Thus, it is found that the calculated variation of resistivity and Hall coefficient with magnetic field intensity is very much greater for an intrinsic semiconductor than for an impurity semiconductor. At the present time experimental investigations of the magnetic field dependence of resistivity and Hall coefficient have not been published, and hence theory and experiment cannot be compared for such conditions.

The authors wish to thank Dr. K. Lark-Horovitz for his advice and encouragement during the progress of this work, and R. Bray, J. W. Cleland, and W. W. Scanlon for discussions of their experimental studies of magnetic field effects.

PHYSICAL REVIEW VOLUME 89, NUMBER 5 MARCH 1, 1953

Tables for Second Born Approximation Scattering from Various Potential Fields

LYMAN MOWER
Department of Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts

(Received October 22, 1952)

By means of a variational approach, the scattering amplitude for electron scattering from various potentials is calculated. Numerical values for the functions involved are tabulated as a function of energy and scattering angle.

RECENTLY variational methods have been applied to scattering problems in nuclear physics; it was of interest to see with what success these methods would meet in problems on the atomic scale. At the same time it was felt that it was important to make available in tabular form the various functions involved. Here, scattering of a particle from various static potentials is considered. The variational principle used is essentially that as introduced by Schwinger and is in the form suited for determining scattering amplitudes as presented by Lax[1] and by Morse and Feshbach.[2] It is as follows:

$$J(\theta)=\frac{-\int e^{i\mathbf{k}_i\cdot\mathbf{r}}U(r)\psi_s^*(r)dv\cdot\int e^{-i\mathbf{k}_s\cdot\mathbf{r}_1}U(r_1)\psi_i(r_1)dv_1}{4\pi\int\psi_s^*(r)U(r)\psi_i(r)dv+\int\int\psi_s^*(r)U(r)(e^{ik|r-r_1|}/|r-r_1|)U(r_1)\psi_i(r_1)dv,\,dv_1}.$$

This quantity $J(\theta)$, the scattering amplitude which we wish to minimize, was calculated for various potentials $U(r)$; namely, an exponential, Yukawa, and a potential of the form $-2(U_0/a_0)(1/r+\lambda/2)\exp(-r/\lambda)$. The trial wave function which was used in the above expression for $J(\theta)$ was $\psi_i=\exp(i\mathbf{k}_i\cdot\mathbf{r})$ and $\psi_s^*=\exp(-i\mathbf{k}_s\cdot\mathbf{r})$, where $\mathbf{k}_i$ is the momentum vector in the direction of the incident wave, $\mathbf{k}_s$ is the momentum vector in the direction of the scattered wave, $|\mathbf{k}_s|^2=|\mathbf{k}_i|^2$, and θ is the angle between the two vectors.

Many of the integrals involved have been calculated previously; the evaluation of those integrals not readily

TABLE I. Algebraic values for $S_i(\theta)$, where $S_i(\theta)$ is essentially the ratio of the second to the first Born approximation for the indicated potentials.

Potential	$S_i(\theta)$
Yukawa $U(r)=U_0(\lambda/r)\exp(-\lambda r)$	$S_y=-U_0(1+x^2c^2)I_y$ [a]
Exponential $U(r)=U_0\lambda^2\exp(-\lambda r)$	$S_e=U_0(1+x^2c^2)^2I_e$
Mixed $U(r)=-U_0(2/a_0)(1/r+\lambda/2)\times\exp(-\lambda r)$	$S_m=U_0(2/\lambda a_0)(1+x^2c^2)^2I_m/(2+x^2c^2)$

where

$$I_y=(2/xcA)\left[\arctan\left(\frac{xc}{A}\right)+\frac{i}{2}\ln\left(\frac{A+x^2c}{A-x^2c}\right)\right],$$

$$I_e=a+ib+dI_y,$$

$$a=4/[c^2A^2(A^2+x^2c^2)]-(2+x^2)^2(4-8x^2-2x^4c^2)/A^4(A^2+x^2c^2)-2/(x^2c^2+4)^2,$$

