Multiple Compton Scattering of Low Energy Gamma-Radiation

R. C. O'ROURKE
Electricity Division, Naval Research Laboratory, Washington, D. C.
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The exact problem of multiple Compton scattering in an infinite plane parallel medium is set up for a monochromatic primary spectral distribution normally incident upon a semi-infinite medium for isotropic scattering. It is treated by a different approach to that previously used by the author, the new approach being based in part on Marshak's method of spherical harmonics. Numerical results are presented for the P_1 approximation to this problem. This result is compared with the corresponding solution obtained previously. The present method shows that the broadening is not as severe as the previous method had indicated.

SECTION 1. BASIC EQUATIONS AND METHOD OF SOLUTION

IN a previous paper[1] the author found approximate solutions to several problems of multiple Compton scattering of low energy ($E \ll 0.5$ Mev) gamma-radiation. The present paper contains a different approach to the problem, based in part on Marshak's[2] method of spherical harmonics. None of the approximations of the first paper are made. The problem is to find accurate expressions for the broadening of a primary monochromatic spectral distribution as a function of the distance into the medium and the angle (measured from the direction of the primary beam).

Referring to (I), the appropriate transport equation which must be solved in the case of Compton scattering with no photoelectric absorption and for an infinite plane-parallel scattering medium (semi-infinite or of finite thickness) is the following

$$\mu\frac{\partial}{\partial\tau}I(\tau,\mu,\sigma)+I(\tau,\mu,\sigma)=\frac{3}{16\pi}\int_{-1}^{+1}d\mu'\int_0^{2\pi}d\varphi'(1+\cos^2\Theta)I(\tau,\mu',\sigma'),\quad(1.01)$$

where $\tau=(8/3)N\pi r_0^2\cdot Z$, Z=actual distance into medium in cm, r_0=classical electron radius in cm, N=number of electrons per cm^3, $\mu=\cos\vartheta$; $\mu'=\cos\vartheta'$, $\sigma=(mc/h)\lambda$, σ_0=primary "wavelength" $(mc/h)\lambda_0$, $\sigma'=\sigma-1+\cos\Theta$, and

$$\cos\Theta=\mu\mu'+(1-\mu^2)^{\frac{1}{2}}\cdot(1-\mu'^2)^{\frac{1}{2}}\cos\varphi'.$$

To begin with, we shall carry out, the problem of isotropic scattering in which $\frac{3}{4}(1+\cos^2\Theta)$ is set to unity. This is done because it is one of the purposes of this paper to check the results of (I), where isotropic scattering was also considered. An appendix shows the modifications necessary in order to solve the nonisotropic problem (1.01). The only limitation on the methods to be presented is the usual one; namely, the exact Klein-Nishina differential cross section, which is energy dependent, cannot be handled by the operational methods below.

$$\mu\frac{\partial}{\partial\tau}I(\tau,\mu,\sigma)+I(\tau,\mu,\sigma)=\frac{1}{4\pi}\int_{-1}^{+1}d\mu'\int_0^{2\pi}d\varphi' I(\tau,\mu',\sigma').\quad(1.02)$$

This transport equation will now be solved for the case of a plane parallel semi-infinite scattering medium with a monochromatic beam of gammas incident normal to the face $z=0$. The unscattered component is singular in wavelength and direction, and one can therefore introduce the spectral density $J(\tau,\mu,\sigma)$ of all gammas scattered at least once as follows:

$$I(\tau,\mu,\sigma)=\pi Fe^{-\tau}\delta(1-\mu)\delta(\sigma-\sigma_0)+J(\tau,\mu,\sigma).\quad(1.03)$$

