THE THEORY OF ELECTROMAGNETIC MASS OF THE PARSON MAGNETON AND OTHER NON-SPHERICAL SYSTEMS.

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INTRODUCTION.

IN the remarkable theory of atomic structure recently proposed by Parson[1] it is assumed that the electron is not spherical, as previously supposed, but a very thin ring, about 1.5×10^{-9} cm. in radius, on which the negative charge revolves at a very high velocity, of the order of that of light. This gives the electron, or magneton, as he calls it, a combination of electrostatic and steady magnetic properties, the steadiness of these magnetic properties being essential to the theory. Their detection by direct experiment with molar magnetic fields was shown by Parson to be very difficult, owing to the fact that the magneton is electrostatically single but magnetically a neutral doublet. Evidence of this sort has, however, been obtained by Grondahl[2] for free electrons within an iron wire, that seems to agree satisfactorily with calculated effects. It might be supposed that evidence would be obtained also from the fact that the inequality of the magnetic energies of a magneton moving along its axis or perpendicular to it would give a tendency for high speed cathode rays to orient themselves in one plane and produce magnetic and other effects. Or one might expect the same influence in a non-rotating monatomic atom such as one of mercury vapor to produce polarization in the light emitted by it. But as Ehrenfest[3] and others have pointed out, such effects would give direct evidence of absolute motion, and therefore are prohibited by the principle of relativity.

Notwithstanding this, the electromagnetic mass and momentum, m_e and M_e, and the magnetic and electrostatic energies T_e and W_e, are very different for motions with the same velocity in different directions in systems not having spherical symmetry, and one may readily prove that $T_e + W_e$ is not constant although $T_e - W_e$ is so. Therefore in such cases we must have some other forms of mass, momentum and energy, due to some other cause than the electromagnetic field. Much as we

[1] "A Magneton Theory of the Structure of the Atom," Smithsonian Miscellaneous Collections, Nov., 1915.

[2] Amer. Phys. Soc., Dec., 1916, PHYS. REV. [2], *9*, 1917.

[3] Ann. d. Phys., *23*, 204–5, 1907.

may hesitate to speculate on what is inside as small a thing as the magneton, there seems to be no other cause for this extra mass than the changes occurring on acceleration in the internal forces that hold the magneton together. This compels us to test the possibilities of various assumptions about them in search of a possible explanation of this "internal mass." One such explanation is that which I outlined in a discussion before the American Physical Society in December, 1916, and more fully in February, 1917, and which is described in an abstract in the Physical Review of 1917. The purpose of the present paper is to give an actual proof of the assertions made there.

The analysis whose results are outlined in this paragraph is designed primarily for the magneton theory; but it is equally necessary for and applicable to any system of static charges and steady currents (that is, if ρ and ρv are independent of t). Let each pair of charge or current elements be imagined connected by a string or rod that will balance their forces on each other when the system is at rest. When the system is accelerated, the non-radiated part of the electromagnetic field from element A upon element B will appear as though A was not where it actually is, but where it would be if it had not been accelerated since the radiation left it which is now arriving at B. If now the internal force on B due to A is governed by this fictitious position of A rather than its real position, but is of the same strength as though the fictitious distance to A were equal to its real distance, then these internal forces will give a mass effect varying with the direction of acceleration in exactly the required way.

This is not the only system of internal forces that will accomplish the result, and for that reason the internal mass may differ from the mass given by this system by any constant amount. A result of this is that the relation between the mass of the classical electron (if it exists) and its radius is not determined so unambiguously as it is supposed to be, and would change by 50 per cent. if a set of forces of this type were substituted for the hydrostatic tension postulated by Lorentz. If the forces in the magneton are of the type suggested here, it appears that the internal forces balancing magnetic actions will give a negative mass always exactly balancing the positive magnetic mass, and that the mass of the magneton is determined solely by its electrostatic properties and corresponding internal forces. It should, however, be remembered that the internal forces are not definitely known, and therefore that the total mass of any electrical system is really uncertain, though the necessity for some internal mass of any non-spherical system is a direct consequence of the principle of relativity.

In any case, there must be an internal mass, m_i, and momentum, M_i, given presumably by the forces required to hold the system in shape against the electrical forces and these must satisfy the equations:

$$m_i = m - m_e, \quad \text{and} \quad M_i = M - M_e. \tag{1}$$

Thus to find the relation between the mass of the magneton and its charge, current, and dimensions, we must investigate these internal forces in detail.

THE MOMENTUM AND ENERGY OF A MOVING ELECTROSTATIC SYSTEM.

Before undertaking the treatment of the more complicated case of a magneton, let us obtain some idea of how all this may happen by considering a simpler hypothetical case of two electrons of the classical type, of charge $+ e$ and radius R held at a distance a which is constant except for Lorentz-Fitzgerald contraction.

