Polarization Dependence of the Integrated Bremsstrahlung Cross Section

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The differential cross section for bremsstrahlung derived in the previous paper is integrated over the direction of the emerging electron. It is found that regions of appreciable polarization occur near both the low energy and high energy ends of the photon spectrum. Calculations are performed for the polarization dependence of the radiation for incident electrons of about 0.1, 0.5, 2.5 Mev with and without approximate shielding corrections calculated for aluminum. The polarization dependence of the differential cross section for pair production and of the cross section integrated over either the electron or positron direction is obtained from the results for bremsstrahlung by the well-known procedure of changes from positive to negative energy state.

I. INTRODUCTION

In the preceding paper[1] the differential cross section for bremsstrahlung is given in Eq. (16) as a function of photon polarization for a specified photon momentum $\mathbf{k}$ and for the electron emerging in a direction within the element of solid angle $d\Omega$. In the present paper this cross section is integrated over the direction of the emerging electron. One obtains, in this way, the angular and energy distribution and the polarization of the photons emitted in the bremsstrahlung process.

The considerations in Sec. III of the preceding paper regarding the nature of the photon polarization are not necessary for the integrated cross section. The symmetry with respect to reflections in the $\mathbf{p}_0\mathbf{k}$ plane is sufficient to eliminate the possibility of an elliptically polarized component of the radiation.

The notation used is the same as that of the preceding paper and reference to equations and references in the preceding paper will be made by inserting I- in front of the appropriate equation or reference number.

II. INTEGRATED BREMSSTRAHLUNG CROSS SECTION

The integration of the cross section given in Eq. (I-15) over the direction of the emerging electron is rather lengthy, and only a brief outline of the method will be given. A few intermediate steps are described in Appendix I.

The integrals to be evaluated are of the general form

$$\int q^{-2m}\Delta^{-n}d\Omega;\quad m=0,1,2;\quad n=-1,0,1,2,$$

where $\Delta=E-p\cos\theta$. If one writes

$$q^2=(\mathbf{p}_0-\mathbf{p}-\mathbf{k})^2=(T^2+p^2)(1-\mathbf{p}\cdot\mathbf{a}), \tag{1}$$

with

$$\mathbf{a}=2\mathbf{T}/(T^2+p^2),\quad \mathbf{T}=\mathbf{p}_0-\mathbf{k}, \tag{1.1}$$

and

$$\Delta=E(1-\mathbf{p}\cdot\mathbf{b}), \tag{2}$$

with

$$\mathbf{b}=\mathbf{k}/kE, \tag{2.1}$$

the integrals

$$I_{m,n}=\int(d\Omega/2\pi)(1-\mathbf{p}\cdot\mathbf{a})^{-m}(1-\mathbf{p}\cdot\mathbf{b})^{-n} \tag{3}$$

are nontrivial for $m=1, 2$ and $n=1, 2$. These integrals are discussed further in Appendix I.

It suffices to consider two conditions of polarization referred to below as Cases I and II. Case I corresponds to a sum of cross sections over polarizations. In Case II the polarization vector is taken to lie in the $\mathbf{p}_0\mathbf{k}$ plane and may be expressed as

$$(2kp_0\sin\theta_0)^{-1}[k(T^2+p^2)\mathbf{a}+E(T^2-p^2-2kE)\mathbf{b}].$$

After considerable algebraic manipulation involving the integrals $I_{m,n}$, the photon cross section for each polarization as a function of θ_0, the angle between the photon and incoming electron directions, is found to be

$$\begin{aligned}d\sigma_{\rm I}=&(Z^2e^6/8\pi)(dk/k)(p/p_0)d\Omega_0\\&\times\{8m^2\sin^2\theta_0(2E_0^2+m^2)/(p_0^2\Delta_0^4)\\&-2(5E_0^2+2EE_0+3m^2)/(p_0^2\Delta_0^2)\\&-2(p_0^2-k^2)/(T^2\Delta_0^2)+4E/(p_0^2\Delta_0)\\&+(L/pp_0)[4E_0m^2\sin^2\theta_0(3km^2-p_0^2E)/(p_0^2\Delta_0^4)\\&+(4E_0^2(E_0^2+E^2)\\&-2m^2(7E_0^2-3EE_0+E^2)+2m^4)/(p_0^2\Delta_0^2)\\&+2k(E_0^2+EE_0-m^2)/(p_0^2\Delta_0)]\\&+(\epsilon^T/pT)[4m^2/\Delta_0^2-6k/\Delta_0\\&\quad-2k(p_0^2-k^2)/(T^2\Delta_0)]-4\epsilon/(p\Delta_0)\},\end{aligned} \tag{4.1}$$

