\partial_\tau{}^*A_\alpha)$, where A_τ is a four-vector potential. Inclusion of such an electromagnetic force would be accomplished by simply replacing the chemical four-vector potential $A\hat{u}_\sigma$ by the electrochemical potential $A\hat{u}_\sigma+A_\sigma$. A general investigation of electrical processes in continuous media will no doubt reveal such quantities, but this will not be undertaken here.

(3) $-S_{\rho\rho}\bar{\delta}_{\beta\alpha}{}^*\bar{\delta}_{\alpha\tau}\partial_\tau{}^*T$ is the force density arising from the presence of entropy in a temperature gradient. This force is exerted on the matter with which the entropy is associated.

The sum of these three force densities constitutes the single force density $\bar{\delta}_{\beta\alpha}{}^*\bar{\delta}_{\alpha\tau}\partial_\sigma{}^*\phi_{\sigma\tau}$ on the right-hand side of Eq. (49), the total force density exerted on an element of continuum relative to the secondary reference frame.

On multiplying Eq. (59) by $-u_\alpha{}^*$ we obtain the kinetic energy equation,

$$\rho(A+c^2)dk'/d\tau+v_\beta\bar{\delta}_{\alpha\beta}{}^*S_\alpha dT/d\tau \\ =-S_{\rho\rho}v_\beta\bar{\delta}_{\alpha\beta}{}^*\bar{\delta}_{\alpha\tau}\partial_\tau{}^*T-\rho v_\beta\bar{\delta}_{\alpha\beta}{}^*\bar{\delta}_{\alpha\tau}\partial_\tau{}^*A \\ +v_\beta\bar{\delta}_{\alpha\beta}{}^*\bar{\delta}_{\alpha\tau}\partial_\sigma{}^*(\phi_{\sigma\tau}-\phi_{\sigma\tau}{}^r). \quad (61)$$

Each term on the right, representing a rate of performance of work per unit volume, is simply the product of the corresponding force density term of Eq. (59) by the relative space velocity $v_\rho\bar{\delta}_{\beta\rho}{}^*$. The last term on the right, attributable to surface forces, may be written as

$$v_\beta\bar{\delta}_{\alpha\beta}{}^*\bar{\delta}_{\alpha\tau}\partial_\sigma{}^*(\phi_{\sigma\tau}-\phi_{\sigma\tau}{}^r)=\partial_\sigma{}^*[(\phi_{\sigma\alpha}-\phi_{\sigma\alpha}{}^r)\bar{\delta}_{\alpha\beta}{}^*v_\beta] \\ -k'(\phi_{\sigma\tau}-\phi_{\sigma\tau}{}^r)\partial_\sigma{}^*u_\tau. \quad (61')$$

Each term on the right side of this equation has an interpretation in the classical limit. The first yields a space divergence which gives the total rate of performance of work by the stress at the surface of the element in excess of that which the element can support. The other is the rate of dissipation accompanying the action of this excess stress. It is of interest that dissipative effects are associated with surfaces stresses, not with body forces.

For reversible processes the dynamical equations, Eqs. (47) and (59), give

$$\rho(1+A/c^2)du_\alpha/d\tau=\bar{\delta}_{\alpha\tau}\partial_\sigma{}^*\phi_{\sigma\tau}{}^r \\ =-S_{\rho\rho}\bar{\delta}_{\sigma\tau}\partial_\tau{}^*T-\rho\bar{\delta}_{\alpha\tau}\partial_\tau{}^*A, \quad (62)$$

or

$$\rho du_\alpha/d\tau=-S_{\rho\rho}\bar{\delta}_{\sigma\tau}\partial_\tau{}^*T+(\rho u_\tau/c)(\partial_\alpha{}^*A\hat{u}_\tau-\partial_\tau{}^*A\hat{u}_\alpha). \quad (62')$$

According to Eq. (62′), in the absence of a temperature gradient an element of continuum moves in a reversible process like a particle of rest mass density ρ in a field whose four-vector potential is $A\hat{u}_\sigma$. As a special case of reversible process we have the condition of unaccelerated motion relative to the secondary frame, for which $du_\alpha/d\tau$ vanishes. Equation (62) shows that in this case $\bar{\delta}_{\alpha\tau}\partial_\sigma{}^*\phi_{\sigma\tau}{}^r=0$. Since the operator $d/d\tau=u_\sigma\partial_\sigma{}^*$ depends on the choice of secondary frame, the vanishing of $du_\alpha/d\tau$ in one secondary frame does not guarantee its vanishing in all others, unless the secondary frame in which $du_\alpha/d\tau$ vanishes is also a rest frame.

