PHYSICAL REVIEW VOLUME 90, NUMBER 6 JUNE 15, 1953

Polarization of Bremsstrahlung Radiation

R. L. GLUCKSTERN, M. H. HULL, JR., AND G. BREIT
Yale University, New Haven, Connecticut*
(Received February 25, 1953)

The dependence of the differential cross section for bremsstrahlung on photon polarization is calculated in the same approximation as the Bethe-Heitler formula for the sum over polarizations. The radiation is found to consist of a mixture of an unpolarized and a linearly polarized component. The relation of the method of intermediate states to the method of transitions between stationary states is explicitly stated.

I. INTRODUCTION

It is well known that x-rays produced when an electron strikes a solid target are polarized. Sommerfeld[1] has treated this process on the basis of nonrelativistic quantum mechanics and showed the correspondence with the classical picture of the radiation being due to the acceleration of a charged particle in the electric field of a nucleus.

The cross section for bremsstrahlung summed over photon polarization has been treated on the Dirac theory of the electron by Heitler[2,3] and by Bethe and Heitler.[4] The cross section for arbitrary photon polarization was obtained by May and Wick[5] by means of the Weizsäcker-Williams method[6] and by May[7] as an extension of the Bethe-Heitler formula for extreme relativistic energies ($\gg 137Z^{-\frac{1}{3}}mc^2$). All of these calculations presuppose the validity of the first Born approximation in the calculation of the effect of the Coulomb field on the electron wave function, namely $Ze^2/\hbar v \ll 1$. Maximon and Bethe[8] and Bess[9] have investigated the change in the Bethe-Heitler formula for large Z using the more exact wave function of Furry[10] and of Sommerfeld and Maue.[11] The exact answer for the first nonvanishing term in the interaction with transverse photons has not been obtained even for the intensity summed over polarizations.

The assumptions involved in the derivation of the differential cross section formula[7,12] for arbitrary photon polarizations are analyzed below in Sec. II. Stationary states of matter are considered as being perturbed by radiation and the first-order effects in $e^2/\hbar c$ are calculated, reducing the problem to the consideration of Einstein's spontaneous emission probability. Comparison with other treatments shows the equivalence of this consideration and that of intermediate states. In Sec. III some symmetry considerations regarding the nature of photon polarization are discussed.

Symbols and Notation

$\mathbf{p}_0$, $\mathbf{p}$ = Initial and final momentum of the electron.
E_0, E = Initial and final total energy of the electron.
$\mathbf{k}$, k = Momentum and energy of the emitted photon.
θ_0, θ = Angle of $\mathbf{p}_0$ and $\mathbf{p}$ with respect to $\mathbf{k}$.
$d\Omega_0$, $d\Omega$ = Element of solid angle in the directions $\mathbf{p}_0$ and $\mathbf{p}$, taken with respect to $\mathbf{k}$.
Ψ_0, Ψ = Dirac wave functions of the initial and final states of the electron in the field of the nucleus.
Φ_0, Φ = Initial and final states of the radiation field.
$\boldsymbol{\alpha}$ = Dirac matrix vector.
α_l = Component of $\boldsymbol{\alpha}$ in the direction of polarization.
a, $a^\dagger$ = Creation and destruction operators for the electron.
$\mathcal{V}$ = Fundamental volume for normalization.
$\mathbf{A}_R$ = Vector potential of the radiation field.
φ_0, φ = Time independent 4-component Dirac spinors for the initial and final electron states.
α, S = Direction of spin and sign of energy.
Z = Nuclear charge.
H = Dirac Hamiltonian ($H = -\boldsymbol{\alpha}\cdot\mathbf{p} - \beta m$).
$\mathbf{q}$ = Momentum transferred to the nucleus ($\mathbf{q} = \mathbf{p}_0 - \mathbf{p} - \mathbf{k}$).
$\epsilon'^2 = -2(kE - \mathbf{k}\cdot\mathbf{p}) = -2k\Delta$.
$\epsilon''^2 = 2(kE_0 - \mathbf{k}\cdot\mathbf{p}_0) = 2k\Delta_0$.
$\Delta_0 = E_0 - p_0 \cos\theta_0$.
$\Delta = E - p \cos\theta$.
γ^μ = Matrix four-vector $(\boldsymbol{\gamma}, \gamma^4)$; $\boldsymbol{\gamma} = -i\beta\boldsymbol{\alpha}$, $\gamma^4 = -\beta$.
γ^l = Component of $\boldsymbol{\gamma}$ in the direction of polarization.
$p_{0\mu}$, p_μ = Momentum energy four vectors $(\mathbf{p}_0, iE_0)$ and $(\mathbf{p}, iE)$ for initial and final electron states.
$\gamma A = \gamma^\mu A_\mu$.