$$\begin{aligned}d\sigma_{\rm II}=&(Z^2e^6/8\pi)(dk/k)(p/p_0)d\Omega_0\\&\times\{8m^2\sin^2\theta_0(2E_0^2+m^2)/(p_0^2\Delta_0^4)\\&-(5E_0^2+2EE_0+5m^2)/(p_0^2\Delta_0^2)\\&-(p_0^2-k^2)/(T^2\Delta_0^2)+2(E+E_0)/(p_0^2\Delta_0)\\&+(L/pp_0)[4E_0m^2\sin^2\theta_0(3km^2-p_0^2E)/(p_0^2\Delta_0^4)\\&+(2E_0^2(E_0^2+E^2)\\&-m^2(9E_0^2-4EE_0+E^2)+2m^4)/(p_0^2\Delta_0^2)\\&+k(E_0^2+EE_0)/(p_0^2\Delta_0)]\\&+(\epsilon^T/pT)[4m^2/\Delta_0^2-7k/\Delta_0-k(p_0^2-k^2)/\\&(T^2\Delta_0)-4]-4\epsilon/(p\Delta_0)+(1/p_0^2\sin^2\theta_0)\\&\times[(2L/pp_0)(2E_0^2-EE_0-m^2-(m^2k/\Delta_0))\\&\quad-4\epsilon^T(\Delta_0-E)^2/(pT)-2\epsilon(\Delta_0-E)/p]\},\end{aligned} \tag{4.2}$$

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[1] Gluckstern, Hull, and Breit (preceding paper), Phys. Rev. **90**, 1026 (1953).

$$
\begin{aligned}
d\sigma_{\rm III}=&(Z^2e^6/8\pi)(dk/k)(p/p_0)d\Omega_0\\
&\times\{-(5E_0^2+2EE_0+m^2)/(p_0^2\Delta_0^2)\\
&-(p_0^2-k^2)/(T^2\Delta_0^2)-2k/(p_0^2\Delta_0)\\
&+(L/pp_0)[(2E_0^2(E_0^2+E^2)\\
&-m^2(5E_0^2-2EE_0+E^2))/(p_0^2\Delta_0^2)\\
&+k(E_0^2+EE_0-2m^2)/(p_0^2\Delta_0)]\\
&+(\epsilon^T/pT)[k/\Delta_0-k(p_0^2-k^2)/(T^2\Delta_0)+4]\\
&-(1/p_0^2\sin^2\theta_0)[(2L/pp_0)\\
&\times(2E_0^2-EE_0-m^2-(m^2k/\Delta_0))\\
&\quad\times 4\epsilon^T(\Delta_0-E)^2/(pT)-2\epsilon(\Delta_0-E)/p]\}.\quad (4.3)
\end{aligned}
$$

Here

$$T=|\mathbf{p}_0-\mathbf{k}|,$$

$$L=\ln[(EE_0-m^2+pp_0)/(EE_0-m^2-pp_0)],$$

$$\epsilon=\ln[(E+p)/(E-p)],\quad \epsilon^T=\ln[(T+p)/(T-p)],$$

$$\Delta_0=E_0-p_0\cos\theta_0.$$

The polarization vector in Case III is perpendicular to both $\mathbf{p}_0$ and $\mathbf{k}$ and $d\sigma_{\rm III}=d\sigma_{\rm I}-d\sigma_{\rm II}$. As a check, $d\sigma_{\rm I}$ was integrated over $d\Omega_0$ and agreed with the integrated form of the Bethe-Heitler bremsstrahlung formula.[2]

The limits of validity of Eqs. (4.1), (4.2) and (4.3) are the usual ones made in treatments of the bremsstrahlung process (references [I-3] and [I-4]). Due to the limitations of the first Born approximation the low energy limit of validity is

$$(Ze^2/\hbar v)\ll 1;\quad (\text{kinetic energy}/\text{mc}^2)\gg(Z/137)^2. \quad (5)$$

As previously mentioned the situation may be improved by using more accurate electron wave functions in Eq. (I-3). This has been done by Sommerfeld [I-1] in the nonrelativistic limit and by Maximon and Bethe [I-8] and Bess [I-9] as an improvement on the Bethe-Heitler relativistic formula.

The other limit of validity occurs for low energy photons or for extremely high energy electrons and is due to screening of the (Ze/r) potential by the atomic electrons. The effect of screening may be taken into account roughly by using[3]

$$V(r)=(Ze/r)\exp(-r/a), \quad (6)$$

where $a=108Z^{-\frac{1}{3}}(\hbar/mc)$ is chosen such that in the high energy limit the total cross section agrees with that obtained using the numerical values of the form factor of a Thomas-Fermi atom.[4] The potential in Eq. (6) simulates screening by reducing the Coulomb interaction for distant collisions. The effect is to replace q^{-2} in the Fourier transform of (Ze/r) by $(q^2+\alpha^2)^{-1}$ with $\alpha=(Z^{\frac{1}{3}}m/108)$. This then changes the factor (dk/kq^4) in Eq. (I-15) to $[dk/k(q^2+\alpha^2)^2]$, leaving the remainder of Eq. (I-15) unchanged. The integration over $d\Omega$ is more complex when shielding is included in this way, but may be carried out exactly. However, it seems sufficient to assume that α can be neglected with respect to q except for very low energy photons. The case of electrons of sufficiently high energy ($E_0\gg 137mc^2$) to make shielding important for all photon energies has been treated by May [I-7]. Except for the corrections due to shielding, the present results, Eqs. (4.1), (4.2), (4.3), agree with those of May in the extreme relativistic low angle range. Equation (4.1) is also in agreement with calculations of Schiff[5] who obtained the angular distribution of high energy photons summed over polarizations.

