tude λ, we have from Eq. 3, reference 5, $I_v \propto I$, where I is the counting rate of the omnidirectional detector.

If we separate $S_Z(E, x)$ to form the product of two functions, one dependent on Z, the other dependent on E and x, then

$$S_Z(E, x) = k_Z S(E, x), \quad (2)$$

where $k_Z \approx A$, the atomic weight of the primary particle.[5] Accordingly,

$$I_v = (\lambda, x, t) = \sum_Z k_Z \int_{E_Z(\lambda,t)}^{\infty} S(E, x) j_Z(E, t) dE. \quad (3)$$

Fonger[8] has fitted the function $S(E, x)$ to the experimental data shown in Fig. 5 for the omnidirectional detector at 0° and above 40° for $x = 680$ g-cm^{-2}. The primary differential spectra are obtained from Kaplon *et al.*[9] The results are as follows:

$$S(E, 680) = \begin{cases} 0 & \text{for } E < E_0, \\ 9.64 \ln[(1+E)/(1+E_0)] & \text{for } E > E_0, \end{cases} \quad (4)$$

where E_0 is 0.83 Bev. The computed detector counting rate was normalized to unity at $\lambda = 0°$.

This function will generate an intensity *vs* latitude curve which falls below the experimental curve in Fig. 5 at intermediate latitudes since we do not take account of the longitude effect.

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[8] W. H. Fonger, thesis, University of Chicago, 1953 (unpublished).

[9] Kaplon, Peters, Reynolds, and Ritson, Phys. Rev. 85, 295 (1952).

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A Note on Meson-Nucleon Scattering

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A relativistic meson-nucleon two-body equation applicable to the elastic nonexchange scattering of negative pions by protons is solved using the lowest order interaction kernel. The scattering matrix which this equation yields is shown to be unitary. The total cross section calculated from this scattering matrix is finite at threshold and relatively independent of the coupling constant. A plot of the cross section as a function of energy is included.

THE relativistic two-body equation[1,2] has proved useful[3] in discussing the behavior of two-nucleon systems. With this equation one may attempt to get approximate solutions by using terms from the expansion of the interaction operator in powers of the coupling constant, but not assuming such an expansion for the meson-nucleon Green's function itself.[4] The three-dimensional approximation to the resulting equation is equivalent to the configuration space or generalized Tamm-Dancoff method.[3] The latter has the disadvantage that self-energy terms cannot be readily recognized and removed. In the case of negative pion proton elastic nonexchange scattering that part of the lowest order kernel which leads to divergences is so simple that with this term alone we may solve the four-dimensional integral equation directly. Renormalization may then be carried out in the usual manner. The scattering matrix for the process calculated from this Green's function can be shown to be unitary. For meson energies above a few hundred Mev the calculated cross section is almost independent of the coupling constant; even at low energies the *s* wave scattering is insensitive to the choice of coupling constant in the usual range of values.

We begin with the relativistic meson-nucleon two-body equation. Using the notation of the previous papers,[2,4,5] this may be written

$$[(\gamma p + M)(k^2 + \mu^2 + \Pi) - I_{\mathrm{MN}}]G_{\mathrm{MN}} = 1. \quad (1)$$

We retain only that part of the interaction operator which gives rise to the above-mentioned divergence difficulty,

$$(x\xi | I_{\mathrm{MN}} | y\eta) = -ig^2\gamma_5\tau^\xi G_+^{(0)}(x-x')\gamma_5\tau^\eta \times \delta(x-\xi)\delta(y-\eta). \quad (2)$$

* National Science Foundation Predoctoral Fellow.

[1] H. A. Bethe and E. E. Salpeter, Phys. Rev. 84, 1232 (1951).

[2] J. Schwinger, Proc. Natl. Acad. Sci. U. S. 37, 452, 455 (1951).

[3] M. Levy, Phys. Rev. 88, 441 (1952); A. Klein, Phys. Rev. 90, 1101 (1953).

[4] S. Deser and P. Martin, preceding paper [Phys. Rev. 90, 1075 (1953)].