The constants $\hbar$ and c will be taken as 1 throughout the paper except where the meaning may be clearer with the explicit use of $\hbar$ and c.

---

* Assisted by the joint program of the U. S. Office of Naval Research and the U. S. Atomic Energy Commission.
1 A. Sommerfeld, Ann. Physik **11**, 257 (1931).
2 W. Heitler, Z. Physik **84**, 145 (1933).
3 W. Heitler, *The Quantum Theory of Radiation* (Oxford University Press, London, 1949), second edition, p. 161.
4 H. A. Bethe and W. Heitler, Proc. Roy. Soc. (London) **A146**, 83 (1934).
5 M. May and G. C. Wick, Phys. Rev. **81**, 628 (1951).
6 C. F. Weizsäcker, Z. Physik **88**, 612 (1934); E. J. Williams, Kgl. Danske Videnskab. Selskab. Mat.-fys. Medd. **13**, No. 4 (1935); Phys. Rev. **45**, 729 (1934).
7 M. May, Phys. Rev. **84**, 265 (1951).
8 L. C. Maximon and H. A. Bethe, Phys. Rev. **87**, 156 (1952).
9 L. Bess, Phys. Rev. **77**, 550 (1950).
10 W. H. Furry, Phys. Rev. **46**, 391 (1934).
11 A. Sommerfeld and A. W. Maue, Ann. Physik **22**, 629 (1935).
12 Gluckstern, Hull, and Breit, Science **114**, 480 (1951).

II. DIFFERENTIAL CROSS SECTION

The radiation field is treated as a perturbation between states of the electron in the field of the nucleus. Thus if Ψ_0 and Ψ are the Dirac wave functions of the initial and final states of the electron in the field of the nucleus, and if Φ_0 and Φ are the initial and final states of the radiation field, the matrix element for the emission of a photon with momentum $\mathbf{k}$ is given by

$$\mathfrak{M}=\{\Phi\Psi,\, e(\boldsymbol{\alpha}\cdot\mathbf{A}_R)\Phi_0\Psi_0\}, \tag{1}$$

where $\boldsymbol{\alpha}$ is the Dirac matrix vector and $\mathbf{A}_R$ is the vector potential of the radiation field. It is convenient to make use of Heisenberg and Pauli's second form of quantum electrodynamics, employing the choice of gauge enabling the elimination of the electrostatic field and leaving the radiation variables only in the description of transverse waves. This is the form of radiation theory which has proved especially useful in nonrelativistic quantum mechanics. The interaction energy with the radiation field may be written as[13]

$$(\boldsymbol{\alpha}\cdot\mathbf{A}_R(\mathbf{r}))=\textstyle\sum_{\text{states}}\alpha_l(\mathcal{V}k/2\pi)^{-\frac{1}{2}}\\ \times\{a\exp(-i\mathbf{k}\cdot\mathbf{r})+ia^\dagger\exp(i\mathbf{k}\cdot\mathbf{r})\}, \tag{2}$$

where α_l is the component of $\boldsymbol{\alpha}$ in the direction of polarization, and $\mathcal{V}$ is the fundamental volume used for normalization. Only the creation operator a contributes to spontaneous emission, and the matrix element becomes

$$\mathfrak{M}=e(\mathcal{V}k/2\pi)^{-\frac{1}{2}}\{\Psi,\,\alpha_l\exp(-i\mathbf{k}\cdot\mathbf{r})\Psi_0\}. \tag{3}$$