The expressions (4.1), (4.2), and (4.3) are therefore approximately valid as long as the electron energy is not too high ($E_0\ll 137Z^{-\frac{1}{3}}mc^2$) and the photon energy is not too low ($k\sim mc^2$). The modified expression for $k\ll mc^2$ may be obtained[6] by taking the limit $k\to 0$ in Eq. (I-15), retaining the first nonvanishing term. The main contribution to the integral over $d\Omega$ occurs near $\theta=\theta_0$ and $\varphi=\varphi_0$, and all quantities are expanded in powers of $\theta-\theta_0$, $\varphi-\varphi_0$, and k/p_0. *The azimuthal angles φ and φ_0 used here should not be confused with the 4-component Dirac spinors φ and φ_0 used in the previous paper.* One obtains the approximation

$$
\begin{aligned}
q^2\cong(k^2/p_0^2)(E_0-p_0\cos\theta_0)^2+[p_0(\theta-\theta_0)-k\sin\theta_0]^2&\\
+p_0^2\sin^2\theta_0(\varphi-\varphi_0)^2,&\quad (7)
\end{aligned}
$$

and similar expressions for the other quantities in Eq. (I-15). If one makes the substitutions,

$$
\begin{aligned}
&u=\theta-\theta_0-k\sin\theta_0/p_0,\quad w=(\varphi-\varphi_0)\sin\theta_0,\\
&\delta=k\Delta_0/p_0,\quad \Delta_0=E_0-p_0\cos\theta_0,\quad \bar{\Delta}_0=E_0\cos\theta_0-p_0,
\end{aligned}\quad (8)
$$

the integral over $d\Omega$ in Eq. (I-15) assumes the form

$$
\begin{aligned}
\int du\int dw[&u^2\bar{\Delta}_0^2+w^2\Delta_0^2-2u\delta\bar{\Delta}_0m^2\sin\theta_0/p_0E_0\\
&+m^2\sin^2\theta_0\delta^2/E_0^2]/[u^2+w^2+(\alpha^2+\delta^2)/p_0^2]^2\quad (9)
\end{aligned}
$$

for Case I and similar forms for Cases II and III. The integral over u and w in Eq. (9) is logarithmically divergent for large u and w even though no such difficulty existed before the approximations $\theta\approx\theta_0$, $\varphi\approx\varphi_0$ were made. In order to avoid getting an infinite answer, cutoffs of order of magnitude unity are used for u and w corresponding to similar cutoffs for θ and φ. This procedure is not intended as an exact evaluation of the effect of shielding, but rather it is hoped that the main effect will be given sufficiently well by the leading term thus obtained. In this approximation the leading term arises from the terms in the numerator proportional to

[2] See W. Heitler, *The Quantum Theory of Radiation* (Oxford University Press, London, 1949), second edition, p. 165, Eq. (16).

[3] See, for example, H. A. Bethe, Proc. Cambridge Phil. Soc. **30**, 538 (1934).

[4] L. H. Thomas, Proc. Cambridge Phil. Soc. **23**, 542 (1926); E. Fermi, Z. Physik **48**, 73 (1928).

[5] L. I. Schiff, Phys. Rev. **83**, 252 (1951).

[6] No account is taken of multiple photon emission and the related infrared catastrophe.

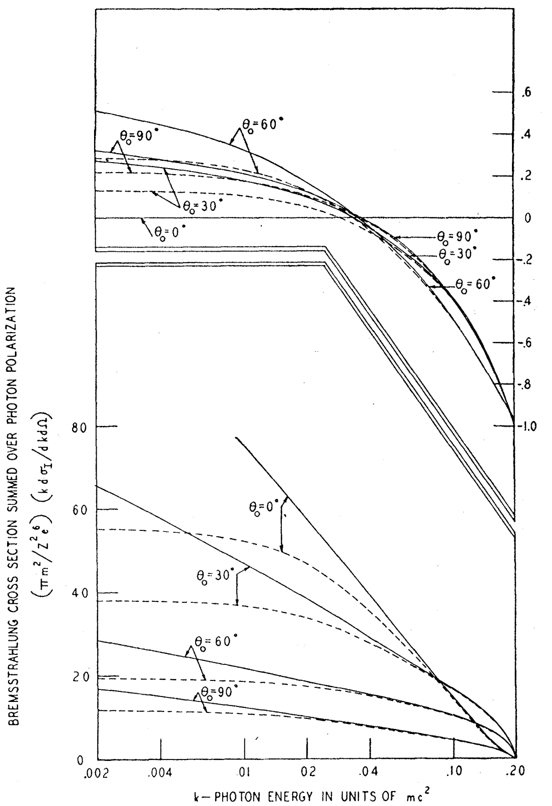

FIG. 1. Bremsstrahlung cross section (lower half of figure) and degree of polarization (upper half of figure) for incident electrons with $E_0=1.2mc^2$ (≈ 0.1 Mev) *vs* photon energy for photon angles $\theta_0=0°, 30°, 60°, 90°$. The solid curves are without shielding, and the dashed curves are with shielding for $Z=13$.

u^2 and w^2 and leads to the following result:

$$(d\sigma_{\rm I})_{k\to 0}=(d\sigma_{\rm II})_{k\to 0}+(d\sigma_{\rm III})_{k\to 0}, \tag{10.1}$$

$$(d\sigma_{\rm II})_{k\to 0}\approx (Z^2e^6/2\pi)(pE_0^2/p_0^3)(dk/k)d\Omega_0\bar{\Delta}_0^2\Delta_0^{-4} \times\{\ln[p_0^2/(\delta^2+\alpha^2)]+O_{\rm II}(1)]\}, \tag{10.2}$$

$$(d\sigma_{\rm III})_{k\to 0}\approx (Z^2e^6/2\pi)(pE_0^2/p_0^3)(dk/k)d\Omega_0\Delta_0^{-2} \times\{\ln[p_0^2/(\delta^2+\alpha^2)]+O_{\rm III}(1)\}, \tag{10.3}$$

where $O_{\rm II}(1)$ and $O_{\rm III}(1)$ stand for terms of order of magnitude unity.