The delta function $\delta(1-\mu)$ is normalized to unity over the whole solid angle. The transport equation for $J(\tau,\mu,\sigma)$ then becomes

$$\mu\frac{\partial}{\partial\tau}J(\tau,\mu,\sigma)+J(\tau,\mu,\sigma)=\frac{F}{4}e^{-\tau}\delta(\sigma-\sigma_0-1+\mu)+\frac{1}{4\pi}\int_{-1}^{+1}d\mu'\int_0^{2\pi}d\varphi' J(\tau,\mu',\sigma'),\quad(1.04)$$

in which the singular factor $(1-\mu)\delta(1-\mu)$ occurred and was considered identically zero, operationally speaking, since it would drop out in what follows below. One now makes a Fourier integral decomposition in wavelength shift $(\sigma-\sigma_0\equiv y)$ where, again, σ_0 is the dimensionless primary wavelength $(mc/h)\lambda_0$.

$$J(\tau,\mu,\sigma)=\frac{1}{2\pi}\int_{-\infty}^{\infty}M(\tau,\mu,\alpha)e^{i\alpha y}d\alpha.\quad(1.05)$$

One must note the important fact that physically the scattered radiation consists only of wavelengths larger than $\lambda_0(\sigma-\sigma_0\geqslant 0)$. This indicates that one should perhaps use a one-sided Laplace transform in wavelength shift which is ideally suited for variables which range from zero to infinity. This is however not practicable since the inversion of such transforms leads to

[1] R. C. O'Rourke, Phys. Rev. **85**, 881 (1952) [referred to hereafter as (I)].
[2] R. Marshak, Phys. Rev. **71**, 443 (1947).

complex integrals which are very difficult to handle. The inversion of Eq. (1.05) is, of course,

$$M(\tau,\mu,\alpha)=\int_{-\infty}^{\infty}J(\tau,\mu,y)e^{-i\alpha y}dy,$$

which is formally an integral over negative values of $\sigma-\sigma_0$. By representing $J(\tau,\mu,\sigma)$ as a Fourier integral one must find the function $M(\tau,\mu,\alpha)$ to be such a function that there is no "violet shift" i.e., $J(\tau,\mu,\sigma)$ should turn out identically zero for $\sigma-\sigma_0\leqslant 0$. One however has no control over the function $M(\tau,\mu,\alpha)$, as will be seen since it is obtained by successive approximations in the spherical harmonic method. In any P_{l_0} approximation one must expect a small violet shift [see (I)]. As will become evident, the violet shift is quite small even in the P_1 approximation, and the amount of violet shift seems to be a critical method of evaluating the success of the spherical harmonic method in handling problems of the type under consideration here.

Proceeding then, one substitute Eqs. (1.05) into (1.04) and obtains

$$\mu\frac{\partial M}{\partial\tau}+M=\frac{F}{4}e^{-\tau}e^{-i\alpha(1-\mu)}+\frac{1}{4\pi}\int_{-1}^{+1}d\mu'\int_{0}^{2\pi}d\varphi'\times e^{-i\alpha(1-\cos\Theta)}M(\tau,\mu',\alpha).\quad(1.06)$$

One now represents the angular dependence of $M(\tau,\mu,\alpha)$ by the method of spherical harmonics as follows,

$$M(\tau,\mu,\alpha)=\sum_{l=0}^{l_0}\frac{2l+1}{2}K_l(\tau,\alpha)P_l(\mu),\quad(1.07)$$

where

$$K_l(\tau,\alpha)=\int_{-1}^{+1}d\mu P_l(\mu)M(\tau,\mu,\alpha).$$

Since only low energy gamma-scattering is being considered one expects that one will not need a large value of l_0 to represent the solution. As in the case of problems of diffusion with no wavelength shift this is fortunate since, here as well, only the P_1 and P_2 approximations can be carried out with limited numerical facilities. This statement is also made here because the transport equation for high energy gamma-scattering is mathematically equivalent to the problem being considered here, if one makes the same approximations that Foldy[3] does in his work. One could not, however, expect to represent the high energy angular dependence (which is predominantly forward) by only a small cut-off value of l_0.