If exact wave functions of the continuous spectrum were used for Ψ_0 and Ψ in Eq. (3), one would have the true cross section except for radiative corrections, the treatment of which is not attempted in the present paper. The success of the Bethe-Heitler formula indicates that the treatment of matter waves by the first Born approximation should be good enough for many purposes, and all of the work below is carried out by means of this simplifying device. One then has

$$\mathcal{V}^{\frac{1}{2}}\Psi_0(\mathbf{r})=\varphi_0\exp(i\mathbf{p}_0\cdot\mathbf{r})-\left(\frac{Ze^2}{2\pi^2\mathcal{V}^{\frac{1}{2}}}\right)\\ \times\int\sum_{\alpha'S'}\frac{\varphi'\exp(i\mathbf{p}'\cdot\mathbf{r})(\varphi',\varphi_0)d\mathbf{p}'}{(E'-E_0)|\mathbf{p}'-\mathbf{p}_0|^2}, \tag{4.1}$$

$$\mathcal{V}^{\frac{1}{2}}\Psi(\mathbf{r})=\varphi\exp(i\mathbf{p}\cdot\mathbf{r})-\left(\frac{Ze^2}{2\pi^2\mathcal{V}^{\frac{1}{2}}}\right)\\ \times\int\sum_{\alpha''S''}\frac{\varphi''\exp(i\mathbf{p}''\cdot\mathbf{r})(\varphi'',\varphi)d\mathbf{p}''}{(E''-E)|\mathbf{p}''-\mathbf{p}|^2}, \tag{4.2}$$

where φ_0, φ, φ', and φ'' are the Dirac 4-vectors for the various electron states, α denotes the spin orientation, and S the sign of the energy in the perturbed states with moments $\mathbf{p}'$ and $\mathbf{p}''$ and energy E' and E''.

If Eqs. (4.1) and (4.2) are inserted into Eq. (3), the lowest order nonvanishing terms are the two linear in Z. The integration over $\mathbf{r}$ in Eq. (3) gives a delta-function for each of these terms, in one case giving $\delta(\mathbf{p}'-\mathbf{p}-\mathbf{k})$ and in the other $\delta(\mathbf{p}''-\mathbf{p}_0+\mathbf{k})$. The $\mathbf{p}'$ and $\mathbf{p}''$ integration then yields an expression of the form

$$\sum_{\alpha\alpha_0}\left|\sum_{\alpha'S'}\frac{(\varphi,\alpha_l\varphi')(\varphi',\varphi_0)}{E'-E_0}+\sum_{\alpha''S''}\frac{(\varphi,\varphi'')(\varphi'',\alpha_l\varphi_0)}{E''-E}\right|^2. \tag{5}$$

The sums over α', α'', S'' may be performed with the aid of the factors $(H'+E)/(E'+E)$ and $(H''+E_0)/(E''+E_0)$ which make the denominators independent of the sign of the energy in the perturbed electron state. Here $H'=-\boldsymbol{\alpha}\cdot\mathbf{p}'-\beta m$ and $H''=-\boldsymbol{\alpha}\cdot\mathbf{p}''-\beta m$. The resulting expression is of the form

$$\sum_{\alpha,\alpha_0}|(\varphi,\,(A+B)\varphi_0)|^2, \tag{6}$$

where

$$A=\alpha_l(H'+E_0)/\epsilon'^2;\quad B=(H''+E)\alpha_l/\epsilon''^2,$$

$$\epsilon'^2=E'^2-E_0^2=(\mathbf{p}+\mathbf{k})^2+m^2-E_0^2=-2(kE-\mathbf{k}\cdot\mathbf{p}), \tag{6.1}$$

$$\epsilon''^2=E''^2-E^2=(\mathbf{p}_0-\mathbf{k})^2+m^2-E^2=2(kE_0-\mathbf{k}\cdot\mathbf{p}_0).$$