If one instead takes the limit $k\to 0$ in Eqs. (4.1), (4.2), and (4.3), which do not include shielding, the result is

$$(d\sigma_{\rm I})'_{k\to 0}=(d\sigma_{\rm II})'_{k\to 0}+(d\sigma_{\rm III})'_{k\to 0}, \tag{11.1}$$

$$(d\sigma_{\rm II})'_{k\to 0}\approx (Z^2e^6/2\pi)(pE_0^2/p_0^3)(dk/k)d\Omega_0\bar{\Delta}_0^2\Delta_0^{-4} \times\{\ln(p_0^2/\delta^2)+O_{\rm II}(1)\}, \tag{11.2}$$

$$(d\sigma_{\rm III})'_{k\to 0}\approx (Z^2e^6/2\pi)(pE_0^2/p_0^3)(dk/k)d\Omega_0\Delta_0^{-2} \times\{\ln(p_0^2/\delta^2)+O_{\rm III}(1)\}. \tag{11.3}$$

As expected, the leading terms in Eqs. (11.1), (11.2), and (11.3) agree with those in Eqs. (10.1), (10.2), and (10.3) for $\alpha=0$. The main effect of the shielding is to replace $\delta^2=k^2\Delta_0^2p_0^{-2}$ by $\delta^2+\alpha^2=(k^2+\alpha^2p_0^2\Delta_0^{-2})\Delta_0^2p_0^{-2}$ in Eqs. (11.1), (11.2), and (11.3). A rough estimate of the effect of the shielding may therefore be obtained by using Eqs. (4.1), (4.2), (4.3), and replacing k^2 by $k^2+\alpha^2p_0^2\Delta_0^{-2}$ in the logarithmically divergent terms,

$$L=\ln[(EE_0-m^2+pp_0)^2/m^2k^2] \tag{12.1}$$

and

$$2\epsilon^T=2\ln[(T+p)^2/(T^2-p^2)] =\ln[(T+p)^4/(4k^2\Delta_0^2)], \tag{12.2}$$

giving

$$L\to\ln[(EE_0-m^2+pp_0)^2/(m^2k^2+m^2\alpha^2p_0^2\Delta_0^{-2})] \tag{13.1}$$

and

$$2\epsilon^T\to\ln[(T+p)^4/(4k^2\Delta_0^2+4\alpha^2p_0^2)]. \tag{13.2}$$

An exact analysis would most likely show that other changes occur, but the most important ones, namely those which make $k(d\sigma/dk)$ finite as $k\to 0$, have been taken into account by the changes indicated in Eqs. (13.1) and (13.2).

Calculations have been performed to determine the polarization effect for $E_0=1.2\,m$, $2m$, $6m$, corresponding to incident electrons of kinetic energy $mc^2/5$, mc^2, $5mc^2$.

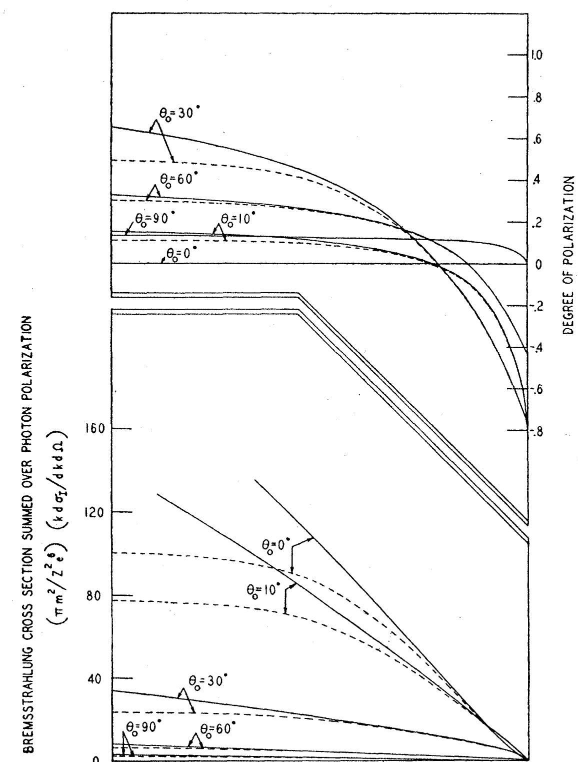

FIG. 2. Bremsstrahlung cross section (lower half of figure) and degree of polarization (upper half of figure) for incident electrons with $E_0=2mc^2$ (≈ 0.5 Mev) *vs* photon energy for photon angles $\theta_0=0°, 10°, 30°, 60°, 90°$. The solid curves are without shielding, and the dashed curves are with shielding for $Z=13$.