Equations (52) and (62) can be used as a starting point for the formulation of the theory of elasticity. The stress which according to Hooke's law is proportional to the symmetric strain is $\phi_{\sigma\tau}{}^r$. The theory is essentially a theory of reversible strains. For example, in the derivation of the Laplace formula for the velocity of sound in a gas, the equations for reversible, adiabatic compressions are employed.

F. TRANSPORT PROCESSES IN A ONE-COMPONENT MEDIUM

In the description of the transport processes of conduction and diffusion the motion of every element of the continuum is referred to the same secondary frame. It is necessary to find a secondary frame relative to which the continuum everywhere satisfies a special condition, which may be formulated as

$$\partial_\sigma^*[(\phi_{\sigma\tau}-\phi_{\sigma\tau}{}^r)v_\mu\bar{\delta}_{\tau\mu}{}^*]=v_\beta\bar{\delta}_{\alpha\beta}{}^*\bar{\delta}_{\alpha\tau}\partial_\sigma{}^*\phi_{\sigma\tau}. \quad (63)$$

It is evident that the fact that Eq. (63) characterizes the motion relative to one secondary frame does not imply that a similar relation will hold for all other secondary frames. This, however, does not vitiate the covariance of Eq. (63) since this expression is the same for all primary observers; any primary observer recording the motion of the continuum relative to the special secondary frame will find that Eq. (63) is satisfied, if the motion is amenable to description by the theories of diffusion or conduction.

According to Eq. (63), the rate at which total work is performed upon each element of continuum by the excess of stress in its surroundings over that in the element is equal to the rate of dynamical work which increases the kinetic energy. Equation (48) gives

$$\partial_\sigma^*[(\phi_{\sigma\tau}-\phi_{\sigma\tau}{}^r)v_\mu\bar{\delta}_{\tau\mu}{}^*]=\rho(A+c^2)dk'/d\tau + (S_\alpha v_\beta\bar{\delta}_{\alpha\beta}{}^*/c^2)dT/d\tau. \quad (64)$$

On the other hand, Eqs. (61′) and (54) give

$$k'(\phi_{\sigma\tau}-\phi_{\sigma\tau}{}^r)\partial_\sigma{}^*u_\tau=v_\beta\bar{\delta}_{\alpha\beta}{}^*\bar{\delta}_{\alpha\tau}\partial_\sigma{}^*\phi_{\sigma\tau}{}^r = -\rho v_\beta\bar{\delta}_{\alpha\beta}{}^*\bar{\delta}_{\alpha\tau}\partial_\tau{}^*A-S_{\rho\rho}v_\beta\bar{\delta}_{\alpha\beta}{}^*\bar{\delta}_{\alpha\tau}\partial_\tau{}^*T, \quad (65)$$

so that the rate of dissipation $k'(\phi_{\sigma\tau}-\phi_{\sigma\tau}{}^r)\partial_\sigma{}^*u_\tau$ is maintained by a lowering of potential levels in the element. If this last equation is combined with Eq. (52), the rate of thermodynamic work performed by the surroundings upon the element is found to be

$$k'\phi_{\sigma\tau}\partial_\sigma{}^*u_\tau=\rho\partial A/\partial\tau^*+S_{\rho\rho}\partial T/\partial\tau^*; \quad (66)$$

whereas combination with Eq. (58) gives the dissipation equation,

$$-S_{\rho\rho}v_\beta\bar{\delta}_{\alpha\beta}{}^*\bar{\delta}_{\alpha\tau}\partial_\tau{}^*T-k'S_\sigma\partial_\sigma{}^*T -\rho v_\beta\bar{\delta}_{\alpha\beta}{}^*\bar{\delta}_{\alpha\tau}\partial_\tau{}^*A\geq 0. \quad (67)$$

Equation (67) will serve as starting point for a covariant theory of transport processes.