We shall denote the electric and magnetic vectors measured in the standard electrostatic and magnetic systems by $\boldsymbol{e}$ and $\boldsymbol{b}$ respectively, these letters being used rather than Lorentz's $\boldsymbol{d}$ and $\boldsymbol{h}$ because their averages over "physically infinitesimal" volumes are the $\boldsymbol{E}$ and $\boldsymbol{B}$ of ordinary electromagnetics, rather than $\boldsymbol{D}$ and $\boldsymbol{H}$. The "time" as measured by the distance, ct, that light has travelled since $t = 0$ will be denoted by l, and the units of momentum and mass will be those required by the use of l, that is, the ordinary ones divided by c and c^2 respectively. The value of $\boldsymbol{v}/c$ for any bit of electricity will be denoted by $\boldsymbol{u}$, and that of the system as a whole by $\boldsymbol{\beta}$. Vectors will be denoted by Clarendon type, and their components when treated as scalars will be in italics.

For the general case of any system we have

$$\left.\begin{aligned} \boldsymbol{M}_e &= \frac{1}{4\pi}\int_\infty \boldsymbol{e} \times \boldsymbol{b}\, d\tau, \\ T_e &= \frac{1}{8\pi}\int_\infty \boldsymbol{b}^2 d\tau, \quad W_e = \frac{1}{8\pi}\int_\infty \boldsymbol{e}^2 d\tau, \end{aligned}\right\} \tag{2}$$

where $\boldsymbol{e} \times \boldsymbol{b}$ denotes the vector product of $\boldsymbol{e}$ and $\boldsymbol{b}$.

The vectors $\boldsymbol{e}$ and $\boldsymbol{b}$ may be found by the equations

$$\left.\begin{aligned} \boldsymbol{e} = -\nabla\varphi - \dot{\boldsymbol{a}}, \quad \boldsymbol{b} = \nabla \times \boldsymbol{a}, \\ \nabla^2\varphi - \ddot{\varphi} = -4\pi\rho, \quad \nabla^2\boldsymbol{a} - \ddot{\boldsymbol{a}} = -4\pi\rho\boldsymbol{u}, \quad \nabla\cdot\boldsymbol{a} + \dot{\varphi} = 0, \end{aligned}\right\} \tag{3}$$

where ∇ is the vector operator $(\boldsymbol{k}_x D_x + \boldsymbol{k}_y D_y + \boldsymbol{k}_z D_z)$, $\boldsymbol{k}_x$, $\boldsymbol{k}_y$ and $\boldsymbol{k}_z$ being unit vectors. Thus $\nabla\varphi$ is the gradient of the scalar potential φ, and $\nabla \times \boldsymbol{a}$ and $\nabla\cdot\boldsymbol{a}$ the curl and divergence, respectively, of the vector potential $\boldsymbol{a}$, the dot between two vectors denoting the scalar product.

$\dot{\varphi}$ denotes $D_l\varphi$. These equations, in a different notation, and many that we shall derive from them, are given in Lorentz's "Theory of Electrons."

For the special case of the two electrons, we may use Lorentz's method of dealing with a static system in motion. Taking x in the direction of $\boldsymbol{\beta}$, we have $\boldsymbol{u} \equiv \boldsymbol{\beta}$, and $\dot{\varphi} = -\beta D_x\varphi$, $\ddot{\varphi} = \beta^2 D_x{}^2\varphi$, etc., so that if we let $k = (1 - \beta^2)^{-1/2}$ and $x' = k(x - \beta l)$ we may put (3) in the form

$$\nabla'^2\varphi = -4\pi\rho, \quad \nabla'^2\boldsymbol{a} = -4\pi\rho\boldsymbol{\beta}, \quad \boldsymbol{a} = \varphi\boldsymbol{\beta}, \tag{4}$$

or by introducing $\rho' = k^{-1}\rho$ and $\varphi' = k^{-1}\varphi$, $\boldsymbol{a}' = k^{-1}\boldsymbol{a}$ we have

$$\nabla'^2\varphi' = 4\pi\rho', \quad \boldsymbol{a}' = \varphi'\boldsymbol{\beta} \tag{5}$$

with the values of ρ' in the x', y, z system the same as ρ at corresponding points in an x, y, z system when everything is stationary. This, of course, is merely a special case of the relativity transformations.

From these equations, by steps given in full by Lorentz, note 14, we may prove

$$W_e = \frac{1}{8\pi}\int_\infty \{k^{-1}(D_{x'}\varphi')^2 + k[(D_y\varphi')^2 + (D_z\varphi')^2]\}d\tau', \tag{6}$$

$$T_e = \frac{1}{8\pi}\beta^2 k\int_\infty \{(D_y\varphi')^2 + (D_z\varphi')^2\}d\tau', \tag{7}$$

$$\left.\begin{aligned} M_{ex} &= \frac{1}{4\pi}\beta k\int_\infty \{(D_y\varphi')^2 + (D_z\varphi')^2\}d\tau', \\ M_{ey} &= \frac{1}{4\pi}\beta\int_\infty D_{x'}\varphi' D_y\varphi' d\tau', \\ M_{ez} &= \frac{1}{4\pi}\beta\int_\infty D_{x'}\varphi' D_z\varphi' d\tau'. \end{aligned}\right\} \tag{8}$$