The photon spectrum summed over polarization $(\pi m^2/Z^2e^6)(kd\sigma_{\rm I})/(dkd\Omega_0)$ is shown in Figs. 1–3 for each value of E_0, and in each case for various values of θ_0, the angle between the photon and incoming electron directions. In each figure the degree of polarization,

$$P=(d\sigma_{\rm III}-d\sigma_{\rm II})/(d\sigma_{\rm I}), \tag{14}$$

is also plotted against the photon energy k for the same values of θ_0. The dotted curves represent the approximate effects of shielding, calculated for $Z=13$ (aluminum).

The main features of the results are that the polarization effect is appreciable both for low and high energy photons, with P having a different sign in the two cases. At the low energy end of the photon spectrum $d\sigma_{\rm II}<d\sigma_{\rm III}$, and the effect is most pronounced for high electron energies with $\theta_0\sim m/E_0$. This can be seen from Eq. (10.2) where in the low energy photon limit $d\sigma_{\rm II}\rightarrow 0$ for the particular angle given by $\cos\theta_0=p_0/E_0$ or $\sin\theta_0=m/E_0$. This also happens to be the angular region in which the bremsstrahlung cross section is important. As has been pointed out by May and Wick [I-5], the fact that an appreciable polarization effect occurs in the range $\theta_0\sim m/E_0$ may be seen by using the Weizsäcker-Williams method [I-6].

The other region of appreciable polarization occurs at the high end of the photon spectrum, as can be seen in Figs. 1–3, with $d\sigma_{\rm II}>d\sigma_{\rm III}$, but this effect decreases with increasing electron energy. The factor (p/p_0) in Eqs. (4.1), (4.2), and (4.3) causes the cross section to tend to zero as $(k_0-k)^{\frac{1}{2}}$ at the upper end of the spectrum $(E=m,\ p=0,\ k_0=E_0-m)$. In addition the first Born approximation limitation given by Eq. (5) is no longer satisfied as $p\rightarrow 0$. However, these limitations are serious only in the immediate vicinity of the high end of the photon spectrum. Condition (5) for aluminum is equivalent to $Z=13$, $k_0-k\gg 0.005mc^2\approx 2.5$ kev. The factor (p/p_0) which tends to zero at the high energy end of the photon spectrum has the values 0.21, 0.18 and 0.12 for $E_0=1.2m$, $2m$, and $6m$, respectively, for a photon energy 95 percent of the available energy. Therefore an appreciable cross section may be obtained for photon energies up to about 95 percent of the maximum.

The reason for the presence of a polarization effect at the high end of the spectrum may be seen from Eq. (I-16), where for $p\rightarrow 0$ the acceleration of the electron on the classical picture is in the direction of motion of the incident electron. In this nonrelativistic limit $d\sigma_{\rm III}=0$ for the direction of polarization perpendicular to $\mathbf{p}_0$ and $\mathbf{k}$, and the degree of polarization approaches -1. The effect diminishes as the other terms in Eq. (I-15) enter for higher energies.

The limiting forms for $d\sigma_{\rm I}$, $d\sigma_{\rm II}$, and $d\sigma_{\rm III}$ near $p=0$

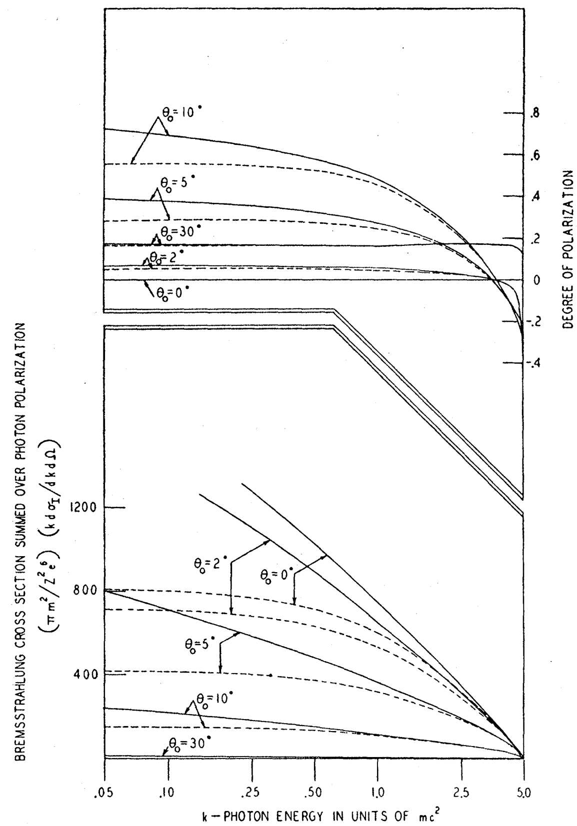

FIG. 3. Bremsstrahlung cross section (lower half of figure) and degree of polarization (upper half of figure) for incident electrons with $E_0=6mc^2$ (≈ 2.5 Mev) *vs* photon energy for photon angles $\theta_0=0°, 2°, 5°, 10°, 30°$. The solid curves are without shielding, and the dashed curves are with shielding for $Z=13$.