The average over the initial spin states and sum over the final spin states is carried out by using the projection operators $(H+E)/2E$ and $(H_0+E_0)/2E_0$ in the usual way. The cross section then becomes

$$d\sigma=(Z^2e^6/8\pi^2)(p/p_0)(kdk/q^4)\\ \times\mathrm{Tr}\{(\tilde{A}^*+\tilde{B}^*)(H+E)(A+B)(H_0+E_0)\}d\Omega d\Omega_0, \tag{7}$$

where the symbols $\tilde{A}^*$ and $\tilde{B}^*$ represent the Hermitian conjugates of the matrices A and B, and the momentum transferred to the nucleus is given by

$$\mathbf{q}=\mathbf{p}_0-\mathbf{p}-\mathbf{k}. \tag{7.1}$$

Equation (7) may also be reached formally by use of Feynman diagrams. The terms A and B in Eq. (7) correspond to the two diagrams in which the photon emitted "after" and "before" the interaction of the electron with the field of the nucleus, and may be written down directly according to Feynman's prescription.[14] However, the relationship to the nonrelativistic emission probability calculations employing exact nonrelativistic functions is not immediately clear in the method of Feynman diagrams.

The evaluation of the trace in Eq. (7) can lead to an unreasonable amount of work unless it is properly arranged and a few of the intermediate steps will be outlined therefore. It is convenient to introduce the matrices

$$\gamma^\mu=(\boldsymbol{\gamma},\gamma^4),\quad \boldsymbol{\gamma}=-i\beta\boldsymbol{\alpha},\quad \gamma^4=-\beta. \tag{8}$$

[13] The notation used is similar to that of G. Breit, Revs. Modern Phys. 4, 504 (1932).

[14] R. P. Feynman, Phys. Rev. 76, 749, 769 (1949).

The functions $H+E$ and H_0+E_0 may be written in terms of the momentum energy four-vectors as

$$H+E=-i(\gamma p+im)\gamma^4,\quad H_0+E_0=-i(\gamma p_0+im)\gamma^4, \tag{9}$$

where γp stands for $\Sigma_\mu \gamma^\mu p_\mu$ and

$$p_\mu=(\mathbf{p}, iE),\quad p_{0\mu}=(\mathbf{p}_0, iE_0). \tag{9.1}$$

The quantities A, B, $\tilde{A}^*$, and $\tilde{B}^*$ in Eq. (7) may similarly be written as

$$A=\gamma^l\gamma^4(\gamma p+im)\gamma^4/\epsilon'^2,\quad \tilde{A}^*=(\gamma p'+im)\gamma^4\gamma^l\gamma^4/\epsilon'^2,$$
$$B=(\gamma p''+im)\gamma^4\gamma^l\gamma^4/\epsilon''^2,\quad \tilde{B}^*=\gamma^l\gamma^4(\gamma p''+im)\gamma^4/\epsilon''^2, \tag{10}$$

where

$$p_\mu'=p_\mu+k_\mu=p_{0\mu}-q_\mu$$

and

$$p_\mu''=p_{0\mu}-k_\mu=p_\mu+q_\mu, \tag{10.1}$$

with

$$k_\mu=(\mathbf{k}, ik)\quad\text{and}\quad q_\mu=(\mathbf{q}, 0).$$

The γ-matrices obey the usual commutation rule

$$\gamma^\mu\gamma^\nu+\gamma^\nu\gamma^\mu=2\delta_{\mu\nu}, \tag{11}$$

and γ^l, the component of $\boldsymbol{\gamma}$ in the direction of polarization, having only a space component, anticommutes with γ^4.