Thus,

$$\left.\begin{aligned} T_e &= \tfrac{1}{2}\boldsymbol{\beta}\cdot\boldsymbol{M}_e, \\ T_e - W_e &= -k^{-1}W_{eo}, \end{aligned}\right\} \tag{9}$$

where W_{eo} is the electrostatic energy of the system at rest, so that $T_e - W_e$ is the same for all directions. (For the general case one may readily prove from the relativity transformations that

$$T_e - W_e = k^{-1}(T_{eo} - W_{eo}).)$$

To evaluate M_e, we might use equations (8) but shall not do so, as there is a shorter way. For, by equation (2), the result must be the same as if we assume all electric and magnetic energy to be transported as indicated by Poynting's vector $\boldsymbol{e} \times \boldsymbol{b}$ and to have momentum equal, in the units of this paper, to its value as energy times the velocity with which it would have to move to give the required transfer. This statement emphasizes the pragmatic value of this assumption, and is independent of what may or may not be said about actual truth of the assumption. Here we shall limit ourselves to a very slow motion, so that the total magnetic energy, being less than β^2 times the electrostatic, may be neglected.

The assumed flow of energy must be fast enough to accomplish three things: first, it must transport the whole electrostatic energy bodily with the system at a velocity $\boldsymbol{\beta}$; second, since energy is developed in the field at the back of each electron by the internal stresses pulling the electricity along against the retarding electric force, it must be transferred forward around the electron to be given up to the internal stresses again; and third, when the direction of motion is that of a, a similar transfer is necessitated by the force holding them together against their repulsion.

The first part gives $\frac{1}{2}(e^2/R)\boldsymbol{\beta}$ for each electron and $(e^2/a)\boldsymbol{\beta}$ for their mutual energy. For the second part we may notice that as the back of the electron sweeps over any element of volume, the work done by the internal stress equals the product of the volume of the element by the tension stress holding the surface of the electron from expanding. If dS is the element of surface, and x its distance forward of the center the energy given to the electromagnetic field per unit time is

$$\frac{1}{2}\frac{e}{R^2}\cdot e\frac{dS}{4\pi R^2}\cdot\frac{-x}{R}\beta$$

and the distance this must be carried forward by the field is $2x$, making the contribution of this element to $\boldsymbol{M}_e$ equal to $(e^2\boldsymbol{\beta}x^2/4\pi R^5)dS$. Now $(x/R)dS$ is the projection of dS on the y, z plane, and therefore $(2x^2/R)dS$ is the volume in a cylinder parallel to the x axis on this element. Integrating, therefore, we have for the total contribution of this sort to $\boldsymbol{M}_e$, $\frac{1}{6}(e^2/R)\boldsymbol{\beta}$, making the total mass of one electron the same as one obtained by direct integration.

The last term may likewise be found to be $(e^2/a)\beta$ in the case of motion parallel to the line of centers, and zero for motion perpendicular to it.

Thus the total electromagnetic momentum is $\frac{4}{3}(e^2/R)\boldsymbol{\beta}$ for both electrons separately, and $(2e^2/a)\boldsymbol{\beta}$ or $(1e^2/a)\boldsymbol{\beta}$ for the mutual momentum, according to the direction of motion. The total magnetic energy, T_e, being $\frac{1}{2}\boldsymbol{\beta}\cdot\boldsymbol{M}_e$ undergoes similar changes with direction. Notwithstanding this, the system must be free to turn in any position without external forces or gain or loss of velocity. The next problem is therefore to find the source of momentum and energy that makes up for these changes in $\boldsymbol{M}_e$ and T_e.

To determine this, let us examine the mutual electromagnetic mass from another viewpoint. In the case of acceleration along the line of centers there is a retarding action of the ether due to the delay in the changes of the electrostatic force by the time a required for propagation from one electron to the other. This makes a difference between the forces acting on them, equal to

$$2\frac{d}{da}\left(\frac{e^2}{a^2}\right)\cdot\tfrac{1}{2}\dot{\mathbf{u}}a^2 = -\frac{2e^2}{a}\dot{\mathbf{u}},$$

which, acting for the time $\beta/\dot{u}$ required to gain the velocity β, will produce an impulse

$$-\frac{2e^2}{a}\boldsymbol{\beta},$$

which must be overcome by the external forces. The mutual electromagnetic mass is therefore the same in this case, that we found by the other analysis.

For acceleration across the line of centers we have a similar effect due in this case to a change of direction and giving a forward component of force on each one equal to

$$\frac{e^2}{a^2}\cdot\frac{\tfrac{1}{2}\dot{\mathbf{u}}a^2}{a} = \tfrac{1}{2}\frac{e^2}{a}\dot{\mathbf{u}}.$$

Also the electric vector radiated from each one, which was zero in the other case, gives a component on the other electron equal to $-(e^2/a)\dot{\mathbf{u}}$. Thus, by a calculation exactly like that of the other case, we have a mutual electromagnetic mass of $1e^2/a$, rather than $2e^2/a$. The most plausible way to satisfy equations (1) is to account for the constancy of the total mass by some effect of the forces that hold the electrons together.