It is interesting that the alternative form of the transport condition appearing in Eq. (64) gives, in the classical limit, exactly the same equation as that which must be assumed in a nonrelativistic transport theory, the equation expressing the assumption that "the entire acceleration of any fluid element is produced by viscous forces alone, acting on the element as though it were rigid."[3] Equation (67) differs from the corresponding nonrelativistic equation in that the latter is limited to isotropic media since it employs the specific Gibbs free energy to represent the chemical potential. The theory developed here will describe transport processes in an anisotropic, one-component chemical continuum in the absence of external body forces. We follow the method used in the nonrelativistic theory.

Equation (67) can be written

$$-[S_{\rho\rho}v_\beta\bar{\delta}_{\alpha\beta}{}^*+k'S_\beta(\bar{\delta}_{\alpha\beta}{}^*-v_\rho\bar{\delta}_{\beta\rho}{}^*v_\sigma\bar{\delta}_{\alpha\sigma}{}^*/c^2)]\bar{\delta}_{\alpha\tau}\partial_\tau{}^*T -\rho v_\beta\bar{\delta}_{\alpha\beta}{}^*\bar{\delta}_{\alpha\tau}\partial_\tau{}^*A\geq 0. \quad (67')$$

We assume the linear relationships,

$$\begin{aligned}\rho v_\beta\bar{\delta}_{\alpha\beta}{}^*&=-a_{\alpha\beta}{}^*\bar{\delta}_{\beta\tau}\partial_\tau{}^*A-b_{\alpha\beta}{}^*\bar{\delta}_{\beta\tau}\partial_\tau{}^*T,\\ S_{\rho\rho}v_\beta\bar{\delta}_{\alpha\beta}{}^*+k'S_\beta(\bar{\delta}_{\alpha\beta}{}^*-v_\rho\bar{\delta}_{\beta\rho}{}^*v_\sigma\bar{\delta}_{\alpha\sigma}{}^*/c^2)&=-b_{\alpha\beta}{}^*\bar{\delta}_{\beta\tau}\partial_\tau{}^*A-c_{\alpha\beta}{}^*\bar{\delta}_{\beta\tau}\partial_\tau{}^*T,\end{aligned} \quad (68)$$

where, according to Onsager's reciprocal relations,[4] the tensors $a_{\alpha\beta}{}^*$, $b_{\alpha\beta}{}^*$, $c_{\alpha\beta}{}^*$ are symmetric. These tensors are pure space tensors in the secondary reference frame. For example,

$$a_{\alpha\beta}{}^*=\left\{\begin{array}{l}\bar{\imath}_\alpha{}^*\bar{\imath}_\beta{}^*a_{11}{}^0+\bar{\imath}_\alpha{}^*\bar{\jmath}_\beta{}^*a_{12}{}^0+\bar{\imath}_\alpha{}^*\bar{k}_\beta{}^*a_{13}{}^0\\ +\bar{\jmath}_\alpha{}^*\bar{\imath}_\beta{}^*a_{21}{}^0+\bar{\jmath}_\alpha{}^*\bar{\jmath}_\beta{}^*a_{22}{}^0+\bar{\jmath}_\alpha{}^*\bar{k}_\beta{}^*a_{23}{}^0\\ +\bar{k}_\alpha{}^*\bar{\imath}_\beta{}^*a_{31}{}^0+\bar{k}_\alpha{}^*\bar{\jmath}_\beta{}^*a_{32}{}^0+\bar{k}_\alpha{}^*\bar{k}_\beta{}^*a_{33}{}^0\end{array}\right\} =L_{\alpha m}{}^*L_{\beta n}{}^*a_{mn}{}^0,$$

where six coefficients, independent of the velocity components v_1', v_2', v_3' of the continuum relative to the secondary frame, appear. In an isotropic medium we should have

$$a_{\alpha\beta}{}^*=a\bar{\delta}_{\alpha\beta}{}^*,\quad b_{\alpha\beta}{}^*=b\bar{\delta}_{\alpha\beta}{}^*,\quad c_{\alpha\beta}{}^*=c\bar{\delta}_{\alpha\beta}{}^*.$$