$$b=x/A^2c^2(1+x^2)-3x(2+x^2)^2/A^4(1+x^2),$$

$$d=-2/A^2c^2+4/A^2-6(2+x^2)^2/A^4,$$

$$I_m=[1+2(x^2+2)/A^2]I_y+(4-x^4c^2)/A^2(A^2+x^2c^2)+ix(2+x^2)/A^2(1+x^2)-I_e/2,$$

and where we have also used the notation:

$$x=2k/\lambda,\qquad A^2=4+4x^2+x^4c^2,$$
$$c=\sin\theta/2,\qquad \cos\theta=\mathbf{k}_i\cdot\mathbf{k}_s/k^2,$$
$$k=|\mathbf{k}_i|=|\mathbf{k}_s|=(mc/\hbar)(2E/mc^2)^{\frac{1}{2}}.$$

E denotes the energy of the incident electrons, m their mass, and $\hbar$ Planck's constant.

[a] See reference 3.

TABLE II. Algebraic values for $S_i(\theta)$ for $\theta=0$ and for the potentials indicated.

Potential	$S_i(\theta)_{\theta=0}$
Yukawa	$S_y=-U_0(1+ix)/2(1+x^2)$
Exponential	$S_e=-2U_0[(15+10x^2+3x^4)/24(1+x^2)^3+ix(3+3x^2+x^4)/3(1+x^2)^3]$
Mixed	$S_m=(U_0/2a_0\lambda)[(75+106x^2+39x^4)/24(1+x^2)^3+ix(12+18x^2+7x^4)/3(1+x^2)^3]$

[1] M. Lax, Phys. Rev. 78, 306 (1950).
[2] P. M. Morse and H. Feshbach, *Methods of Theoretical Physics*, to be published.

TABLE III. Numerical values of S_y and S_e as functions of θ and x.