To proceed, then one substitutes Eq. (1.07) into Eq. (1.06) multiplies by $P_m(\mu)$ and integrates over $-1\leqslant\mu\leqslant+1$. Using the recurrence relation

$$(2l+1)\mu P_l(\mu)=lP_{l-1}(\mu)+(l+1)P_{l+1}(\mu),$$

one is led to the result

$$lK'_{l-1}+(l+1)K'_{l+1}+(2l+1)K_l=\frac{F}{4}(2l+1)e^{-\tau}\int_{-1}^{+1}d\mu e^{-i\alpha(1-\mu)}P_l(\mu)+\sum_{m=0}^{l_0}\mathfrak{A}_{lm}(\alpha)K_m(\tau,\alpha),\quad(1.08)$$

where the prime means derivative with respect to τ, and where

$$\mathfrak{A}_{lm}(\alpha)=\frac{(2l+1)(2m+1)}{4}\frac{1}{2\pi}\int_{-1}^{+1}d\mu\int_{-1}^{+1}d\mu'\int_{0}^{2\pi}d\varphi'\times e^{-i\alpha(1-\cos\Theta)}P_l(\mu)P_m(\mu').$$

All of the integrals can be carried out in closed form and are actually quite simple. To see this, introduce the well-known expansion,[4]

$$e^{i\alpha\cos\Theta}=\sum_{n=0}^{\infty}i^n(2n+1)j_n(\alpha)P_n(\cos\Theta),\quad(1.09)$$

and the addition theorem for spherical harmonics,

$$P_n(\cos\Theta)=P_n(\mu)P_n(\mu')+2\sum_{K=1}^{n}\frac{(n-K)!}{(n+K)!}P_n{}^K(\mu)P_n{}^K(\mu')\cos K\varphi';\quad(1.10)$$

the second term gives nothing since one integrates over φ' from 0 to 2π. One has then

$$\frac{4e^{i\alpha}\mathfrak{A}_{lm}}{(2l+1)(2m+1)}=\sum_{n=0}^{\infty}i^n(2n+1)j_n(\alpha)\int_{-1}^{+1}d\mu\int_{-1}^{+1}d\mu\times P_l(\mu)P_n(\mu)P_m(\mu')P_n(\mu')$$
$$=\frac{4}{(2l+1)(2m+1)}\sum_{n=0}^{\infty}i^n(2n+1)j_n(\alpha)\delta_{ln}\delta_{mn}.\quad(1.11)$$

Therefore only diagonal elements survive,

$$\mathfrak{A}_{lm}(\alpha)=e^{-i\alpha}i^l(2l+1)j_l(\alpha)\delta_{lm}.\quad(1.12)$$

Similarly, for the other integral in Eqs. (1.08), one can use the same method,

$$\int_{-1}^{+1}e^{i\alpha\mu}P_l(\mu)d\mu=2i^lj_l(\alpha).\quad(1.13)$$

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[3] L. Foldy, Phys. Rev. 81, 395 (1951).

[4] J. Stratton, *Electromagnetic Theory* (McGraw-Hill Book Company, Inc., New York, 1941), p. 409.

Finally, the system of equations (1.08) reduces to

$$lK'_{l-1}+(l+1)K'_{l+1}+(2l+1)K_l = \tfrac{1}{2}Fi^l(2l+1)e^{-i\alpha}j_l(\alpha)e^{-\tau} + i^l(2l+1)e^{-i\alpha}j_l(\alpha)K_l, \quad (1.14)$$

and then the formal solution would be:

$$J(\tau,\mu,y)=\sum_{l=0}^{l_0}\frac{2l+1}{2}P_l(\mu)\cdot\frac{1}{2\pi}\int_{-\infty}^{\infty}K_l(\tau,\alpha)e^{i\alpha y}d\alpha.$$