The terms arising in Eq. (7) separate naturally into those proportional to $(\epsilon')^{-4}$, $(\epsilon'\epsilon'')^{-2}$, and $(\epsilon'')^{-4}$. For example, the term proportional to $(\epsilon')^{-4}$ is

$$-(\epsilon')^{-4}\,\mathrm{Tr}\{(\gamma p_0+im)\gamma^4(\gamma p'+im)\gamma^l \times(\gamma p+im)\gamma^l(\gamma p'+im)\gamma^4\}. \tag{12}$$

In the evaluation of the trace, the commutation rule is conveniently written in the form

$$(\gamma A)(\gamma B)=2\textstyle\sum_\mu A_\mu B_\mu-(\gamma B)(\gamma A). \tag{11.1}$$

Since a trace is invariant under a cyclic permutation of its factors, one can arrange for the second factor $\gamma p'+im$ to occur first, having first removed the free factors γ^4, γ^l by means of Eq. (11.1). Employing next relations such as

$$(im+\gamma p)(im-\gamma p)=-m^2-\textstyle\sum_\mu p_\mu p_\mu=0,$$
$$(im+\gamma p')(im-\gamma p')=-m^2-\textstyle\sum_\mu p_\mu' p_\mu'=-\epsilon'^2, \tag{13}$$
$$\mathrm{Tr}[1]=4,\quad \mathrm{Tr}[(\gamma A)(\gamma B)]=4\textstyle\sum_\mu A_\mu B_\mu,$$

one obtains the term displayed in Eq. (12) after some simple manipulation as

$$8p_l^2(4E_0^2-q^2)(\epsilon')^{-4}+8(p_lq_l-E_0k)(\epsilon')^{-2}+2(\epsilon'')^2(\epsilon')^{-2}. \tag{14.1}$$

In a similar manner the term containing the factor $(\epsilon'')^{-4}$ is found to be the result of changing p to p_0, ϵ' to ϵ'', E to E_0, k to $-k$, q to $-q$ in Eq. (14.1). The part of the trace containing $(\epsilon'\epsilon'')^{-2}$ is similarly found to be

$$16p_lp_0(4EE_0-q^2)(\epsilon'\epsilon'')^{-2}+8(p_{0l}q_l-kE)(\epsilon'')^{-2}-8(p_lq_l-kE_0)(\epsilon')^{-2}-8k^2q^2+4. \tag{14.2}$$

Combining these three contributions, one obtains

$$d\sigma=(Z^2e^6/4\pi^2)(p/p_0)(dk/kq^4)d\Omega d\Omega_0 \times\{(4E_0^2-q^2)(p_l^2/\Delta^2)+(4E^2-q^2)(p_{0l}^2/\Delta_0^2)-2(4EE_0-q^2)(p_lp_{0l}/\Delta\Delta_0)-k^2[\Delta/\Delta_0+\Delta_0/\Delta-2-(q^2/\Delta\Delta_0)]\}, \tag{15}$$

where

$$\epsilon'^2=-2k\Delta,\quad \Delta=E-p\cos\theta,$$
$$\epsilon''^2=2k\Delta_0,\quad \Delta_0=E_0-p_0\cos\theta_0, \tag{15.1}$$

and θ and θ_0 are the angles of $\mathbf{p}$ and $\mathbf{p}_0$ measured with respect to $\mathbf{k}$. This expression was reported previously[12] and was also given by May[7] who then went on to treat the extreme relativistic case. If a sum over polarization is taken, Eq. (15) goes over directly into the Bethe-Heitler differential cross section for bremsstrahlung.