[5] S. F. Edwards, Phys. Rev. 90, 284 (1953).

The superscript 0 on the nucleon Green's function is used to indicate that it is the Green's function for a noninteracting field. In the absence of an external field, we may replace the inherently one-body radiative corrections M and Π by m and 0, so that our equation becomes, in the integral form appropriate to scattering:

$$G_{\text{MN}}=G_+^{(0)}\Delta_+^{(0)}+G_+^{(0)}\Delta_+^{(0)}I_{\text{MN}}G_{\text{MN}}, \tag{3a}$$

$$G_{\text{MN}}(x,y;\xi_i,\eta_-)=G_+^{(0)}(x,y)\Delta_+^{(0)}(\xi_i,\eta_-)$$
$$-ig^2\int G_+^{(0)}(x,\xi')\Delta_+^{(0)}(\xi_i,\xi_k')\gamma_5\tau_kG_+^{(0)}(\xi',\xi'')\gamma_5\tau_j$$
$$\times G_{\text{MN}}(\xi'',y;\xi_j''\eta_-)d\xi'd\xi'', \tag{3b}$$

where we have written the isotopic spin indices, 1, 2, 3, + or −, explicitly as subscripts on the arguments of the $\Delta_+^{(0)}$ function.

The solution to this equation may be found formally. It is

$$G_{\text{MN}}=G_+^{(0)}\Delta_+^{(0)}-ig^2G_+^{(0)}\Delta_+^{(0)}\gamma_5\tau G_+^{(1)}\gamma_5\tau G_+^{(0)}\Delta_+^{(0)}, \tag{4a}$$

or, with indices,

$$G_{\text{MN}}(x,y;\xi_i,\eta_-)=G_+^{(0)}(x,y)\Delta_+^{(0)}(\xi_i,\eta_-)$$
$$-ig^2\int d\xi'd\xi''G_+^{(0)}(x,\xi')\Delta_+^{(0)}(\xi_i,\xi_j')$$
$$\times\gamma_5\tau_jG_+^{(1)}(\xi',\xi'')\gamma_5\tau_k$$
$$\times G_+^{(0)}(\xi'',y)\Delta_+^{(0)}(\xi_k'',\eta_-). \tag{4b}$$

$G_+^{(1)}$ is the solution of the one-body equation

$$(\gamma p+m+ig^2T\rho[\gamma_5\tau G_+^{(0)}\gamma_5\tau\Delta_+^{(0)}])G_+^{(1)}=1. \tag{5}$$

It is the nucleon Green's function obtained by using the first approximation to the mass operator. In momentum space, operation with $\gamma p+M^{(1)}$ becomes merely multiplication by a function of p, whence the inverse operator, $G_+^{(1)}$, may be determined by taking Fourier transforms. We obtain[6]

$$G_+^{(1)}(x,x')=\frac{1}{(2\pi)^4}\int d^4pe^{ip(x-x')}G^{(1)}(p), \tag{6a}$$

$$G^{(1)}(p)=\left[\gamma p+m+\frac{ig^2}{(2\pi)^4}\int\gamma_5\tau_jG_+^{(0)}(p-k)\times\gamma_5\tau_j\Delta_+^{(0)}(k)d^4k\right]^{-1}. \tag{6b}$$

The integral in $G_+^{(1)}(p)$ may be evaluated by procedures identical with those used in the corresponding electrodynamic calculation[7] with trivial modifications due to the use of a $\gamma_5\tau$-vertex operator instead of a γ_μ. Incorporating the renormalization terms appropriately, we have

$$\delta M=\frac{3g^2}{16\pi^2}\int_0^1du\bigg\{[(u-1)\gamma p-m]\times\ln\left[\frac{(1-u)\mu^2+um^2+u(1-u)p^2}{(1-u)\mu^2+u^2m^2}\right]+(\gamma p+m)\frac{2m^2u^2(1-u)}{m^2u^2+(1-u)\mu^2}\bigg\} \tag{7}$$