Substituting Eq. (68) into Eq. (67′) gives

$$a_{\alpha\beta}{}^*\bar{\delta}_{\alpha\sigma}\partial_\sigma{}^*A\,\bar{\delta}_{\beta\tau}\partial_\tau{}^*A+2b_{\alpha\beta}{}^*\bar{\delta}_{\alpha\sigma}\partial_\sigma{}^*A\,\bar{\delta}_{\beta\tau}\partial_\tau{}^*T +c_{\alpha\beta}{}^*\bar{\delta}_{\alpha\sigma}\partial_\sigma{}^*T\bar{\delta}_{\beta\tau}\partial_\tau{}^*T\geq 0. \quad (69)$$

Since this relation must be valid for any choice of $\bar{\delta}_{\alpha\tau}\partial_\tau{}^*A$ and $\bar{\delta}_{\alpha\tau}\partial_\tau{}^*T$, the determinant

$$\begin{vmatrix} a_{11}{}^0, & a_{12}{}^0, & a_{13}{}^0, & b_{11}{}^0, & b_{12}{}^0, & b_{13}{}^0\\ a_{21}{}^0, & a_{22}{}^0, & a_{23}{}^0, & b_{21}{}^0, & b_{22}{}^0, & b_{23}{}^0\\ a_{31}{}^0, & a_{32}{}^0, & a_{33}{}^0, & b_{31}{}^0, & b_{32}{}^0, & b_{33}{}^0\\ b_{11}{}^0, & b_{12}{}^0, & b_{13}{}^0, & c_{11}{}^0, & c_{12}{}^0, & c_{13}{}^0\\ b_{21}{}^0, & b_{22}{}^0, & b_{23}{}^0, & c_{21}{}^0, & c_{22}{}^0, & c_{23}{}^0\\ b_{31}{}^0, & b_{32}{}^0, & b_{33}{}^0, & c_{31}{}^0, & c_{32}{}^0, & c_{33}{}^0 \end{vmatrix}\geq 0, \quad (70)$$

and each of its principal minors must be non-negative. The conditions of reversibility of transport processes, under which the equality holds in Eq. (69), appear as

$$\bar{\delta}_{\alpha\tau}\partial_\tau{}^*T=0\quad\text{and}\quad\bar{\delta}_{\alpha\tau}\partial_\tau{}^*A=0, \quad (71)$$

which may be compared to the more general conditions of Eq. (57). According to Eqs. (68) and (62), we must

[3] B. Leaf, Phys. Rev. **70**, 752 (1946), Eq. (42).

[4] L. Onsager, Phys. Rev. **37**, 405 (1931); **38**, 2265 (1931).

also have

$$v_\beta\bar{\delta}_{\alpha\beta}^*=0, \quad du_\alpha/d\tau=0, \quad S_\alpha=0, \tag{72}$$

so that in this case the secondary frame to which the transport is referred is the rest frame for the entire continuum. The continuum is static throughout a reversible transport process, which is, therefore, a quasi-static process.

Let us write

$$b_{mn}{}^0=\sigma_{mk}{}^0 a_{kn}{}^0, \tag{73}$$

so that

$$b_{\alpha\beta}^*=L_{\alpha m}^*L_{\beta n}^*\sigma_{mk}{}^0a_{kn}{}^0 = (L_{\alpha m}^*L_{\nu k}^*\sigma_{mk}{}^0)(L_{\nu j}^*L_{\beta n}^*a_{jn}{}^0)=\sigma_{\alpha\nu}^*a_{\nu\beta}^*.$$