SECTION 2. THE SEMI-INFINITE MEDIUM-P_1 APPROXIMATION

One could apply a Laplace transformation to the system (1.14) in the variable τ and reduce it to a system of algebraic equations for the transforms of the $K_l(\tau,\alpha)$. However, for the P_1 and P_2 approximations it is easier to solve the system directly. In the P_1 approximation (i.e., $l_0=1$) one cuts the system off by keeping only terms in $K_0(\tau,\alpha)$ and $K_1(\tau,\alpha)$. This method of cut-off can only be judged by its success in the numerical sense of doing both the P_1 and P_2 approximations (and higher if possible) and seeing how the successive solutions to $K_l(\tau,\alpha)$ compare with each other.[5]

In the P_1 approximation, then, the system (1.14) becomes simply

$$\begin{aligned}K_1'+K_0&=\mathfrak{a}_{00}(K_0+\tfrac{1}{2}Fe^{-\tau}),\\ K_0'+3K_1&=\mathfrak{a}_{11}(K_1+\tfrac{1}{2}Fe^{-\tau}),\end{aligned}\quad (2.01)$$

where

$$\mathfrak{a}_{00}(\alpha)=e^{-i\alpha}j_0(\alpha),\quad \mathfrak{a}_{11}(\alpha)=3ie^{-i\alpha}j_1(\alpha).$$

Uncoupling these equations leads one to the following equation:

$$K_0''-\Omega^2K_0=\tfrac{1}{2}F(\Omega^2-3)e^{-\tau}, \quad (2.02)$$

where

$$\Omega^2(\alpha)=(1-\mathfrak{a}_{00})(3-\mathfrak{a}_{11})\equiv\Lambda_1+i\Lambda_2,$$

$$\Lambda_1(\alpha)=3[1-\cos\alpha j_0(\alpha)-\sin\alpha j_1(\alpha)+\sin 2\alpha j_0(\alpha)j_1(\alpha)],$$

$$\Lambda_2(\alpha)=3[\sin\alpha j_0(\alpha)-\cos\alpha j_1(\alpha)+\cos 2\alpha j_0(\alpha)j_1(\alpha)].$$

The solution of Eq. (2.02) is simply

$$K_0(\tau,\alpha)=\tfrac{1}{2}F[A(\alpha)e^{-\Omega\tau}+B(\alpha)e^{\Omega\tau}+e^{-\tau}/(1-\Omega^2)]\cdot(\Omega^2-3), \quad (2.03)$$

where

$$\Omega(\alpha)\equiv\Theta_1(\alpha)+i\Theta_2(\alpha),$$

$$\Theta_1(\alpha)=2^{-\frac{1}{2}}[\Lambda_1+(\Lambda_1^2+\Lambda_2^2)^{\frac{1}{2}}]^{\frac{1}{2}},\quad \Theta_2(\alpha)=\Lambda_2[2\Theta_1(\alpha)]^{-1}.$$

The boundary conditions for the determination of $A(\alpha)$, $B(\alpha)$ are the following:

$$\int_0^1 J(0,\mu,\sigma)\mu d\mu=0,\ \text{for all }\sigma; \quad (2.04)$$

$$\lim_{\tau\to\infty}\int_{-1}^{+1}J(\tau,\mu,\sigma)d\mu=0,\ \text{for all }\sigma. \quad (2.05)$$

These are to be used for a semi-infinite plane parallel medium. The first condition simply states that the net flux of gammas scattered at least once "into" the medium must vanish at $\tau=0$. This is the exact boundary condition because the incident beam and the unscattered beam are equal at $\tau=0$, and the flux of the latter has been separated completely from the problem. The second boundary condition (2.05) simply assumes that the spectral distribution averaged over angles approaches zero properly as $\tau\to\infty$, for all finite σ. Since photoelectric absorption is being neglected, one of course cannot assign physical significance to wavelengths larger than those wavelengths at which photons would be lost by absorption. This is always a difficulty in treating a semi-infinite medium. As far as broadening is concerned, however, the results of (I) have indicated already that at a distance τ_1 into a semi-infinite medium one has a broadened spectral distribution which does not differ greatly from the corresponding solution for a finite medium of the same thickness τ_1. A quantitative answer to the above difficulty can of course only be obtained by solving the present problem for a finite scattering medium (see below). One would then replace Eq. (2.05) by the exact condition

$$\int_{-1}^{0}J(\tau_1,\mu,\sigma)\mu d\mu=0,\ \text{for all }\sigma\ (\text{vacuum for }\tau>\tau_1). \quad (2.06)$$