where

$$\delta M(p)=M^{(1)}(p)-m. \tag{7'}$$

Since G_{MN} is the vacuum expectation value of the ordered product[4] $-[\psi(x)\bar{\psi}(y)\varphi(\xi)\varphi(\eta)]_+\epsilon(x-y)$, the **S** matrix elements are obtained by integrating G_{MN} between eigenvalues of the momenta conjugate to ϕ, $\bar{\psi}$, and ψ, namely, $\chi=n_u\partial_u\phi$, $-in_u\gamma_u\psi=n_u\partial_u\bar{\psi}$, and $i\bar{\psi}n_u\gamma_u=n_u\partial_u\psi$. The **S** matrix element between eigenfunctions $\psi_i'(y)$ and $\chi_i'(\eta)$ on a space-like surface prior to the interaction and $\bar{\psi}_f'(x)$ and $\chi_f'(\xi)$ on such a surface subsequent to the interaction is given by the expression[8]

$$(\bar{\psi}_f'\chi_f'|\mathbf{S}|\psi_i'\chi_i')=-\int_{\sigma_{-\infty}}d\sigma_y\int_{\sigma_{-\infty}}d\sigma_\eta\int_{\sigma_\infty}d\sigma_x\int_{\sigma_\infty}d\sigma_\xi\,\bar{\psi}_f'(x)\chi_f'(\xi)\times\gamma_0G_{\text{MN}}(x,y;\xi,\eta)\gamma_0\chi_i'(\eta)\psi_i'(y). \tag{8}$$

In particular, if at large times we assume the fields to be adiabatically decoupled, the eigenvalues of the momenta may be taken to be plane waves satisfying noninteracting field equations, and we may determine matrix elements from the meson state with momentum and charge denoted by $(\mathbf{k},i)$ and the proton state of momentum and spinor index described by $(\mathbf{p},\lambda)$ to the state described by $(\mathbf{k}',i')$ and $(\mathbf{p}',\lambda')$. The **S** matrix element then becomes

$$(\psi_{\lambda'\mathbf{p}'}'\chi_{\mathbf{k}'i'}'|\mathbf{S}-\mathbf{1}|\psi_{\lambda\mathbf{p}}'\chi_{\mathbf{k}i}')=ig^2\int d^4xd^4x'\bar{\psi}_{\lambda'\mathbf{p}'}'(x)\chi_{i'\mathbf{k}'}'(x)\gamma_5\tau_{i'}\times G_+^{(1)}(x-x')\gamma_5\tau_i\chi_{i\mathbf{k}}'(x')\psi_{\lambda\mathbf{p}}'(x'). \tag{9}$$

Expressing all quantities in terms of their Fourier transforms and performing the usual integrations, we get

$$(p'k'|\mathbf{S}-\mathbf{1}|pk)=C_{\mathbf{pk}}C_{\mathbf{p}'\mathbf{k}'}(2\pi)^4\delta(p'+k'-p-k)\times ig^2\bar{u}_{\lambda'\mathbf{p}'}(e_{i'}\cdot\tau)^\dagger\gamma_5G^{(1)}(p+k)\gamma_5(e_i\cdot\tau)u_{\lambda\mathbf{p}}. \tag{10}$$

The quantity $C_{\mathbf{pk}}$ is the normalization factor for a

[6] We are indebted to Dr. S. D. Drell for furnishing us helpful information about K. Brueckner's recent discussion of this Green's function and its bearing on the two nucleon problem.

[7] R. Karplus and N. M. Kroll, Phys. Rev. **77**, 536 (1950).

[8] J. Schwinger, Lectures on quantum field theory, Harvard, 1952 (unpublished).