Then, according to Eq. (68),

$$\begin{aligned}k'S_\beta(\bar{\delta}_{\alpha\beta}^*-v_\rho\bar{\delta}_{\beta\rho}^*v_\sigma\bar{\delta}_{\alpha\sigma}^*/c^2) &= -(\sigma_{\alpha\nu}^*-S_{\rho\rho}\bar{\delta}_{\alpha\nu}/\rho)a_{\nu\beta}^*\bar{\delta}_{\beta\tau}\partial_\tau^*A \\ &\quad -(c_{\alpha\beta}^*-S_{\rho\rho}b_{\alpha\beta}^*/\rho)\bar{\delta}_{\beta\tau}\partial_\tau^*T \\ &= (\rho\sigma_{\alpha\beta}^*-S_{\rho\rho}\bar{\delta}_{\alpha\beta}^*)v_\beta-(c_{\alpha\beta}^*-\sigma_{\alpha\mu}^*\sigma_{\mu\nu}^*a_{\nu\beta}^*)\bar{\delta}_{\beta\tau}\partial_\tau^*T.\end{aligned}$$

Therefore,

$$k'Q_\beta(\bar{\delta}_{\alpha\beta}^*-v_\rho\bar{\delta}_{\beta\rho}^*v_\sigma\bar{\delta}_{\alpha\sigma}^*/c^2)=T(\rho\sigma_{\alpha\beta}^*-S_{\rho\rho}\bar{\delta}_{\alpha\beta}^*)v_\beta -\lambda_{\alpha\beta}^*\bar{\delta}_{\beta\tau}\partial_\tau^*T, \tag{74}$$

which, for $v_\beta\bar{\delta}_{\alpha\beta}^*=0$, gives in the classical limit Fourier's law of heat conduction. Here $\lambda_{\alpha\beta}^*=L_{\alpha m}^*L_{\beta n}^*\lambda_{mn}{}^0$, where $\lambda_{mn}{}^0$ is the heat conductivity tensor for an anisotropic medium.

$$\lambda_{mn}{}^0=T(c_{mn}{}^0-\sigma_{mk}{}^0\sigma_{kj}{}^0a_{jn}{}^0). \tag{75}$$

Solving Eq. (73) for $\sigma_{mk}{}^0$ we find

$$\sigma_{mk}{}^0=b_{mn}{}^0A_{kn}{}^0/|a_{ij}|,$$

where $A_{kn}{}^0$ is the cofactor of $a_{kn}{}^0$ in the determinant $|a_{ij}|$. Therefore,

$$|a_{ij}{}^0|\lambda_{mn}{}^0/T=|a_{ij}{}^0|c_{mn}{}^0-b_{mj}{}^0A_{kj}{}^0b_{kn}{}^0 = \begin{vmatrix} a_{11}{}^0, & a_{12}{}^0, & a_{13}{}^0, & b_{1n}{}^0 \\ a_{21}{}^0, & a_{22}{}^0, & a_{23}{}^0, & b_{2n}{}^0 \\ a_{31}{}^0, & a_{32}{}^0, & a_{33}{}^0. & b_{3n}{}^0 \\ b_{m1}{}^0, & b_{m2}{}^0, & b_{m3}{}^0, & c_{mn}{}^0 \end{vmatrix},$$

which is a minor of the determinant of Eq. (70). Accordingly, $\lambda_{11}{}^0$, $\lambda_{22}{}^0$, $\lambda_{33}{}^0$ are all non-negative. Clearly, $\lambda_{mn}{}^0$ is a symmetric tensor.

Equations (68) and (73) give

$$\rho v_\beta\bar{\delta}_{\alpha\beta}^*=-a_{\alpha\beta}^*(\bar{\delta}_{\beta\tau}\partial_\tau^*A+\sigma_{\rho\beta}^*\bar{\delta}_{\rho\tau}\partial_\tau^*T).$$

Therefore,

$$\bar{\delta}_{\rho\beta}^*\bar{\delta}_{\rho\tau}\partial_\tau^*A+\sigma_{\rho\beta}^*\bar{\delta}_{\rho\tau}\partial_\tau^*T=-r_{\alpha\beta}^*\rho v_\alpha; \tag{76}$$

where

$$r_{\alpha\beta}^*=L_{\alpha m}^*L_{\beta n}^*r_{mn}{}^0, \quad r_{mn}{}^0=A_{mn}{}^0/|a_{ij}|. \tag{77}$$