$x \backslash \theta$	0° Re	0° Im	30° Re	30° Im	60° Re	60° Im	90° Re	90° Im	120° Re	120° Im	150° Re	150° Im	180° Re	180° Im
							(a) $S_y/-U_0=(1+x^2c^2)I_y$.							
0.8	0.3049	0.2439	0.3159	0.2537	0.3455	0.2800	0.3843	0.3154	0.4214	0.3501	0.4476	0.3751	0.4570	0.3841
1.0	0.2500	0.2500	0.2638	0.2652	0.3000	0.3061	0.3463	0.3597	0.3891	0.4120	0.4186	0.4489	0.4290	0.4621
1.2	0.2049	0.2459	0.2208	0.2671	0.2616	0.3230	0.3118	0.3953	0.3566	0.4632	0.3864	0.5105	0.3968	0.5273
1.6	0.1404	0.2247	0.1590	0.2580	0.2039	0.3426	0.2543	0.4459	0.2953	0.5374	0.3208	0.5984	0.3293	0.6196
2.0	0.1000	0.2000	0.1198	0.2449	0.1644	0.3540	0.2094	0.4782	0.2426	0.5819	0.2619	0.6481	0.2681	0.6706
2.4	0.0740	0.1775	0.0943	0.2334	0.1363	0.3618	0.1741	0.4981	0.1995	0.6055	0.2134	0.6715	0.2177	0.6936
2.8	0.0566	0.1584	0.0769	0.2244	0.1152	0.3675	0.1460	0.5090	0.1650	0.6150	0.1749	0.6783	0.1779	0.6992
3.0	0.0500	0.1500	0.0703	0.2208	0.1066	0.3696	0.1342	0.5120	0.1505	0.6161	0.1588	0.6776	0.1614	0.6978
3.2	0.0445	0.1423	0.0646	0.2178	0.0989	0.3713	0.1235	0.5135	0.1376	0.6154	0.1447	0.6749	0.1468	0.6943
3.6	0.0358	0.1289	0.0555	0.2130	0.0857	0.3735	0.1055	0.5133	0.1159	0.6099	0.1209	0.6653	0.1224	0.6833
4.0	0.0294	0.1176	0.0485	0.2096	0.0750	0.3743	0.0906	0.5099	0.0985	0.6008	0.1022	0.6523	0.1033	0.6690
4.4	0.0246	0.1081	0.0429	0.2071	0.0661	0.3738	0.0785	0.5042	0.0845	0.5896	0.0872	0.6375	0.0880	0.6528
4.8	0.0208	0.0998	0.0385	0.2054	0.0586	0.3723	0.0685	0.4970	0.0731	0.5772	0.0752	0.6217	0.0758	0.6360
5.2	0.0178	0.0927	0.0348	0.2041	0.0522	0.3698	0.0602	0.4889	0.0638	0.5642	0.0654	0.6057	0.0658	0.6193
5.6	0.0155	0.0865	0.0316	0.2032	0.0468	0.3668	0.0533	0.4806	0.0561	0.5510	0.0573	0.5898	0.0576	0.6023
6.0	0.0135	0.0811	0.0290	0.2024	0.0421	0.3631	0.0474	0.4712	0.0496	0.5379	0.0506	0.5745	0.0508	0.5860
							(b) $S_e/U_0=(1+x^2c^2)^2I_e$.							
0.8	−0.214	−0.322	−0.226	−0.347	−0.259	−0.416	−0.302	−0.517	−0.342	−0.625	−0.371	−0.707	−0.379	−0.738
1.0	−0.146	−0.291	−0.158	−0.325	−0.187	−0.420	−0.219	−0.559	−0.243	−0.705	−0.254	−0.815	−0.257	−0.855
1.2	−0.102	−0.256	−0.112	−0.300	−0.135	−0.418	−0.152	−0.588	−0.152	−0.760	−0.141	−0.884	−0.136	−0.930
1.6	−0.056	−0.203	−0.063	−0.259	−0.070	−0.414	−0.050	−0.613	−0.003	−0.809	+0.043	−0.931	+0.062	−0.972
2.0	−0.034	−0.165	−0.039	−0.233	−0.031	−0.417	−0.018	−0.634	+0.094	−0.802	+0.157	−0.898	+0.180	−0.928
2.4	−0.023	−0.138	−0.026	−0.218	−0.005	−0.422	+0.065	−0.629	+0.151	−0.764	+0.214	−0.831	+0.238	−0.858
2.8	−0.017	−0.119	−0.018	−0.210	+0.015	−0.426	+0.095	−0.611	+0.178	−0.712	+0.233	−0.755	+0.251	−0.767
3.0	−0.015	−0.111	−0.015	−0.208	+0.023	−0.426	+0.105	−0.598	+0.185	−0.684	+0.234	−0.718	+0.251	−0.727
3.2	−0.013	−0.104	−0.012	−0.207	+0.030	−0.426	+0.112	−0.584	+0.186	−0.656	+0.231	−0.682	+0.247	−0.688
3.6	−0.010	−0.092	−0.007	−0.206	+0.041	−0.422	+0.120	−0.553	+0.183	−0.602	+0.219	−0.617	+0.231	−0.619
4.0	−0.008	−0.083	−0.004	−0.206	+0.049	−0.416	+0.121	−0.520	+0.175	−0.522	+0.203	−0.559	+0.212	−0.560
4.4	−0.007	−0.076	−0.001	−0.208	+0.054	−0.405	+0.120	−0.487	+0.163	−0.507	+0.186	−0.509	+0.193	−0.509
4.8	−0.006	−0.069	+0.002	−0.210	+0.058	−0.393	+0.116	−0.456	+0.151	−0.467	+0.169	−0.467	+0.174	−0.466
5.2	−0.005	−0.064	+0.004	−0.211	+0.059	−0.379	+0.110	−0.427	+0.139	−0.432	+0.153	−0.429	+0.157	−0.428
5.6	−0.004	−0.059	+0.006	−0.212	+0.060	−0.364	+0.104	−0.400	+0.127	−0.401	+0.138	−0.397	+0.142	−0.395
6.0	−0.003	−0.056	+0.008	−0.213	+0.059	−0.350	+0.098	−0.375	+0.117	−0.374	+0.126	−0.369	+0.128	−0.367