Proceedings with the semi-infinite medium one substitutes Eqs. (1.05) and (1.07) into the boundary conditions (2.04) and (2.05) and obtains $B(\alpha)\equiv 0$ and

$$A(\alpha)=\frac{1}{3+2\Omega-\mathfrak{a}_{11}}\left[\frac{2\mathfrak{a}_{11}}{3-\Omega^2}-\frac{5-\mathfrak{a}_{11}}{1-\Omega^2}\right]. \quad (2.07)$$

The results in the P_1 approximation are then (after some tedious but simple algebra)

$$K_0(\tau,\alpha)=\tfrac{1}{2}F\left[\frac{\Delta_1+i\Delta_2}{\Delta_7}e^{-\Omega\tau}+\frac{\Delta_3+i\Delta_4}{\Delta_7\Delta_8}e^{-\Omega\tau}+\frac{\Delta_5+i\Delta_6}{\Delta_7\Delta_8}e^{-\tau}\right], \quad (2.08)$$

where the Δ's are defined below;

$$\Delta_1=Q_1(5-3C_1)-3C_2Q_2,$$
$$\Delta_2=-3C_2Q_1-Q_2(5-3C_1),$$
$$\Delta_3=2(5-C_1)(Q_1Q_3+\Lambda_2Q_2)-2C_2(Q_2Q_3-\Lambda_2Q_1),$$
$$\Delta_4=2(5-C_1)(\Lambda_2Q_1-Q_2Q_3)-2C_2(Q_1Q_3+\Lambda_2Q_2),$$
$$\Delta_5=Q_3(\Lambda_1-3)-\Lambda_2^2,$$
$$\Delta_6=Q_3\Lambda_2+2(\Lambda_1-3),$$
$$\Delta_7=Q_1^2+Q_2^2,$$
$$\Delta_8=Q_3^2+\Lambda_2^2,$$
$$Q_1(\alpha)=3-C_1+2\Theta_1,$$
$$Q_2(\alpha)=2\Theta_2-C_2,$$
$$Q_3(\alpha)=1-\Lambda_1;$$

[5] M. Wang and E. Guth, Phys. Rev. 84, 1092 (1951).

and, finally,

$$\mathfrak{A}_{11}(\alpha)\equiv C_1+iC_2,$$
$$C_1(\alpha)=(3\sin\alpha/\alpha)(\sin\alpha/\alpha-\cos\alpha),$$
$$C_2(\alpha)=\cot\alpha C_1(\alpha).$$

Finally, one needs $K_1(\tau,\alpha)$ since in this P_1 approximation the spectral density $J(\tau,\mu,\sigma)$ is given by

$$J(\tau,\mu,y)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\{\tfrac{1}{2}K_0(\tau,\alpha)+\tfrac{3}{2}\mu K_1(\tau,\alpha)\}e^{i\alpha y}d\alpha. \quad (2.09)$$

From the original Eqs. (2.01) one finds

$$K_1(\tau,\alpha)=1/(3-\mathfrak{A}_{11})\{-K_0'+\tfrac{1}{2}F\mathfrak{A}_{11}e^{-\tau}\}, \quad (2.10)$$

or

$$K_1(\tau,\alpha)=\tfrac{1}{2}F[e^{-(\Theta_1+i\Theta_2)\tau}(\mathfrak{M}_1+i\mathfrak{M}_2)+e^{-\tau}(\mathfrak{M}_3+i\mathfrak{M}_4)], \quad (2.11)$$

where the $\mathfrak{M}$'s are defined below.