may be easily obtained from Eq. (I-15) and are

$$(d\sigma_{\rm I})_{p\rightarrow 0}=(d\sigma_{\rm II})_{p\rightarrow 0}+(d\sigma_{\rm III})_{p\rightarrow 0}, \tag{15.1}$$

$$(d\sigma_{\rm II})_{p\rightarrow 0}\approx(Z^2e^6/4\pi)(p/p_0)(dk/k)d\Omega_0 \times(p_0^2\sin^2\theta_0/mk^2\Delta_0^4)(k^2\Delta_0+4m^3-2mk\Delta_0), \tag{15.2}$$

$$(d\sigma_{\rm III})_{p\rightarrow 0}\approx(Z^2e^6/4\pi)(p/p_0)(dk/k)d\Omega_0 \times(p_0^2\sin^2\theta_0/mk^2\Delta_0^4)(k^2\Delta_0), \tag{15.3}$$

$$P=(mk\Delta_0-2m^3)/(k^2\Delta_0-mk\Delta_0+2m^3). \tag{15.4}$$

For small values of θ_0, the degree of polarization, in the limit $p\rightarrow 0$, is -0.99, -0.87, and -0.43 for $E_0=1.2m$, $2m$ and $6m$, respectively.

The results including the shielding calculated from the corrections indicated in Eqs. (13.1) and (13.2), are shown as the dotted curves in the figures. The effect on the cross sections is to flatten out $k(d\sigma/dk)$ near $k=0$, and the effect on the degree of polarization is to decrease the values calculated without shielding at the low energy end of the photon spectrum.

The formulas given in Eqs. (4.1), (4.2), and (4.3) with the changes indicated in Eqs. (13.1) and (13.2) give the bremsstrahlung cross section with approximate corrections for shielding by the atomic electrons. In practice one would also like to know the effect of multiple electron scattering on the bremsstrahlung angular distribution in order to be able to predict the cross section for targets of finite thickness. This has been done approximately by Schiff[5] and May [I-7]. The experimental measurements of Lanzl and Hanson[7] and Rosengren[8] indicate that the effects of multiple electron scattering should not be neglected for practical target thicknesses.

III. CROSS SECTION FOR PAIR PRODUCTION

As has been pointed out by several authors [I-3], [I-4], pair production differs from bremsstrahlung only in that the energy of the initial state of the electron is negative. The cross section for pair production may be obtained from that of bremsstrahlung by the changes:

$E_0 \to E_-$, the total energy of the electron,

$-E \to E_+$, the total energy of the positron,

$\mathbf{p}_0 \to \mathbf{p}_-$, the momentum of the electron,

$-\mathbf{p} \to \mathbf{p}_+$, the momentum of the positron.

The final state of the system is now an electron-positron pair instead of a photon and an electron, and the factor describing the density of final states must be changed accordingly. The differential cross section for the production of an electron-positron pair of energy E_-, E_+ and momenta $\mathbf{p}_-$, $\mathbf{p}_+$ by a photon of momentum $\mathbf{k}$ may then be obtained from Eq. (I-15) as

$$\begin{aligned} d\sigma = &(Z^2e^6/4\pi^2)(p_+p_-dE_-/k^3q^4)d\Omega_+d\Omega_- \\ &\times\{(q^2-4E_-^2)(p_{+t}^2/\Delta_+^2)+(q^2-4E_+^2)(p_{-t}^2/\Delta_-^2) \\ &-2(q^2+4E_+E_-)(p_{+t}p_{-t}/\Delta_+\Delta_-) \\ &+k^2[(q^2/\Delta_+\Delta_-)-\Delta_+/\Delta_--\Delta_-/\Delta_+-2]\}, \end{aligned} \quad (16)$$

where

$$\mathbf{q}=\mathbf{p}_-+\mathbf{p}_+-\mathbf{k}, \quad \Delta_-=E_--p_-\cos\theta_-, \qquad \Delta_+=E_+-p_+\cos\theta_+, \quad (16.1)$$

and θ_- and θ_+ are the angles of $\mathbf{p}_-$ and $\mathbf{p}_+$ measured with respect to $\mathbf{k}$.

This expression was obtained by Berlin and Madanky[9] and was discussed by Wick[10] in the high energy limit by means of the Weizsäcker-Williams virtual photon method [I-6]. As in the case of bremsstrahlung, Wick[10] and May [I-7] showed that for high energies, an appreciable polarization effect was obtained only at angles of the order $\theta\sim m/k$.

Since the terms in the braces in Eqs. (4.1), (4.2), and (4.3) do not depend on the sign of p, the pair production cross section may be integrated over $d\Omega_+$ by replacing E_0 by E_- and E by $-E_+$ in Eqs. (4.1), (4.2), and (4.3), giving

$$\begin{aligned} d\sigma_{\rm I} = &(Z^2e^6/8\pi)(p_+p_-dE_-/k^3)d\Omega_- \\ &\times\{-4m^2\sin^2\theta_-(2E_-^2+m^2)/(p_-^2\Delta_-^4) \\ &+(5E_-^2-2E_+E_-+3m^2)/(p_-^2\Delta_-^2) \\ &+(p_-^2-k^2)/(T^2\Delta_-^2)+2E_+/(p_-^2\Delta_-) \\ &+(L/p_-p_+)[2E_-m^2\sin^2\theta_-(3km^2+p_-^2E_+)/ \\ &(p_-^2\Delta_-^4)+(2E_-^2(E_-^2+E_+^2) \\ &-m^2(7E_-^2+3E_-E_++E_+^2)+m^4)/(p_-^2\Delta_-^2) \\ &+k(E_-^2-E_-E_+-m^2)/(p_-^2\Delta_-)] \\ &-(\epsilon_+{}^T/p_+T)[2m^2/\Delta_-^2-3k/\Delta_- \\ &-k(p_-^2-k^2)/(T^2\Delta_-)]-2\epsilon_+/(p_+\Delta_-)\}. \end{aligned} \quad (17.1)$$