available is that used by Dalitz.[3] The results are tabulated below and are compared with their corresponding first Born approximation. Using the above trial wave functions, the results are of the form $J(\theta)=f(\theta)/[1-S(\theta)]$, where $f(\theta)$ is the first Born approximation and the function $S(\theta)=U_0I(\theta)/f(\theta)$ represents essentially the ratio of the second Born approximation to the first. In order to obtain the total cross section for elastic scattering, use was made of the relationship existing between the scattering amplitude and the total cross section; namely,[1,2]

$$Q=(4/k)\times\text{imaginary part of } J(\theta)_{\theta=0}.$$

The algebraic results are summarized in Table I. In order to facilitate the numerical calculation of the total cross section for elastic scattering we list the algebraic value of $S_i(\theta)_{\theta=0}$ in Table II. Numerical values for the functions $S_i(\theta)$ as a function of θ are found in Tables III and IV for different values of the variables $x=2k/\lambda$.

For the case of the elastic scattering of electrons from monatomic hydrogen assuming no exchange or polarization, we may set $2U_0/\lambda a_0=1$; $\lambda=2/a_0$; thus $x=2k/\lambda=ka_0=(1/\alpha)(2E/mc^2)^{\frac{1}{2}}$ (as in Mott and Massey[4]), where α is the fine structure constant. We tabulate the numerical values of the function $|1/[1-S_m(\theta)]|^2$ in

TABLE IV. $S_m/(2U_0/\lambda a_0)=(1+x^2c^2)^2I_m/(2+x^2c^2)$.

x	0° Re	0° Im	30° Re	30° Im	60° Re	60° Im	90° Re	90° Im	120° Re	120° Im	150° Re	150° Im	180° Re	180° Im
0.8	0.375	0.399	0.394	0.422	0.446	0.486	0.514	0.574	0.578	0.661	0.624	0.724	0.640	0.747
1.0	0.286	0.385	0.309	0.420	0.366	0.513	0.439	0.639	0.505	0.761	0.549	0.847	0.564	0.878
1.2	0.221	0.361	0.245	0.406	0.305	0.528	0.377	0.688	0.436	0.836	0.474	0.938	0.486	0.974
1.6	0.139	0.307	0.164	0.373	0.222	0.545	0.280	0.751	0.320	0.927	0.342	1.039	0.348	1.076
2.0	0.094	0.261	0.118	0.347	0.170	0.557	0.211	0.785	0.235	0.960	0.245	1.066	0.248	1.100
2.4	0.067	0.225	0.091	0.329	0.134	0.566	0.162	0.796	0.174	0.960	0.179	1.054	0.182	1.084
2.8	0.050	0.197	0.073	0.317	0.109	0.572	0.126	0.793	0.132	0.942	0.133	1.024	0.134	1.050
3.0	0.044	0.185	0.066	0.314	0.098	0.572	0.112	0.788	0.116	0.929	0.117	1.006	0.117	1.031
3.2	0.038	0.175	0.061	0.311	0.089	0.572	0.099	0.782	0.102	0.914	0.103	0.987	0.103	1.010
3.6	0.031	0.157	0.052	0.306	0.073	0.570	0.080	0.764	0.081	0.883	0.081	0.948	0.081	0.968
4.0	0.025	0.142	0.045	0.304	0.061	0.565	0.065	0.744	0.066	0.852	0.066	0.909	0.066	0.927
4.4	0.021	0.130	0.039	0.303	0.052	0.558	0 054	0.723	0.054	0.820	0.054	0.872	0.054	0.889
4.8	0.017	0.119	0.035	0.302	0.044	0.549	0.045	0.701	0.045	0.790	0.045	0.838	0.045	0.853
5.2	0.015	0.110	0.031	0.302	0.038	0.539	0.039	0.680	0.039	0.762	0.038	0.805	0.038	0.819
5.6	0.013	0.103	0.028	0.301	0.033	0.529	0.033	0.660	0.033	0.735	0.033	0.775	0.033	0.788
6.0	0.011	0.096	0.025	0.300	0.029	0.518	0.029	0.639	0.029	0.708	0.029	0.748	0 029	0.759