In rapid steady motion along the line of centers the distance a is contracted to ak^{-1}, while the electrostatic repulsion remains the same as it is at rest; and therefore the other force, of attraction, may plausibly be assumed independent of this distance. Hence in longitudinal acceleration it may be assumed the same for both in spite of the difference of the electrostatic forces, and we may say

$$m_{li} = 0, \quad m = m_{le} = \frac{4}{3}\frac{e^2}{R} + 2\frac{e^2}{a}. \tag{10}$$

Therefore, since

$$m_{te} = \frac{4}{3}\frac{e^2}{R} + \frac{e^2}{a} \tag{11}$$

we must have

$$m_{ti} = \frac{e^2}{a}. \tag{12}$$

This is accounted for if we assume that the internal attraction is propagated with the velocity c in such a way that for each one at a time l it is always directed to the position where the other one would be if it had kept, since the time $l - a$, the velocity it had then. This position differs from the real one by the distance $\frac{1}{2}\dot{\mathbf{u}}a^2$, and since the magnitude of the force is e^2/a^2, this effect would give exactly the required drag, $-(e^2/a)\dot{\mathbf{u}}$, on the system. That means that this force is propagated with the velocity of radiation but without the perpendicular components characteristic of radiation.

While this assumption is of no interest for its own sake, since the case is a purely hypothetical one, it is of interest in showing the type of assumptions that may be made about the internal forces of the magneton or any other system to account for the uniformity of its mass in all directions which is demanded by equation (1) in the name of the principle of relativity and the experiments on cathode and β rays. Moreover it points toward the conclusion that every set of forces which carry energy with them must also give rise to an amount of mass in any direction equivalent to the energy carried by this set of forces when moving in that direction. Although this mass is not explained on an electromagnetic basis, it is correlated with other forces very much as electromagnetic mass is correlated with electromagnetic forces, as a result of the propagation of the forces by the ether with the velocity of light, though the assumed laws of propagation are distinctly different in the two cases.

An interesting consequence of the necessity for this "internal" mass, as we may call it, is a certain indefiniteness in the total mass of a classical electron. For if the internal stress is really a simple hydrostatic tension, as Poincaré and Lorentz have supposed, then there is no internal mass at all. But if the stress is a system of bonds between opposite elements of the charge, like the bond between the two electrons that we have assumed in the hypothetical case, then the internal energy is three times that of the hydrostatic tension and there is along with it an internal mass of $\frac{1}{3}e^2/R$, making the whole mass equal to e^2/R, instead of $\frac{2}{3}$ that amount.

This hypothesis in the case of a single classical electron would be a complication that is unjustified except perhaps by considerations of stability of shape of the electron.[1] Nevertheless the existence of the mass of this type is certainly necessary in cases such as the magneton,

[1] See Lorentz, l.c., Chapter 5, pp. 214–5. Lorentz notes here the probability that the internal stress in the electron may be more than a simple hydraulic tension, which would give a stable volume but not a stable shape.

in which the internal stresses cannot be purely hydrostatic.[1] In such cases therefore we must always recognize the dependence of the total mass on the nature of the internal stresses, and the resulting uncertainty in any inferences from the mass of the system as to the dimensions.

A General Theorem on "Internal" Masses.

Let us consider now the more general case of any system in which the charge densities and current densities at all points are constant, except for the changes required by relativity. This includes the case of the magneton or any atom that is not undergoing any change. In such cases we have, so long as the system is at rest, no displacement currents or time changes of ρ and $\rho\mathbf{u}$.

There are many possible sets of internal forces that will keep such a system in equilibrium when it undergoes the changes demanded by relativity on acquiring a high velocity, and it may be that there are many of them that would give a constant mass with some reasonable laws governing their behavior when the system is accelerated. The purpose of this paper, however, is not to investigate how many there are, or what laws each one would require, but rather in addition to pointing out the constancy of the mass of a system to find some plausible explanation of the non-electromagnetic part.

Therefore let us prove the following theorem: Let the internal forces of the system whose densities of charge and current do not vary with time be two sets of pairs of forces between each pair of volume elements, one of which sets exactly balances the electrostatic forces between these elements when the system is permanently at rest while the other set corresponds in the same way to the magnetic ones; let these forces be independent of the "apparent distances" between the elements; but vary with apparent direction exactly like the electromagnetic forces to which they correspond; when the system is accelerated, let each force of such a pair act at each instant as though the other element had kept its velocity constant since the instant when radiation would have had to start from it to reach the element in question at the present time; then the resultant of all these forces will combine with the electromagnetic field to produce a mass which is the same in all directions when the system is accelerated from rest.

As a corollary to this we may prove that under these conditions the mass due to the magnetic field is always exactly balanced by a negative mass due to the corresponding internal forces.

[1] The impossibility of assuming in the magneton such a distribution of charge and current as to give no internal forces seems to be proved by the excessive thinness of the ring found.