Substitution of Eq. (76) into Eq. (67′) gives, with the help of Eq. (74),

$$r_{\alpha\beta}^*(\rho v_\alpha)(\rho v_\beta)+(\lambda_{\alpha\beta}^*/T)(\bar{\delta}_{\alpha\sigma}\partial_\sigma^*T)(\bar{\delta}_{\beta\tau}\partial_\tau^*T)\geq 0. \tag{78}$$

These equations describe the transport processes in terms of the symmetric resistivity tensor $r_{mn}{}^0$. The quantity $r_{\alpha\beta}^*(\rho v_\alpha)(\rho v_\beta)$ is the analog of the Joule electrical dissipation rate in an anisotropic medium per unit volume in the secondary frame. Both determinants, $|r_{mn}{}^0|$ and $|\lambda_{mn}{}^0|$, and their principal minors are non-negative.

The condition for a transport process given in Eq. (63) requires that a secondary reference frame exist such that when the motion of every element of the continuum is referred to this same frame the condition should hold. All the equations for transport processes are formulated for motion relative to this frame. If this frame should coincide with the momentary rest frame of some element of the continuum, then $v_\beta\bar{\delta}_{\alpha\beta}^*=0$; the transport process in that element would consist merely of heat conduction. According to Eqs. (67) and (78), for such an element the dissipation is given by

$$-S_\sigma\partial_\sigma T=(\lambda_{\alpha\beta}/T)(\bar{\delta}_{\alpha\sigma}\partial_\sigma T)(\bar{\delta}_{\beta\tau}\partial_\tau T)\geq 0,$$

and, according to Eq. (65), $(\phi_{\sigma\tau}-\phi_{\sigma\tau}{}^r)\partial_\sigma u_\tau=0$. If the special secondary frame should coincide with the rest frame of every element of the continuum, then the entire transport process would consist only of heat conduction throughout the continuum and these equations would be valid everywhere. But consider some element whose motion satisfies the transport condition relative to a special secondary frame which does not coincide with its rest frame. Its dissipation will be given by Eqs. (58), (67′), (69), or (78). Its dissipation is also given relative to its rest frame as

$$-S_\sigma\partial_\sigma T+(\phi_{\sigma\tau}-\phi_{\sigma\tau}{}^r)\partial_\sigma u_\tau\geq 0, \tag{58′}$$

according to Eq. (58). It should be noted that, although in the reversible case the dissipation relative to every secondary frame vanishes, in an irreversible process the magnitude of the dissipation given in Eq. (58) depends on the choice of secondary frame. (The equation is nevertheless relativistically covariant since for a given secondary frame its form is the same for all primary observers.)

The theory of viscosity differs in method from the transport theories of diffusion and conduction. In the theory of viscosity the motion of each element of continuum is referred to its own rest frame. The transport condition of Eq. (63) need not be satisfied and no special secondary frame is employed. The dissipation rate is given by Eq. (58′). The linear relations assumed in this theory were stated in Paper I, Eq. (66), as

$$\phi_{\sigma\tau}-\phi_{\sigma\tau}{}^r=\eta_{\sigma\tau\mu\nu}\Sigma_{\mu\nu}, \tag{79}$$

$$\Sigma_{\mu\nu}=\tfrac{1}{2}\bar{\delta}_{\mu\alpha}\bar{\delta}_{\nu\beta}(\partial_\alpha u_\beta+\partial_\beta u_\alpha), \tag{80}$$

where the coefficients of viscosity $\eta_{\sigma\tau\mu\nu}$ relate the excess stress tensor to the symmetric part $\Sigma_{\mu\nu}$ of the tensor which represents rate of strain relative to the rest frame. Every index of the viscosity tensor is a space index; all time components vanish in the rest frame. Substitution of Eq. (79) into Eq. (58′) gives

$$-S_\sigma\partial_\sigma T+\eta_{\sigma\tau\mu\nu}\Sigma_{\sigma\tau}\Sigma_{\mu\nu}\geq 0, \quad (81)$$