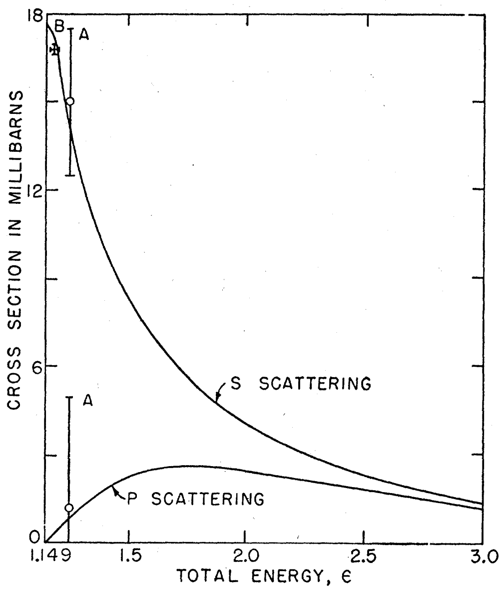

FIG. 1. S and p scattering cross sections as functions of total energy ϵmc^2 in the center-of-mass system. ϵmc^2 is related to T, the kinetic energy of the incident meson in the nucleon rest system, by $T=c^2(2m)^{-1}[m^2\epsilon^2-(m+\mu)^2]$. The experimental data are those of Anderson *et al.* (A) and Barnes *et al.* (B). The experimental error for the latter was not available.

meson-nucleon function and is given by[9]

$$C_{\mathbf{pk}}=\left(\frac{1}{(2\pi)^6}\frac{m}{E(\mathbf{p})}\frac{1}{2\omega(\mathbf{k})}d\mathbf{p}d\mathbf{k}\right)^{\frac{1}{2}}. \tag{11}$$

The meson amplitudes and Dirac spinors have been normalized so that

$$(e^*\cdot e)=1, \quad \text{and} \quad \sum_{\lambda=1}^{4}\bar{u}_{\lambda\mathbf{p}}u_{\lambda\mathbf{p}}=1. \tag{12}$$

Unlike the Born approximation to the $\mathbf{S}$ matrix, the approximate $\mathbf{S}$ matrix here obtained is unitary. We demonstrate this fact directly by proving that the matrix $\mathbf{S}-\mathbf{1}$ satisfies the relation

$$(a|(\mathbf{S}-\mathbf{1})^\dagger(\mathbf{S}-\mathbf{1})|a)=-2\,\mathrm{Re}(a|\mathbf{S}-\mathbf{1}|a). \tag{13}$$

The evaluation of $(a|(\mathbf{S}-\mathbf{1})^\dagger(\mathbf{S}-\mathbf{1})|a)$ by summation over intermediate states is accomplished with the use of the identities

$$\sum_{\lambda=1}^{2}u_{\lambda\mathbf{p}}\bar{u}_{\lambda\mathbf{p}}=\frac{m+\gamma_0E(\mathbf{p})}{2m} \quad \text{and} \quad \sum_{i=1}^{3}(e_i\cdot\tau)^\dagger(e_i\cdot\tau)=3 \tag{14}$$

and integration over $\mathbf{p}'$ and $\mathbf{k}'$ in the frame in which $\mathbf{p}'+\mathbf{k}'=0$ and $p_0'+k_0'=P_0$. If, in this frame, $\mathbf{q}$ is the momentum for which $P_0=E(\mathbf{q})+\omega(\mathbf{q})$ and VT is the magnitude of the space-time region in which the interaction takes place, then the l.h.s. of Eq. (13) becomes

$$C_{\mathbf{pk}}^2\frac{3g^4}{4\pi}VT\bar{u}_{\lambda\mathbf{p}}(e_i\cdot\tau)^\dagger\gamma_5G^{(1)}(P_0)^\dagger\left(\frac{-m+\gamma_0E(\mathbf{q})}{E(\mathbf{q})+\omega(\mathbf{q})}|\mathbf{q}|\right)\times G^{(1)}(P_0)\gamma_5(e_i\cdot\tau)u_{\lambda\mathbf{p}}. \tag{15}$$