First, to prove that the effects of the electric and magnetic fields are independent we may find the electromagnetic momentum of such a system for an infinitesimal velocity $\boldsymbol{\beta}$, by using the relativity transformations with all terms involving $\boldsymbol{\beta}^2$ neglected. Thus, letting primes denote quantities as measured by an observer moving with the system, and therefore equal to the values the corresponding real quantities would have if the system were at rest, we have

$$\begin{cases} \boldsymbol{e}' = \boldsymbol{e} + \boldsymbol{\beta} \times \boldsymbol{b}, & \boldsymbol{b}' = \boldsymbol{b} - \boldsymbol{\beta} \times \boldsymbol{e}, \\ \boldsymbol{u}' = \dfrac{\boldsymbol{u} - \beta}{1 - \boldsymbol{\beta}\cdot\boldsymbol{u}}, & \rho' = \rho(1 - \boldsymbol{\beta}\cdot\boldsymbol{u}). \end{cases}$$

Since to the moving observer the system appears steady we have

$$\boldsymbol{M}_e' = \frac{1}{4\pi}\int_\infty \boldsymbol{e}' \times \boldsymbol{b}' d\tau' = 0, \tag{13}$$

for any system as symmetrical as the magneton.
But in any case,

$$\begin{aligned} \boldsymbol{M}_e' &= \frac{1}{4\pi}\int_\infty [\boldsymbol{e} \times \boldsymbol{b} + (\boldsymbol{\beta} \times \boldsymbol{b}) \times \boldsymbol{b} - \boldsymbol{e} \times (\boldsymbol{\beta} \times \boldsymbol{e})] d\tau' \\ &= \boldsymbol{M}_e - \frac{\boldsymbol{\beta}}{4\pi}\int_\infty [\boldsymbol{b}_t^2 + \boldsymbol{e}_t^2] d\tau' + \frac{1}{4\pi}\int_\infty [\boldsymbol{k}_{b_t}\beta b_t b_l + \boldsymbol{k}_{e_t}\beta e_t e_l] d\tau', \end{aligned} \tag{14}$$

where the subscripts t and l denote the directions perpendicular and parallel to $\boldsymbol{\beta}$, respectively.

But since $\boldsymbol{M}_e - \boldsymbol{M}_e'$ represents the electromagnetic momentum due to the velocity $\boldsymbol{\beta}$, this equation shows that the steady electric and magnetic fields have no mutual mass.[1] This might be guessed from the fact that they have no mutual energy but does not follow from it without proof.

The electric mass alone is therefore the same as if no currents existed, and the magnetic mass the same as if there were no electrostatic charges.[2]

[1] Such a theorem has been proved previously by Comstock, Phil. Mag. (6), 15, pp. 1–21, Jan., 1908, but the proof given here, based on relativity, seems worthy of attention since it is considerably shorter and more direct and its range of application is different from that of Comstock's theorem.

[2] Equation 14 enables us to evaluate approximately the ratios of the masses of the electrostatic and magnetic fields of the magneton for each of the principal directions. As we shall prove below, the ratio of the radius of the cross section of the ring to that of the ring itself is extremely minute, and most of the energy and momentum of the field are concentrated very closely around the ring. This enables us to use the formulas for a straight wire as a first approximation. In this way one may readily prove that the momentum of the static charge is approximately the velocity of the magneton times its electrostatic energy in the case of motion along the axis, or times 3/2 the electrostatic energy for motion perpendicular to the axis. Similarly the steady current has a momentum approximately equal to the velocity of the system times the magnetic energy when moving along the axis, or 3/2 of this value when moving perpendicularly to it.

To prove the theorem we now have to find first some system of internal forces that will balance all those of electrostatic origin and at the same time obey the other conditions. Then we may do the same for the magnetic forces.

Such a system for the electrostatic forces must give a force of attraction of magnitude

$$\frac{\rho\rho' d\tau d\tau'}{r^2}$$

between any two elements $d\tau$ and $d\tau^1$ whose distance is r when the system is permanently at rest; and the direction of each of them must be toward the position the other element would occupy if its velocity had not changed for a time r before the instant in question. This will put these elements in the condition of the two electrons considered above, and thus make the total direct and indirect electric mass of every pair of elements constant.

To find the magnetic mass, we may treat the system as if every element were electrostatically neutral, but had the same current density that it really has. $\boldsymbol{b}$ may then be found by equation (4), where

$$\boldsymbol{a} = \int_{\infty} \frac{[\boldsymbol{q}]d\tau}{r}, \tag{15}$$

the brackets indicating that the value of $\boldsymbol{q}$, or $\rho\boldsymbol{u}$ (the current density vector), for the element $d\tau$ is a "retarded" one, taken for a time an amount r before the time in question.

When the system is permanently at rest, $\boldsymbol{q}$ does not change with time, and $[\boldsymbol{q}] = \boldsymbol{q}$, so that the retarded values present no complications. But if the system is in motion with a low velocity $\boldsymbol{\beta}$ we must take account of the fact that the distance to the retarded position of each element is not that of the present position, and that the effective volume also is different. By exactly the reasoning given by Lorentz for the case of an electron in motion, we may show that this gives

$$\boldsymbol{a} = \int_{\infty} \frac{\boldsymbol{q}d\tau}{[r(1 - \beta_r)]}, \tag{16}$$

where β_r is the component of $\boldsymbol{\beta}$ in the direction from $d\tau$ to the point where $\boldsymbol{a}$ is to be found.

To find $\boldsymbol{b}$, or $\nabla \times \boldsymbol{a}$, for cases where β_r is small we may remember that if $\boldsymbol{\beta}$ is constant, $[r(1 - \beta_r)] = r +$ (terms involving β^2), so that $\boldsymbol{b}$ is the same as when the system is at rest except for such terms. In other cases we must have $\boldsymbol{b}$ equal to this part plus terms involving $\dot{\boldsymbol{\beta}}$; and since the static part produces no resultant force on the whole system, the part involving $\dot{\boldsymbol{\beta}}$ must be held responsible for the magnetic mass. We shall neglect all terms involving $\boldsymbol{\beta}^2$ and $\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}}$.