where $\frac{1}{2}\eta_{\sigma\tau\mu\nu}\Sigma_{\sigma\tau}\Sigma_{\mu\nu}$ is the Rayleigh dissipation rate for an anisotropic medium. In the case of a transport process consisting only of heat conduction this viscous dissipation must vanish, since as we have seen, the entire dissipation in this case is $-S_\sigma\partial_\sigma T\geq 0$. Accordingly, in this case $\Sigma_{\mu\nu}$ must vanish, and $\phi_{\sigma\tau}=\phi_{\sigma\tau}{}^r$ everywhere. Because of the symmetry of the stress and rate-of-strain tensors in Eqs. (79) and (81), only 21 independent viscosity coefficients exist in the general anisotropic case. In the isotropic case, in which the excess stress and the rate-of-strain tensors can be diagonalized simultaneously, the number is reduced to 2, according to the usual arguments; here $\phi_{\sigma\tau}{}^r=-p\bar{\delta}_{\sigma\tau}$, where p is the hydrostatic pressure.

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The Tamm-Dancoff Formalism and the Symmetric Pseudoscalar Theory of Nuclear Forces

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The general method of deducing the Tamm-Dancoff equal-times formalism, as generalized by Lévy, from the relativistic two-body equation of Bethe-Salpeter and Schwinger is given. Only processes which are finite *ab initio* are considered. The essence of the procedure is the relation between a set of conventional matrix elements of the Tamm-Dancoff formalism and the Feynman diagram which summarizes them; this relationship provides a convenient guide for enumerating all matrix elements of a specified type and precludes the possibility of omission of any members of the set. Rules are also given for writing down any matrix element. The method is then applied to the derivation of the fourth-, sixth-, and eighth-order adiabatic potentials on the symmetrical pseudoscalar-pseudoscalar theory. Some discrepancies with the results of Lévy are noted: In connection with the fourth-order potential these are first, that a more careful treatment of the energy denominators of the leading two-pair terms brings to light contributions that cancel with all other two-pair matrix elements that are of relative order μ/M compared to the leading ones; second, that the one-pair terms do not vanish but yield a repulsive interaction which substantially alters the qualitative picture of the fourth-order potential; third, that for the no-pair terms the result should agree with the previously calculated fourth-order potential for the pseudoscalar-pseudovector theory. The sixth- and eighth-order results are also in disagreement with Lévy. Finally, an analysis of the problem of many-particle forces is given and explicit results obtained for the leading terms of the three- and four-particle forces as well as for certain smaller contributions to the three-particle interaction.

I. INTRODUCTION

In a pair of extremely interesting papers recently published Lévy[1] has derived a three-dimensional equation for the relative motion of two particles with an interaction kernel that, in principle, can be computed to any order in the coupling constant; he has used his formalism for the most thorough examination of the nuclear forces predicted by weak coupling theory so far attempted and from the results has give a plausible account of the low energy properties of the deuteron.

Lévy's approach is a hybrid one. It consists, first of all, in an extension of the Fock space method of Tamm[2] and Dancoff[3] to include higher order processes involving multiple meson exchange and pair creation, with the proviso, however, that all infinite matrix elements associated with "radiative" corrections be omitted. It is then possible to eliminate all amplitudes except that for the two bound nucleons and to obtain an equation for the latter, which is interpreted as the wave function of the two-particle system in momentum space. To incorporate radiative corrections Lévy turns to the relativistic two-body equation[4–6] (henceforth called R.E.). He shows that by an appropriate iteration suggested by the solution for an instantaneous interaction the finite terms of the R.E. can be placed in a one-to-one correspondence with those of the T.D. (Tamm-Dancoff) formalism. It is then possible to carry out all required renormalizations *before* the reduction to equal times for the two particles is effected, and the finite residues can be incorporated into the three-dimensional interaction kernel.

The present work, begun after the author's reading of L1, was motivated by the belief that the demonstration given there of the equivalence between the T.D. formalism and the appropriately reduced R.E., though undoubtedly concerned with a true result, lacked cogency in certain details and completeness. It was felt, moreover, that since the R.E. was required for the

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