$$\begin{aligned} d\sigma_{\rm II} = &(Z^2e^6/8\pi)(p_+p_-dE_-/k^3)d\Omega_- \\ &\times\{-8m^2\sin^2\theta_-(2E_-^2+m^2)/(p_-^2\Delta_-^4) \\ &+(5E_-^2-2E_+E_-+5m^2)/(p_-^2\Delta_-^2) \\ &+(p_-^2-k^2)/(T^2\Delta_-^2)-2(E_--E_+)/(p_-^2\Delta_-) \\ &+(L/p_-p_+)[4E_-m^2\sin^2\theta_-(3km^2+p_-^2E_+)/ \\ &(p_-^2\Delta_-^4)+(2E_-^2(E_-^2+E_+^2) \\ &-m^2(9E_-^2+4E_-E_++E_+^2)+2m^4)/(p_-^2\Delta_-^2) \\ &+k(E_-^2-E_-E_+)/(p_-^2\Delta_-)] \\ &-(\epsilon_+{}^T/p_+T)[4m^2/\Delta_-^2-7k/\Delta_- \\ &-k(p_-^2-k^2)/(T^2\Delta_-)-4]-4\epsilon_+/(p_+\Delta_-) \\ &+(1/p_-^2\sin^2\theta_-)[(2L/p_+p_-)(2E_-^2+E_-E_+ \\ &-m^2-(m^2k/\Delta_-)) \\ &+4\epsilon_+{}^T(\Delta_-+E_+)^2/(p_+T)-2\epsilon_+(\Delta_-+E_+)/p_+]\}. \end{aligned} \quad (17.2)$$

$$\begin{aligned} d\sigma_{\rm III} = &(Z^2e^6/8\pi)(p_+p_-dE_-/k^3)d\Omega_- \\ &+\{(5E_-^2-2E_-E_++m^2)/(p_-^2\Delta_-^2) \\ &+(p_-^2-k^2)/(T^2\Delta_-^2)+2k/(p_-^2\Delta_-) \\ &+(L/p_-p_+)[(2E_-^2(E_-^2+E_+^2) \\ &-m^2(5E_-^2+2E_-E_++E_+^2))/(p_-^2\Delta_-^2) \\ &+k(E_-^2-E_-E_+-2m^2)/(p_-^2\Delta_-)] \\ &-(\epsilon_+{}^T/p_+T)][k/\Delta_--k(p_-^2-k^2)/(T^2\Delta_-)+4] \\ &-(1/p_-^2\sin^2\theta_-)[(2L/p_+p_-) \\ &\times(2E_-^2+E_-E_+-m^2-(m^2k/\Delta_-)) \\ &+4\epsilon_+{}^T(\Delta_-+E_+)^2/(p_+T)-2\epsilon_+(\Delta_-+E_+)/p_+]\}. \end{aligned} \quad (17.3)$$

Here

$$T=|\mathbf{p}_--\mathbf{k}|,$$

$$L=\ln[(E_-E_++m^2+p_-p_+)/(E_-E_++m^2-p_-p_+)], \quad (17.4)$$

$$\epsilon_+=\ln[(E_++p_+)/(E_+-p_+)],$$

$$\epsilon_+{}^T=\ln[(T+p_+)/(T-p_+)], \quad \Delta_-=E_--p_-\cos\theta_-.$$

The cross sections in Eqs. (17.1), (17.2), and (17.3) give the energy and angular distribution of the electron

[7] L. H. Lanzl and A. O. Hanson, Phys. Rev. **83**, 959 (1951).
[8] J. W. Rosengren, University of California Radiation Laboratory Report UCRL 1999, Nov. 3, 1952, unpublished.
[9] T. H. Berlin and L. Madanky, Phys. Rev. **78**, 623 (1950).
[10] G. C. Wick, Phys. Rev. **81**, 467 (1951).

for: (I) average over photon polarization, (II) electron created in the plane of polarization, (III) electron emitted perpendicular to the direction of polarization. For pair production $d\sigma_{\text{I}}=(d\sigma_{\text{II}}+d\sigma_{\text{III}})/2$.

Since Eq. (16) is symmetric in the electron-positron pair, Eqs. (17.1), (17.2), and (17.3) hold equally well terms of the angular distribution of the emitted positrons instead of electrons. The approximate expression may be obtained by interchanging $+$ and $-$ signs in Eqs. (17.1), (17.2), and (17.3).