To find $d\boldsymbol{b}$, the part of $\boldsymbol{b}$ due to the element $d\tau$, we must compute

$$\nabla \times (d\boldsymbol{a}) = \nabla \times \frac{\boldsymbol{q}d\tau}{[r(1-\beta_r)]}, \tag{17}$$

where $\boldsymbol{q}$ has the proper value for the element $d\tau$ and the symbol ∇ refers to differentiation with respect to the position of the other end of r. Now $[r] = r + [r\beta_r] + \frac{1}{2}[r^2\dot{\beta}_r] = r + r\beta_r - \frac{1}{2}r^2\dot{\beta}_r$, and $[1-\beta_r] = 1 - \beta_r + r\dot{\beta}_r$, so that the denominator of $d\boldsymbol{a}$ is $r(1+\frac{1}{2}r\dot{\beta}_r)$.

To find $\nabla \times (d\boldsymbol{a})$ let us use polar coördinates with the present position of the element as origin and direction of $\boldsymbol{q}$ as the pole, and with φ measured from the plane that includes the directions of $\dot{\boldsymbol{\beta}}$ and $\boldsymbol{q}$. With these coördinates

$$\begin{aligned}\nabla \times (d\boldsymbol{a}) = \boldsymbol{k}_r &\left\{\frac{1}{r\sin\theta}D_\theta(\sin\theta da_\phi) - \frac{1}{r\sin\theta}D_\phi(da_\theta)\right\} \\ &+ \boldsymbol{k}_\theta\left\{\frac{1}{r\sin\theta}D_\phi(da_r) - \frac{1}{r}D_r(rda_\phi)\right\} \\ &+ \boldsymbol{k}_\phi\left\{\frac{1}{r}D_r(rda_\theta) - \frac{1}{r}D_\theta(da_r)\right\},\end{aligned} \tag{18}$$

where

$$d\boldsymbol{a} = \frac{\boldsymbol{q}d\tau}{r(1+\frac{1}{2}r\dot{\beta}_r)} = \frac{\boldsymbol{q}d\tau}{r} - \tfrac{1}{2}\boldsymbol{q}\dot{\beta}_r d\tau. \tag{19}$$

The part $\boldsymbol{q}d\tau/r$ gives the value $\boldsymbol{b}$ would have if $\dot{\boldsymbol{\beta}} = 0$, that is, if the system were unaccelerated. This part is known to give no resultant force on the system, so that the part to be considered is

$$d\boldsymbol{a}' = -\tfrac{1}{2}\boldsymbol{q}\dot{\beta}_r d\tau. \tag{20}$$

Since the direction of $d\boldsymbol{a}'$ is always that of $\boldsymbol{q}$,

$$da_\phi' \equiv 0, \quad da_r' = -\tfrac{1}{2}q\dot{\beta}_r\cos\theta\, d\tau, \quad da_\theta' = +\tfrac{1}{2}q\dot{\beta}_r\sin\theta\, d\tau.$$

Let us now subdivide $d\boldsymbol{a}'$ into two parts: $d\boldsymbol{a}''$, due to $\dot{\boldsymbol{\beta}}''$ the component of $\dot{\boldsymbol{\beta}}$ parallel to $\boldsymbol{q}$; and $d\boldsymbol{a}'''$, due to $\dot{\boldsymbol{\beta}}'''$, the rest of $\dot{\boldsymbol{\beta}}$. Thus we have

$$da_\phi'' = 0. \quad da_r'' = -\tfrac{1}{2}q\dot{\beta}''\cos^2\theta\, d\tau, \quad da_\theta'' = \tfrac{1}{2}q\dot{\beta}''\sin\theta\cos\theta\, d\tau,$$

so that all derivatives of $d\boldsymbol{a}''$ containing da_ϕ or D_ϕ vanish leaving

$$\nabla \times (d\boldsymbol{a}'') = \boldsymbol{k}_\phi\left\{\frac{da_\theta''}{r} + \frac{1}{2r}q\dot{\beta}''d\tau\frac{d\cos^2\theta}{d\theta}\right\}$$

or

$$d\boldsymbol{b}'' = -\tfrac{1}{2}\boldsymbol{k}_\phi\frac{q\dot{\beta}''d\tau}{r}\sin\theta\cos\theta. \tag{21}$$

Since when the system is unaccelerated

$$d\boldsymbol{b} = \boldsymbol{k}_\phi \frac{q d\tau}{r^2} \sin\theta$$

the effect of the introduction of $d\boldsymbol{b}''$ is like that of moving the element $d\tau$ in the direction opposite to that of $\boldsymbol{q}$ by an amount $\frac{1}{2}\dot{\boldsymbol{\beta}}'' r^2$ only inasmuch as this motion decreases θ by $\frac{1}{2}\dot{\beta}'' r \sin\theta$ thus changing $d\boldsymbol{b}$ by an amount

$$\boldsymbol{k}_\phi q d\tau \left\{ -\frac{\cos\theta}{r^2} \cdot \tfrac{1}{2}\dot{\beta}'' r \sin\theta \right\} = d\boldsymbol{b}''.$$

The complete effect of this imagined motion of $d\tau$ would include also a change of r and produce 3 times as large a change in the magnetic field. Thus the force on one element due to another accelerated in its own direction is affected by the acceleration of the system only inasmuch as the effective direction—not distance—from each one to the other has its changes due to acceleration delayed by the time r.