$$\mathfrak{M}_1(\alpha)=\Gamma(\alpha)\{(3-C_1)(\Theta_1\omega_1-\Theta_2\omega_2)-C_2(\Theta_2\omega_1+\Theta_1\omega_2)\},$$
$$\mathfrak{M}_2(\alpha)=\Gamma(\alpha)\{(3-C_1)(\Theta_2\omega_1+\Theta_1\omega_2)+C_2(\Theta_1\omega_1-\Theta_2\omega_2)\},$$
$$\mathfrak{M}_3(\alpha)=\Gamma(\alpha)\{(3-C_1)\omega_3-C_2\omega_2\},$$
$$\mathfrak{M}_0(\alpha)=\Gamma(\alpha)\{(3-C_1)\omega_4+C_2\omega_3\},$$

where,

$$\omega_1=\Delta_1\Delta_8+\Delta_3,\quad \omega_3=\Delta_5+C_1\Delta_8\Delta_7,$$
$$\omega_2=\Delta_2\Delta_8+\Delta_4,\quad \omega_4=\Delta_6+C_2\Delta_8\Delta_7,$$
$$\Gamma(\alpha)=[\Delta_7\Delta_8\{(3-C_1)^2+C_2^2\}]^{-1}.$$

Finally, for the solution one needs only the real parts of $K_0(\tau,\alpha)e^{i\alpha y}$ and $K_1(\tau,\alpha)e^{i\alpha y}$, which are: $(y\equiv\sigma-\sigma_0)$

$$\mathrm{Re}\{K_0e^{i\alpha y}\}=\tfrac{1}{2}F(e^{-\Theta_1\tau}/\Delta_7\Delta_8)[\omega_1\cos(\Theta_2\tau-\alpha y)+\omega_2\sin(\Theta_2\tau-\alpha y)]+\tfrac{1}{2}F(e^{-\tau}/\Delta_8)[\Delta_5\cos\alpha y-\Delta_6\sin\alpha y], \quad (2.12)$$

TABLE I. Values of the integrals $\mathcal{J}^{(0)}(\tau,y)$ and $\mathcal{J}^{(1)}(\tau,y)$ for $\tau=1$, $\tau=2$ as defined in Eq. (2.15).

y	$\mathcal{J}^{(0)}/2$	$\frac{3}{2}\mathcal{J}^{(1)}$	$\frac{\mathcal{J}^{(0)}}{2}+\frac{3}{2}\mathcal{J}^{(1)}$
	$\tau=1$		
−0.50	−0.01	0.23	0.22
0.00	+0.07	0.31	0.38
0.25	+0.14	0.26	0.40
0.5	+0.14	0.16	0.30
1.0	+0.13	−0.04	0.09
2.0	+0.12	−0.16	−0.04
3.0	+0.11	+0.07	+0.18
	$\tau=2$		
−0.50	0.00	0.08	0.08
0.00	0.04	0.14	0.18
0.25	0.08	0.15	0.23
0.50	0.09	0.12	0.21
1.0	0.08	0.04	0.12
3.0	0.08	0.05	0.13
5.0	0.07	0.01	0.08

and

$$\mathrm{Re}\{K_1e^{i\alpha y}\}=\tfrac{1}{2}Fe^{-\Theta_1\tau}[\mathfrak{M}_1\cos(\Theta_2\tau-\alpha y)+\mathfrak{M}_2\sin(\Theta_2\tau-\alpha y)]+\tfrac{1}{2}Fe^{-\tau}[\mathfrak{M}_3\cos\alpha y-\mathfrak{M}_4\sin\alpha y]. \quad (2.13)$$