[3] R. H. Dalitz, Proc. Roy. Soc. (London) **A206**, 509 (1951).
[4] N. F. Mott and H. S. W. Massey, *The Theory of Atomic Collisions* (Oxford University Press, London, 1949), p. 184.

TABLE V. $|1/(1-S_m)|^2=1/[(1-\mathrm{Re}S_m)^2+(\mathrm{Im}S_m)^2]$.[a]

$x \backslash \theta$	0°	30°	60°	90°	120°	150°	180°
0.8	1.819	1.834	1.840	1.767	1.626	1.501	1.452
1.0	1.521	1.528	1.504	1.385	1.213	1.085	1.040
1.2	1.357	1.360	1.313	1.161	0.983	0.364	0.825
1.6	1.196	1.193	1.108	0.923	0.757	0.661	0.632
2.0	1.124	1.114	1 001	0.808	0.663	0.586	0.563
2.4	1.085	1.070	0.935	0.748	0.624	0.560	0.542
2.8	1.062	1.042	0.892	0.718	0.610	0.556	0.540
3.0	1.054	1.031	0.876	0.709	0.608	0.558	0.543
3.2	1.047	1.022	0.863	0.703	0.609	0.562	0.548
3.6	1.037	1.007	0.844	0.699	0.616	0.574	0.561
4.0	1.030	0.995	0.833	0.701	0.626	0.588	0.577
4.4	1.025	0.985	0.826	0.706	0.638	0.604	0.594
4.8	1.021	0.977	0.823	0.713	0.651	0.620	0.610
5.2	1.018	0.970	0.822	0.721	0.665	0.636	0.627
5.6	1.015	0.965	0.823	0.730	0.678	0.651	0.643
6.0	1.013	0.960	0.826	0.740	0.692	0.666	0.658

[a] Where $\mathrm{Re}S_m$ and $\mathrm{Im}S_m$ denote the real and imaginary part of $S_m(\theta)$, respectively.

Table V. In Table VI we tabulate

$$Q=\frac{(4/x)\,\mathrm{Im}S_m(\theta)_{\theta=0}}{|1-S_m(\theta)_{\theta=0}|^2}=\pi a_0^2\frac{(12+18x^2+7x^4)}{3(1+x^2)^3}\left|\frac{1}{1-S_m(0)}\right|^2,$$

which represents the numerical values for the total cross section for elastic scattering as a function of the energy parameter x. In Table VII we tabulate

$$\begin{aligned}I(\theta)/a_0^2&=[f(\theta)]^2|1/[1-S_m(\theta)]|^2\\&=\tfrac{1}{4}[(2+x^2c^2)/(1+x^2c^2)^2]^2|1/[1-S_m(\theta)]|^2,\end{aligned}$$

which represents the square of the scattering amplitude as a function of θ for various values of x.

In Fig. 1 the total cross section for elastic scattering as derived from the first Born approximation and that as derived from the above variational principle are compared as a function of the energy parameter x.

TABLE VI. $Q=4\,\mathrm{Im}S_m/x|1-S_m|^2$.[a]

x	Q
0.8	3.627
1.0	2.345
1.2	1.633
1.6	0.919
2.0	0.587
2.4	0.407
2.8	0.299
3.0	0.260
3.2	0.229
3.6	0.181
4.0	0.146
4.4	0.121
4.8	0.102
5.2	0.0865
5.6	0.0745
6.0	0.0649

[a] $\mathrm{Im}S_m$ denotes the imaginary part of $S_m(\theta)_{\theta=0}$, and $|1-S_m|^2$ is also evaluated at $\theta=0$.