$G^{(1)}(P_0)^\dagger$ is the adjoint of the Fourier transform of $G^{(1)}(P_0)$. This is to be compared with $-2\,\mathrm{Re}(a|\mathbf{S}-\mathbf{1}|a)$,

$$-2\,\mathrm{Re}(a|\mathbf{S}-\mathbf{1}|a)=-2VT\,\mathrm{Re}\left[ig^2\bar{u}_{\lambda\mathbf{p}}(e_i\cdot\tau)^\dagger\gamma_5\times\frac{\delta M^*(P_0)}{|-\gamma_0P_0+M(P_0)|^2}\gamma_5(e_i\cdot\tau)u_{\lambda\mathbf{p}}\right], \tag{16}$$

where δM^* is the complex conjugate of δM. Only the imaginary part of the mass operator need be evaluated to demonstrate the desired equality. This imaginary part arises from integration over the logarithm in Eq. (7). The sign of the imaginary part may be determined by explicitly including a negative imaginary addition to the mass. The integral in $\delta M(P_0)$ is evaluated in the appendix. The imaginary part is stated in Eq. (A.3). The substitution of Eq. (A.3) into Eq. (16) makes the latter identical with Eq. (15).

The cross section σ is obtained from (15) by multiplying by the density of final states and dividing by the relative velocities in the center-of-mass system. Since the scattering matrix has been expressed in terms of γ_0 and the unit matrix, there is no mixing of small and large components of the spin wave functions, and σ divides into cross sections for s and p wave scattering. Denoting δM evaluated with $\gamma_0=1$ and -1 by δM_+ and δM_-, respectively, we find

$$\sigma_s=\frac{4}{3}\frac{g^2}{|\mathbf{q}|\epsilon}\frac{[1+E(\mathbf{q})/m](\mathrm{Im}\,\delta M_-)}{(\mathrm{Im}\,\delta M_-/m)^2+(1+\epsilon+\mathrm{Re}\,\delta M_-/m)^2} \tag{17a}$$

and

$$\sigma_p=\frac{4}{3}\frac{g^2}{|\mathbf{q}|\epsilon}\frac{[1-E(\mathbf{q})/m](\mathrm{Im}\,\delta M_+)}{(\mathrm{Im}\,\delta M_+/m)^2+(1-\epsilon+\mathrm{Re}\,\delta M_+/m)^2}, \tag{17b}$$

where $\mathbf{q}$ is defined as in Eq. (15). A factor of 2, arising from the isotopic spin matrices which enter explicitly, and an additional factor of 2/3, required to select only those final states in which a negative meson is present, account for the 4/3 in the expressions above. The dependence of the cross section on the kinetic energy of the incoming meson is shown in Fig. 1. The value of 10 has arbitrarily been selected for the coupling constant $g^2/4\pi$. Two experimental values have been included.[10,11]

Naturally, a calculation based on one leading term of an expansion of the interaction in powers of the coupling

[9] $E(\mathbf{p})=(m^2+\mathbf{p}^2)^{\frac{1}{2}}$ and $\omega(\mathbf{k})=(\mu^2+\mathbf{k}^2)^{\frac{1}{2}}$.

[10] Anderson, Fermi, Nagel, and Yodh, Phys. Rev. **86**, 793 (1952).

[11] Barnes, Roberts, and Tinlot, Summer School Newsletter No. 3 (unpublished).

constant is not much sounder than a similar assumption for the Green's function. However, the fact that the partial **S** matrix here obtained is unitary makes us feel that this covariant calculation presents some progress over the Born approximation applied to this term.

APPENDIX

In this section we evaluate the integral appearing in Eq. (9),

$$\delta M=\frac{3g^2}{16\pi^2}\int_0^1 du\Big\{[(u-1)\gamma P-m]\times\ln\left[\frac{(1-u)\mu^2+um^2+u(1-u)P^2}{(1-u)\mu^2+u^2m^2}\right]+(\gamma P+m)\frac{2m^2u^2(1-u)}{m^2u^2+(1-u)\mu^2}\Big\}. \quad (9)$$