For the r component of $\dot{\beta}'''$ we have

$$\dot{\beta}''' \cos\varphi \sin\theta,$$

so that

$$da_\phi''' = 0, \quad da_r''' = -\tfrac{1}{2}\dot{\beta}''' q \cos\theta \sin\theta \cos\varphi \, d\tau,$$

$$da_\theta''' = +\tfrac{1}{2} q\dot{\beta}''' \sin^2\theta \cos\varphi \, d\tau.$$

Here φ derivatives do not vanish, and we have

$$\begin{aligned} d\boldsymbol{b}''' &= \tfrac{1}{2}\boldsymbol{k}_r \frac{q\dot{\beta}''' d\tau}{r} \sin\theta \sin\varphi + \tfrac{1}{2}\boldsymbol{k}_\theta \frac{q\dot{\beta}''' d\tau}{r} \cos\theta \sin\varphi \\ &\qquad + \tfrac{1}{2}\boldsymbol{k}_\phi \frac{q\dot{\beta}''' d\tau}{r} \cos^2\theta \cos\varphi \qquad (22) \\ &= \tfrac{1}{2} \cdot \frac{q\dot{\beta}''' d\tau}{r} \{\sin\varphi(\boldsymbol{k}_r \sin\theta + \boldsymbol{k}_\theta \cos\theta) + \cos\varphi(\boldsymbol{k}_\phi \cos^2\theta)\}. \end{aligned}$$

Now let us compute the change in the vector $d\boldsymbol{b}$ produced by the changes of direction involved in a displacement of the element $d\tau$ in the direction opposite to $\dot{\beta}'''$, that is, of the observing point in the direction of $\dot{\beta}'''$, by an amount $\frac{1}{2}\dot{\beta}''' r^2$. These changes are of two sorts, first, those of $\sin\theta$, and second those of the direction $\boldsymbol{k}_\phi$.

For the first, we have an increase of θ by an amount $\frac{1}{2}\dot{\beta}''' r \cos\theta \cos\varphi$, which changes $d\boldsymbol{b}$ by exactly $d\boldsymbol{b}_\phi'''$.

For the second, we have a change of the direction of $\boldsymbol{k}_\phi$ by an angle $\frac{1}{2}\dot{\beta}''' r(\sin\varphi/\sin\theta)$, so that this vector is changed by the introduction of a component of this magnitude directed away from the axis of the coördinate system, so that its r component has this magnitude times $\sin\theta$, and its θ component this magnitude times $\cos\theta$. Substituting these components for the $\boldsymbol{k}_\phi$ of the expression for $d\boldsymbol{b}$ we find increments in $d\boldsymbol{b}$ exactly equal to the r and θ components of $d\boldsymbol{b}'''$.

Thus these changes of $d\boldsymbol{b}$ due to the acceleration component perpendicular to q can also be identified with those produced by a delay by the time r in the changes of direction but not of distance from the point at which $d\boldsymbol{b}$ is evaluated to the element $d\tau$. The same statement therefore applies to all magnetic forces between any pair of volume elements.

Thus it appears that the magnetic forces on any element in such a system are always the same as if each other element was situated at the distance it actually is at but in the direction it would have if it had not been accelerated since the last radiation left it that has already arrived at the element in question. This is just the law postulated for the corresponding internal forces. Hence it appears that the part of the mass due to the magnetic field will not only have all dependence on direction of acceleration compensated by the internal forces, but will actually be balanced entirely by them. This completes the proof of the theorem and corollary stated above.[1]

This result is of especial interest as applied to the Parson magneton, since it signifies that it is not unreasonable to assume that the magnetic energy of the magneton can not be deduced in any way from its mass, but that the mass is due entirely to its electrostatic energy and the internal forces associated with it. As one may readily see, if the internal forces of either kind are assumed independent of the distance between the elements under all circumstances, the energy of each force, reckoned from zero distance is equal in absolute value to the mutual electrostatic or magnetic energy, but with a plus sign in the former case and a minus in the latter. Hence in this special case we may apply the conclusion so often drawn, though not proved, from relativity, that the mass of each part of the system is equal, in the units used here, to its energy.

[1] A curious paradox arises from the fact that in the case of motion of a magneton perpendicular to its axis the electromagnetic momentum (from equation 14) per unit length of the ring is twice as great in the part of the ring lying parallel to the line of motion as in the part perpendicular to it, and yet during acceleration the former part is subject to no retarding force from the magnetic field. At first sight this would lead one to suspect an error on one side or the other. But the proof of (14) is very direct, and the other assertion rests simply on the fact that the force exerted by any magnetic field on an element of current is perpendicular to the current.

For the solution of this problem it appears that the retarding force on the part of the ring accelerated perpendicular to itself is much greater for acceleration perpendicular to the axis than along it. While the two parts of the ring could exist independently so far as the laws of electrostatics are concerned, the fact that a steady current is impossible without a closed circuit makes it necessary to consider the whole ring in any complete analysis of the magnetic mass.