In the nonrelativistic limit Eq. (15) becomes

$$d\sigma_{\mathrm{N.R.}}=(Z^2e^6/\pi^2)(p/p_0)(dk/kq^4)d\Omega d\Omega_0(p_l-p_{0l})^2. \tag{16}$$

Since on the classical picture $\mathbf{p}-\mathbf{p}_0$ is in the direction of the average acceleration of the electron, Eq. (16) states that the radiation has its electric vector parallel to the direction of acceleration, as expected from the preservation of form of classical equations of motion in quantum theory.

III. POLARIZATION FOR FIXED ELECTRON RECOIL DIRECTION

In an adjoining paper Eq. (15) will be used to obtain the intensity of radiation with specified directions of polarization l and photon propagation $\mathbf{k}/k$, integrating $d\sigma$ of Eq. (15) over all directions of the recoil electron momentum $\mathbf{p}$. The present note is being concluded by a few observations concerning the intensity of the elementary process, averaging over spin directions of the electron in the initial and final states but with a specified direction of the electron recoil momentum $\mathbf{p}$.

Introducing an azimuthal angle φ_l for the direction l measured in a plane perpendicular to $\mathbf{k}$ Eq. (15) can be verified to be of the form

$$d\sigma=[\mathcal{A}+\mathcal{B}\cos2\varphi_l+\mathcal{C}\sin2\varphi_l]d\Omega d\Omega_0, \tag{17}$$

i.e.,

$$d\sigma=[\mathcal{A}+\mathcal{D}\cos(2\varphi_l-2\varphi_l^0)]d\Omega d\Omega_0, \tag{17.1}$$

where

$$\mathcal{D}=(\mathcal{B}^2+\mathcal{C}^2)^{\frac{1}{2}},\quad \tan(2\varphi_l^0)=\mathcal{C}/\mathcal{B}. \tag{17.2}$$

Replacing $\cos2\epsilon$ by $2\cos^2\epsilon-1$ in Eq. (17.1), one sees an intensity variation with φ_l of the form

$$\mathcal{A}-\mathcal{D}+2\mathcal{D}\cos^2(\varphi_l-\varphi_l^0), \tag{18}$$

which is such as would be obtained if the radiation consisted of a superposition of unpolarized radiation of relative intensity $\mathfrak{C}-\mathfrak{D}$ and of linearly polarized radiation of relative intensity $2\mathfrak{D}$. The direction of the electric vector of the linearly polarized component is $\varphi_l=\varphi_l^0$. Intensity measurements with a device capable of measuring the intensity having a given direction of linear polarization cannot give more specific information regarding the composition of the radiation than is contained in Eq. (18). As is well known linearly polarized light may be resolved into circularly or elliptically polarized components and it is impossible to claim on the basis of Eq. (17) that the radiation cannot be analyzed in other terms than the unpolarized intensity $\mathfrak{C}-\mathfrak{D}$ and the linearly polarized $2\mathfrak{D}$. Thus, for example, Eq. (18) is consistent with supposing a part or all of $\mathfrak{C}-\mathfrak{D}$ to be circularly polarized. In optics one is able to distinguish between circular and unpolarized radiations by means of a quarter-wave plate while a Nicol prism by itself corresponds only to Eq. (18). A quarter-wave plate between crossed Nicols when rotated through 90° distinguishes between right- and left-handed polarization. It is thus possible to distinguish between $\mathfrak{C}-\mathfrak{D}$ being composed of unpolarized radiation and consisting at least partly of photons having a preferentially right- or left-handed polarization. Since the $\mathbf{p}$, $\mathbf{p}_0$ plane is one of symmetry and since averages of intensity over electron spin directions are taken, one suspects that there can be no net preference for a sense of rotation around $\mathbf{k}$. A proof of the correctness of this surmise will now be given.