The considerations on the validity of Eqs. (16), (17.1), (17.2), and (17.3) remain essentially unchanged. The expressions are valid for $(Ze^2/\hbar v_+\ll 1$ and $(Ze^2/\hbar v_-)\ll 1$. Shielding becomes important for photon energies of the order of $137Z^{-\frac{1}{3}}mc^2$ and can be taken into account approximately by replacing $(p_+p_-dE_-/k^3q^4)$ in Eq. (16) by $[p_+p_-dE_-/k^3(q^2+\alpha^2)^2]$, as previously discussed. Since the photon energy is now always greater than $2\,mc^2$, the previous considerations for low energy photons are no longer needed. Equations (16), (17.1), (17.2), and (17.3) are therefore approximately valid as long as the photon energy is not too large ($k\ll 137Z^{-\frac{1}{3}}mc^2$). The extreme relativistic range is treated by May [I-7] together with the similar limit for bremsstrahlung as previously mentioned.

APPENDIX I

The integral,

$$I_{m,n}=\int (d\Omega/2\pi)(1-\mathbf{p}\cdot\mathbf{a})^{-m}(1-\mathbf{p}\cdot\mathbf{b})^{-n},$$

in Eq. (3) must be evaluated for the cases $I_{0,1}$, $I_{1,0}$, $I_{0,2}$, $I_{2,0}$, $I_{1,1}$, $I_{1,2}$, $I_{2,1}$, $I_{2,2}$, $I_{2,-1}$. The integral $I_{2,-1}$ may be expressed in terms of the others by choosing $\mathbf{a}$ as the polar axis for the integration over $d\Omega$ and performing the azimuthal integration, giving

$$I_{2,-1}=\int d(\cos\theta_{pa})(1-\mathbf{p}\cdot\mathbf{a})^{-2}\{1-[(\mathbf{p}\cdot\mathbf{a})(\mathbf{a}\cdot\mathbf{b})/a^2]\}$$
$$=\{(\mathbf{a}\cdot\mathbf{b})/a^2\}I_{1,0}+\{1-[(\mathbf{a}\cdot\mathbf{b})/a^2]\}I_{2,0}. \quad (18)$$

The integrals for $m=0$ or $n=0$ may be readily evaluated by choosing $\mathbf{b}$ or $\mathbf{a}$ as the polar axis for the integration over $d\Omega$, giving

$$I_{1,0}=\int_{-1}^{1} d(\cos\theta_{pa})(1-\mathbf{p}\cdot\mathbf{a})^{-1}$$
$$=(pa)^{-1}\ln[(1+pa)/(1-pa)], \quad (19.1)$$

$$I_{2,0}=\int_{-1}^{1} d(\cos\theta_{pa})(1-\mathbf{p}\cdot\mathbf{a})^{-2}=2(1-p^2a^2)^{-1}, \quad (19.2)$$

$$I_{0,1}=(pb)^{-1}\ln[(1+pb)/(1-pb)], \quad (19.3)$$

$$I_{0,2}=2(1-p^2b^2)^{-1}. \quad (19.4)$$

The integrals $I_{1,1}$, $I_{1,2}$, $I_{2,1}$, $I_{2,2}$ may be easily carried out with the aid of the identity

$$(\alpha\beta)^{-1}=\int_0^1 [\alpha x+\beta(1-x)]^{-2}dx, \quad (20)$$

and those obtained by differentiating Eq. (20) with respect to α and β. For example,

$$I_{1,1}=\int (d\Omega/2\pi)(1-\mathbf{p}\cdot\mathbf{a})^{-1}(1-\mathbf{p}\cdot\mathbf{b})^{-1}$$
$$=\int_0^1 dx\int (d\Omega/2\pi)[1-\mathbf{p}\cdot\mathbf{g}]^{-2},$$

where $\mathbf{g}=\mathbf{a}x+\mathbf{b}(1-x)$. If $\mathbf{g}$ is now used as the polar axis, the integration over $d\Omega$ may be performed, giving

$$I_{1,1}=2\int_0^1 X^{-1}dx,$$

where X is a quadratic in x given by

$$X=1-p^2g^2=1-p^2b^2-2xp^2(\mathbf{a}\cdot\mathbf{b}-b^2)-x^2p^2(\mathbf{a}-\mathbf{b})^2.$$

Using $\mathbf{a}=2\mathbf{T}/(T^2+p^2)$, $\mathbf{b}=\mathbf{k}/kE$ from Eqs. (1.1) and (2.1) and $2\mathbf{k}\cdot\mathbf{T}=p_0^2-T^2-k^2$, one obtains

$$I_{1,1}=2\int_0^1 X^{-1}dx=Ek(T^2+p^2)(pp_0)^{-1}(T^2-p^2)^{-1}L, \quad (21.1)$$

where

$$L=\ln[(EE_0-m^2+pp_0)/(EE_0-m^2-pp_0)], \quad (21.2)$$

as defined by Heitler.[11] In a similar way one also obtains

$$I_{1,2}=\int_0^1 4(1-x)X^{-2}dx, \quad (21.3)$$

$$I_{2,1}=\int_0^1 4xX^{-2}dx, \quad (21.4)$$

$$I_{2,2}=\int_0^1 16x(1-x)X^{-3}dx-\int_0^1 4x(1-x)X^{-2}dx. \quad (21.5)$$

These integrals, involving quadratic forms in x, are elementary but lead to long algebraic expressions for $I_{1,2}$, $I_{2,1}$, $I_{2,2}$.

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[11] See reference 2, p. 165, Eq. (16′).