Gyroscopic Effects in the Magneton.

The question of the existence or non-existence of gyroscopic effects when the magneton is rotated about an axis lying in its plane is given considerable importance by the experiments of Barnett[1] on the magnetization of a bar of iron by rotation, and of Einstein and de Haas[2] on the converse of this effect. Although both effects are very small and require extremely delicate measurements, these experiments are quantitative as well as qualitative and indicate that the gyroscopic effects of the magnetic particle are of the order of magnitude of those of an electron revolving in an orbit. As Parson[3] has pointed out there are serious difficulties in the w y of the assumption that magnetism is due to an orbital motion of a electron or ring of electrons of the classical type. For aside from the well-known difficulties connected with radiation, there is the fact that non-coplanar orbits in an atom are unstable, and yet absolutely necessary if most elements, especially light ones, are not to be strongly paramagnetic. Hence arises the necessity for an examination of the gyroscopic properties of the magneton.

Such a calculation is reported in a paper which I presented at the meeting of the American Physical Society in February, 1917, and of which an abstract will be found in the Physical Review of 1917. The net result was that the gyroscopic properties of the magneton are exactly those of a classical electron in an orbit having the same magnetic moment, provided that the assumptions of the present paper are correct.

The Explicit Formula for the Mass.

The radius of the cross section of the ring, or "small radius," R', as we shall call it, is extremely minute compared to the radius, R, of the ring itself. This simplifies greatly the calculation of the absolute value of the electrostatic energy, which would otherwise be quite difficult. One reason is that it makes the charge distribute itself practically uniformly around the cross section. Another is that the potential at any point on the center of the cross section will be practically the same as if all the charge were on the center line of the ring except for a length on each side of this point that is so short that it may be considered straight. While the truth of these assumptions is not self-evident, it may be proved in the calculation. For calculation, let the assumed straight piece have a length $2xR$, where x is a very small fraction and let $a = R'/R$. The line density of electricity on the ring is $e/2\pi R$. The potential at a point on the center line of the ring is

[1] Science, *30*, p. 413, 1909; Phys. Rev., *6*, 239–70, Oct., '15.
[2] Deutsch. Phys. Gesell., Verh., 17.8., pp. 152–70, Apr. 30, '15.
[3] L.c.

$$V = \frac{e}{2\pi R}\left[2\int_0^x \frac{Rd\theta}{\sqrt{R^2\theta^2 + R'^2}} + 2\int_x^\pi \frac{Rd\theta}{2R\sin\dfrac{\theta}{2}}\right], \tag{23}$$

where the first integral gives the effect of the straight cylinder (treated as having a surface charge) and the other gives that of the rest of the ring. The justification for our assumptions must be the cancellation of all x's from the result. By direct integration this becomes

$$V = \frac{e}{\pi R}\left[\log\frac{x + \sqrt{x^2 + a^2}}{a} + \log\frac{\tan\dfrac{\pi}{4}}{\tan\dfrac{x}{4}}\right]$$
$$= \frac{e}{\pi R}\log\left[8a^{-1}\frac{1 + \dfrac{1}{4}\dfrac{a^2}{x^2} + \cdots}{1 + \frac{1}{48}x^2 + \cdots}\right].$$

The x terms may now be dropped with no appreciable error, since the fraction a is so inconceivably small that a/x and x can both be negligible at the same time. This leaves

$$V = \frac{e}{\pi R}\log\frac{8R}{R'}.$$

Since the potential is constant within and on the surface of the ring, the electrostatic energy is

$$\tfrac{1}{2}eV = \frac{e^2}{2\pi R}\log\frac{8R}{R'}. \tag{24}$$

(It may be worth noticing that the magnetic energy may be found by Rayleigh's self-inductance formula[1] as

$$\frac{e^2}{2\pi R}\frac{v^2}{c^2}\log\frac{8R}{R'}$$

or v^2/c^2 times the electrostatic energy, as we should expect.)

Thus the mass of the magneton in ordinary units is

$$m = \frac{eV}{c^2} = \frac{e^2}{\pi c^2 R}\log\frac{8R}{R'}. \tag{25}$$

For the small radius R', this gives the formula

$$R' = 8Re^{-\frac{\pi Rmc^2}{e^2}}. \tag{26}$$

Since $e^2/mc^2 = 2.82 \times 10^{-13}$ cm., this makes R' almost incredibly small if R is as large as the value 1.5×10^{-9} cm., assigned by Parson. This,

[1] Roy. Soc. Proc., 32, p. 104, 1881; *A* 86, p. 562, 1912.

however, is very uncertain, so no definite value of R' can be assigned at this time.

SUMMARY.

1. It is shown that the relativity principle requires certain assumptions about the internal forces in the magneton, or any other non-spherical system, producing a sort of mass which, added to the electromagnetic mass, will make the total mass constant.

2. Making these assumptions in the most plausible way, the mass of the magneton is $2/c^2$ times its electrostatic energy, with no reference to its magnetic properties.

3. This result leads to a formula for the mass from which the small radius could be determined if the large radius were accurately known.

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