TABLE VII. $I(\theta)/a_0^2=(1/4)[(2+x^2c^2)/(1+x^2c^2)^2]^2|1/[1-S_m(\theta)]|^2$.

$x \backslash \theta$	0°	30°	60°	90°	120°	150°	120°
0.8	1.819	1.618	1.185	0.783	0.521	0.389	0.350
1.0	1.521	1.259	0.800	0.477	0.245	0.167	0.146
1.2	1.357	1.034	0.535	0.256	0.125	0.080	0.069
1.6	1.196	0.747	0.267	0.092	0.040	0.024	0.020
2.0	1.124	0.554	0.141	0.0399	0.016	0.0096	0.008
2.4	1.085	0.413	0.078	0.0196	0.0078	4.61(−3)	3.90(−3)
2.8	1.062	0.307	0.045	0.0107	4.23(−3)	2.52(−3)	2.14(−3)
3.0	1.054	0.265	0.035	8.2(−3)[a]	3.23(−3)	1.64(−3)	1.65(−3)
3.2	1.047	0.228	0.028	6.35(−3)	2.51(−3)	1.51(−3)	1.29(−3)
3.6	1.037	0.170	0.018	4.01(−3)	1.60(−3)	0.97(−3)	0.83(−3)
4.0	1.030	0.127	0.012	2.67(−3)	1.07(−3)	0.65(−3)	0.56(−3)
4.4	1.025	0.096	8.30(−3)	1.85(−3)	0.75(−3)	0.46(−3)	0.39(−3)
4.8	1.021	0.073	5.05(−3)	1.33(−3)	0.54(−3)	0.33(−3)	0.287(−3)
5.2	1.018	0.056	4.35(−3)	0.97(−3)	0.40(−3)	0.249(−3)	0.21(−3)
5.6	1.015	0.439	3.26(−3)	0.72(−3)	0.30(−3)	0.189(−3)	0.16(−3)
6.0	1.013	0.034	2.50(−3)	0.57(−3)	0.24(−3)	0.147(−3)	0.126(−3)

[a] (−3) denotes that the entry should be multiplied by 10^{-3}.

Here we have denoted the cross section derived from the first Born approximation as

$$Q(\text{Born})=\pi a_0^2(12+18x^2+7x^4)/3(1+x^2)^3,$$

and the cross section derived from the variational principle as

$$Q(\text{Variational})=\text{the above }Q.$$

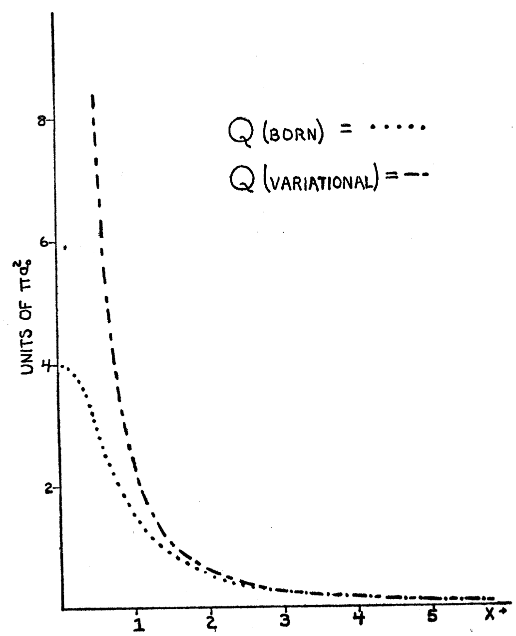

FIG. 1 A comparison of the total cross section of the elastic scattering of electrons from atomic hydrogen not including exchange or polarization using the first Born approximation and the above variational principle

I wish to thank Professor P. M. Morse for suggesting the need for the above tabulations and for aid in their development, and also to thank Elgie Ginsburgh for her painstaking calculations.