The spectral distribution is finally given by the following Fourier integral, in the P_1 approximation,

$$J(\tau,\mu,y)=\frac{1}{\pi}\int_0^{\infty}\mathrm{Re}\{[\tfrac{1}{2}K_0+\tfrac{3}{2}\mu K_1]e^{i\alpha y}\}d\alpha \quad (2.14)$$

or the following

$$J(\tau,\mu,y)=\tfrac{1}{2}F[\tfrac{1}{2}\mathcal{J}^{(0)}(\tau,y)+\tfrac{3}{2}\mu\mathcal{J}^{(1)}(\tau,y)], \quad (2.15)$$

$$\mathcal{J}^{(0)}(\tau,y)=\frac{1}{\pi}\int_0^{\infty}d\alpha\left[\frac{e^{-\Theta_1\tau}}{\Delta_7\Delta_8}\{\omega_1\cos(\Theta_2\tau-\alpha y)+\omega_2\sin(\Theta_2\tau-\alpha y)\}+\frac{e^{-\tau}}{\Delta_8}\{\Delta_5\cos\alpha y-\Delta_6\sin\alpha y\}\right],$$

$$\mathcal{J}^{(1)}(\tau,y)=\frac{1}{\pi}\int_0^{\infty}d\alpha[e^{-\Theta_1\tau}\{\mathfrak{M}_1\cos(\Theta_2\tau-\alpha y)+\mathfrak{M}_2\sin(\Theta_2\tau-\alpha y)\}+e^{-\tau}\{\mathfrak{M}_3\cos\alpha y-\mathfrak{M}_4\sin\alpha y\}].$$

These Fourier integrals $\mathcal{J}^{(0)}(\tau,y)$ and $\mathcal{J}^{(1)}(\tau,y)$ have been evaluated for $\tau=1$, $\tau=2$. The results are presented in Table I. Then from Eq. (2.15) one can obtain $J(\tau,\mu,\sigma)$ for any angle ϑ where $(\mu=\cos\vartheta)$. Figure 1 shows the spectral distribution for $\mu=1$, $F=1$, for both $\tau=1$ and $\tau=2$ along with the corresponding results obtained in (I). One sees that the two methods agree quite well for $\tau=2$ in giving the magnitude of the maximum intensity but disagree as to the location of the maximum and the shape of the distribution. The present method yields a rather poor result in the P_1 approximation for small τ-values (say $\tau<2$) in the practical sense that the Fourier integrals are more difficult to handle, and the results indicate rather large oscillations in the tails of the spectral distributions, i.e., at wavelength shifts beyond the maximum (see Fig. 1). For $\tau=2$ these oscillations damp out and approximate a monotonic decreasing function as one expects. We will forego a detailed discussion of the numerical aspects of the problem at this time since they are not complete.

The net flux across a surface of unit area per unit σ interval is given by

$$\pi\mathfrak{F}(\tau,\sigma)=2\pi\int_{-1}^{+1}J(\tau,\mu,\sigma)\mu d\mu+\pi F\delta(y)e^{-\tau}=\pi F\mathcal{J}^{(1)}(\tau,y)+\pi F\delta(y)e^{-\tau}. \quad (2.16)$$

Thus, only $\mathcal{G}^{(1)}(\tau, y)$ survives in the calculation of net flux. This is true, of course, in all higher orders of approximation $l_0=1, 2, \cdots$ and is one of the attractive features of the spherical harmonic method. At the same time one may have to calculate all the $K_l(\tau, \alpha)$, $l=0, 1, \cdots l_0$ in order to determine $K_1(\tau, \alpha)$ from which $\mathcal{G}^{(1)}(\tau, y)$ follows.

The corresponding results for a finite medium of thickness τ_1 can be readily solved by using the boundary conditions (2.04) and (2.06) along with Eqs. (2.01) and (2.03). The results are not recorded here because of their length and a lack of numerical facilities which makes it impossible to evaluate the Fourier integral solutions at the present time.