Elliptically polarized photons in the general case can be described by means of Eqs. (135), (136), page 103 of the continuation[15] of a previously quoted paper.[13] The propriety of physical identification in terms of elliptic polarizations follows from an examination of absorption or emission effects as on page 102 of the same reference. Changing φ to $(\pi/2)-\varphi$, the right- and left-handed ellipses of Fig. 2 on page 102 are seen to interchange. Calling the radiation variables thus obtained a_1''', a_2''' and comparing them with variables a_1^{iv}, a_2^{iv} obtained by changing i to $-i$ in the formulas expressing the a_s'', $a_s''^\dagger$ in terms of the a_s', $a_s'^\dagger$, one finds

$$a_s^{iv}=ia_s''',\quad (a_s^{iv})^\dagger=-i(a_s''')^\dagger,\quad (s=1,2) \tag{19}$$

which shows that these variables are equivalent, since the factors i, $-i$ correspond only to a contact transformation involving one s at a time. It is thus seen that one can analyze the radiation in terms of elliptic polarizations with reversed directions of rotation simply by reversing the sign of i in the defining formulas for the a''.

The calculation of the intensity of elliptic polarization takes place through the introduction of the variables a_1'', a_2'' with the result that the α_l in Eq. (5) becomes replaced by expressions of the type $\alpha_l\cos\varphi-i\alpha_m\sin\varphi$ with m standing for a direction perpendicular to $\mathbf{k}$ and l. According to the discussion of elliptic polarizations in connection with Eq. (19) the change to the opposite direction of rotation can be made by changing from $\alpha_l\cos\varphi-i\alpha_m\sin\varphi$ to $\alpha_l\cos\varphi+i\alpha_m\sin\varphi$. The intensity difference between the two directions of rotation can be ascertained therefore by substituting $\alpha_l+i\lambda\alpha_m$ for α_l in Eq. (5) and ascertaining the difference caused by a change of sign of λ. It may be shown in several ways that the sign of λ does not matter. One way of doing so is to transform the sum in Eq. (5) by means of projection operators which gives an expression of the form $F/(4E_0E)$, with

$$F=\sum_{\gamma,\gamma_0}|(\varphi')^*(H+E)[\alpha_\lambda(H'+E_0)/\epsilon'^2+(H''+E)\alpha_\lambda/\epsilon''^2](H_0+E_0)\varphi_0|^2, \tag{20}$$

where

$$\gamma=\alpha,S;\quad \gamma_0=\alpha_0,S_0; \tag{20.1}$$

and

$$\alpha_\lambda=\alpha_l+i\lambda\alpha_m. \tag{20.2}$$

On employing completeness relations for φ and φ_0 one obtains an expression for F in the form of a trace of products of four-row matrices. The coefficient of $i\lambda$ is also such an expression and the matrices occurring in the product are Dirac's α_1, α_2, α_3, and β. The trace of a product of any number of such matrices is real or zero. Therefore the coefficient of $i\lambda$ is real or zero. If the coefficient were not zero the quantity F would have a nonvanishing imaginary part which contradicts Eq. (20). Hence the coefficient of λ is zero and a change in the sign of λ does not affect F. By working with Eq. (5) one can show by explicit calculation that a change in the direction of the electron spin in the initial, intermediate, and final states leads to a change in the sense of rotation of the polarization which can be represented by a change in the sign of λ. It is probable therefore that polarized electrons can produce elliptically polarized bremsstrahlung. The lack of ellipticity for unpolarized electrons is the result of compensation of right by left-handed components somewhat as in the Zeeman pattern of optical lines. The part $\mathfrak{C}-\mathfrak{D}$ of Eq. (18) has no net right-handedness. This part of the intensity is thus neutral to all tests for polarization. The whole radiation may be considered therefore as consisting of an unpolarized part with intensity proportional to $\mathfrak{C}-\mathfrak{D}$ and a linearly polarized part with intensity proportional to $2\mathfrak{D}$. The proofs as given refer specifically to the approximations used in obtaining Eq. (5). The symmetry involved is that of the time reversal transformation.[16]

[15] G. Breit, Revs. Modern Phys. 5, 91 (1933).

[16] Unpublished work of L. C. Biedenharn and G. Breit.