The integration is carried out in the reference frame in which the spatial components of P are zero and the dimensionless quantities $\alpha=\mu/m$ and $\epsilon=P_0/m$ are introduced. The real part of δM is obtained by straightforward integration with absolute value signs placed on the argument of the logarithm:

$$\begin{aligned}\mathrm{Re}\,\delta M=(3g^2/16\pi^2)m\{&-1+2\alpha^2+(1+3\alpha^2-2\alpha^4\\&+(\alpha^2-1)/\epsilon^2)\ln\alpha+\alpha^3(2\alpha^2-7)(4-\alpha^2)^{-\frac{1}{2}}\\&\times\tan^{-1}\alpha(4-\alpha^2)^{\frac{1}{2}}-\tfrac{1}{2}B(\epsilon)\ln F(\epsilon)\\&+\tfrac{1}{2}\gamma_0\epsilon[1-3\alpha^2+(1-\alpha^2)/\epsilon^2-(1+4\alpha^2\\&-3\alpha^4+2\alpha^2/\epsilon^2-(1-\alpha^2)^2/\epsilon^4)\ln\alpha\\&-\alpha^3(3\alpha^2-10)(4-\alpha^2)^{-\frac{1}{2}}\tan^{-1}\alpha(4-\alpha^2)^{-\frac{1}{2}}\\&+(1-(\alpha^2-1)/\epsilon^2)\tfrac{1}{2}B(\epsilon)\ln F(\epsilon)]\}.\end{aligned} \quad (A.1)$$

Here

$$B(\epsilon)=(1-2(1+\alpha^2)/\epsilon^2+(1-\alpha^2)^2/\epsilon^4)^{\frac{1}{2}} \quad (A.1a)$$

FIG. 2. Path of integration for the variable u.

and

$$F(\epsilon)=\frac{1-(1+\alpha^2)/\epsilon^2-B(\epsilon)}{1-(1+\alpha^2)/\epsilon^2+B(\epsilon)}. \quad (A.1b)$$

To obtain the imaginary part of δM, we note that the polynomial $\alpha^2(1-u)+u-\epsilon^2u(1-u)$ which appears in the numerator of the logarithm in Eq. (9) has two roots in the region of integration $0\leqslant u\leqslant 1$,

$$u=(\epsilon^2-1+\alpha^2)/2\epsilon^2\pm\tfrac{1}{2}B(\epsilon). \quad (A.2)$$

and that it is negative between these roots. The values of the logarithm appropriate to the three segments of the interval may be inferred from the path of integration shown in Fig. 2, which follows from the choice of Green's functions for outgoing waves. The imaginary part of the mass operator is therefore

$$\mathrm{Im}\delta M=-(3g^2/16\pi)m\times\{-1+\tfrac{1}{2}\gamma_0\epsilon[1+(1-\alpha^2)/\epsilon]\}B(\epsilon). \quad (A.3)$$

The numerical value of $\alpha^{-1}=M/\mu=6.70$ may be used to simplify the expressions in Eqs. (A.1) and (A.3) to

$$\begin{aligned}\delta M=\frac{3g^2m}{16\pi^2}\{&-2.998+1.862/\epsilon^2-\tfrac{1}{2}B(\epsilon)\ln F(\epsilon)\\&+\gamma_0\epsilon[1.504+0.531/\epsilon^2-0.910/\epsilon^4\\&+\tfrac{1}{4}(1+0.9778/\epsilon^2)B(\epsilon)\ln F(\epsilon)]\\&+i\pi[-1+\tfrac{1}{2}\gamma_0\epsilon(1+0.9778/\epsilon^2)]B(\epsilon)\}.\end{aligned} \quad (A.4)$$

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A Covariant Meson-Nucleon Equation

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A relativistic meson-nucleon two-body equation is derived in a form suitable for carrying out renormalization. Methods for determining the interaction kernel and classifying its terms are discussed. A reduction of the equation to three dimensions is carried out and the approximations involved in this procedure are examined. The resulting equation agrees with a corresponding one derived by Tamm-Dancoff methods.

INTRODUCTION

SEVERAL investigations[1] have been undertaken recently with the aim of improving the Born approximation results for pion-nucleon scattering. It is the purpose of this note to point out that such a boson-fermion system may be described to advantage by means of a covariant two-body equation akin to the one employed in the two-nucleon problem.[2–4] We shall derive such an equation for the pion-nucleon Green's

* National Science Foundation Predoctoral Fellow.
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