APPENDIX

For the problem of anisotropic scattering one must solve Eq. (1.01) as it stands. Only a few modifications are introduced and these will be indicated here. One introduces Eqs. (1.03) and (1.01) and obtains

$$\mu\frac{\partial J}{\partial\tau}+J=\frac{3}{16\pi}Fe^{-\tau}\delta(\sigma-\sigma_0-1+\mu)$$

$$+\frac{3}{16\pi}\int_{-1}^{+1}d\mu'\int_0^{2\pi}d\varphi'(1+\cos^2\Theta)J(\tau, \mu', \sigma'). \quad (A.01)$$

One again makes a Fourier integral decomposition (1.05) which gives

$$\mu\frac{\partial}{\partial\tau}M(\tau, \mu, \alpha)+M$$

$$=\frac{3}{16}Fe^{-\tau}(1+\mu^2)e^{-i\alpha(1-\mu)}$$

$$+\frac{3}{16\pi}\int_{-1}^{+1}d\mu'\int_0^{2\pi}d\varphi'(1+\cos^2\Theta)$$

$$\times e^{-i\alpha(1-\cos\Theta)}M(\tau, \mu', \alpha). \quad (A.02)$$

Introducing the "spherical harmonic" expansion (1.07) for $M(\tau, \mu, \alpha)$ leads one to

$$lK'_{l-1}+(l+1)K'_{l+1}+(2l+1)K_l$$

$$=\frac{3}{16}Fe^{-\tau}(2l+1)\int_{-1}^{+1}d\mu(1+\mu^2)e^{-i\alpha(1-\mu)}P_l(\mu)$$

$$+\sum_{m=0}^{l_0}\mathfrak{A}_{lm}(\alpha)K_m(\tau, \alpha), \quad (A.03)$$

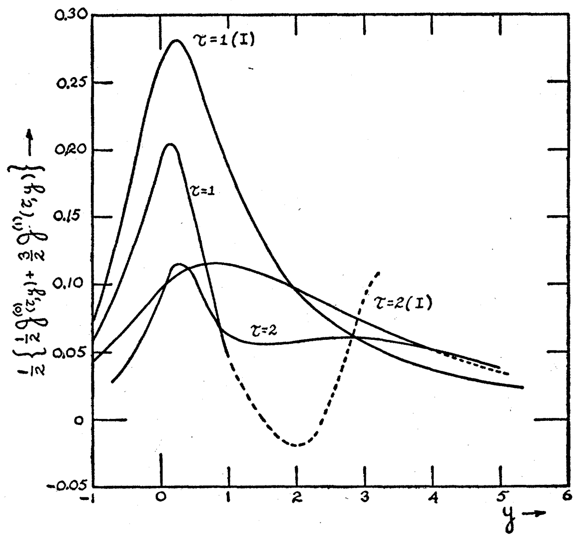

FIG. 1. Spectral distribution of all gammas scattered more than once are shown for distances $\tau=1$ and $\tau=2$. The unscattered spectral component would be a delta-function at $y=0$. The corresponding solutions obtained in an earlier paper are drawn for comparison. The primary intensity is unity. The solutions are for $\mu=1$ in both cases.

where

$$\frac{4}{(2l+1)(2m+1)}e^{i\alpha}\mathfrak{A}_{lm}(\alpha)$$

$$=\frac{3}{8\pi}\int_{-1}^{+1}d\mu\int_{-1}^{+1}d\mu'\int_0^{2\pi}d\varphi'(1+\cos^2\Theta)$$

$$\times e^{i\alpha\cos\Theta}P_l(\mu)P_m(\mu').$$

For the small values of l, m needed in the P_1 and P_2 approximations these integrals can be readily evaluated just as above. The only difference is that $\mathfrak{A}_{lm}(\alpha)$ is not diagonal and therefore the system of Eqs. (A.03) will differ slightly from (1.14). This case of anisotropic scattering has not been studied (numerically) and should not until the P_2 approximation to the above isotropic scattering problems is better understood (i.e., in the numerical sense of the convergence of the method which can be estimated by the decrease of the "violet shift" in passing from the P_1 to the P_2 approximation). In the P_2 approximation one can again uncouple the system of equations. One obtains again a second-order differential equation for $K_0(\tau, \alpha)$ just as above and $K_1(\tau, \alpha)$, $K_2(\tau, \alpha)$ can be expressed in terms of $K_0(\tau, \alpha)$ and the entire solution carried through in the same manner as in the P_1 approximation.

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