Theory of the Magnetoresistive Effect in Semiconductors*

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Experiments show that the application of a magnetic field increases the resistivity of a semiconductor and produces a decrease in the magnitude of the Hall coefficient. Existing theoretical treatments predict much smaller effects than are actually observed in semiconductors. The present calculation has been carried out to see if theory is brought closer to experiment by considering (1) scattering of conduction electrons by impurity ions as well as by the lattice, and (2) conduction by both holes and electrons at high temperatures. The calculation shows that the presence of impurity scattering decreases the magnitude of the effects produced by a magnetic field and thus increases the gap between theoretical and experimental values. It is noted that the discrepancy decreases with falling temperatures and is no longer present for data measured on a Ge sample at 20°K. The calculated magnetic field effects are very much greater for an intrinsic semiconductor than for an impurity semiconductor. The fractional changes in resistivity and Hall coefficient are given, for several different values of the electron-hole mobility ratio, as functions of a parameter containing magnetic field strength and temperature. The absence of experimental values of the magnetic field effects at high temperatures prevents comparison of theory and experiment for the intrinsic semiconductor.

I. INTRODUCTION

THE magnetoresistive effect is studied by measuring the resistance of a sample in the presence of a magnetic field which is perpendicular to the electric current density. Just as for metals, it is observed that the application of the magnetic field increases the resistivity of a semiconductor[1–7] and produces a decrease in the magnitude of the Hall coefficient.[2–5] The first theoretical treatment of this effect is that of Gans;[8] this early work has been extended by Harding,[9] Sangupta,[10] and Davis.[11] More recently, theories have been developed by Seitz[12] and by Pearson and Suhl[7] to apply to cubic crystals in various orientations relative to the electric and magnetic fields. It is generally found that theory predicts a much smaller magnetoresistive effect than is actually observed in semiconductors. The present calculation has been carried out to find the changes produced in theoretical values by considering (1) scattering of conduction electrons by impurity ions as well as by the lattice, and (2) conduction by both holes and electrons in high temperature semiconductors.

The effect of magnetic field upon the resistivity and Hall coefficient is calculated from the electric current density equations.[8] Assume that no temperature gradients exist, electric current flows in the X direction (j_x), a magnetic field is applied in the Z direction (H), and that there is no current flow in the Y direction. The sample is assumed to be ohmic, homogeneous, and isotropic. The electric current densities j_x and j_y are related to the electric field intensities E_x and E_y by the equation[13]

$$j_x = AE_x - BE_y, \tag{1.1}$$

$$j_y = BE_x + AE_y = 0, \tag{1.2}$$

where, if current is carried by electrons only,

$$A = -\frac{4\pi e^2}{3}\int_0^\infty lv^3(1+s^2)^{-1}\frac{\partial f}{\partial \epsilon}dv, \tag{1.3}$$

$$B = -\frac{4\pi e^2}{3}\int_0^\infty slv^3(1+s^2)^{-1}\frac{\partial f}{\partial \epsilon}dv, \tag{1.4}$$

in which

$$s = elH/mv, \tag{1.5}$$

and e denotes the magnitude of electronic charge, l the mean free path, m the effective mass of conduction electrons, f the classical distribution function

$$f = n\{m/(2\pi kT)\}^{\frac{3}{2}}\exp(-\epsilon/kT), \tag{1.6}$$

n the number of conduction electrons per unit volume, and ϵ the kinetic energy $mv^2/2$.

Equations (1.1) and (1.2) yield for the conductivity σ and Hall coefficient R

$$\sigma = j_x/E_x = A + (B^2/A), \tag{1.7}$$

$$R = E_y/(j_x H) = -B/\{H(A^2+B^2)\}. \tag{1.8}$$

If the current is carried only by holes, Eq. (1.7) is unchanged and the negative sign in Eq. (1.8) becomes positive.

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[13] All derivations in this paper are carried out in electromagnetic units.

TABLE I. Values of $J_1(\beta,\gamma)$,[a] and $F(\beta)$.[b]

γ[c] = \ β =	0	2/3	3/2	4	9	24
0.00	1.00000	0.71609	0.59156	0.42182	0.28701	0.15698
0.01	0.99041	0.71391	0.59024	0.42129	0.28680	0.15693
0.04	0.95649	0.70723	0.58642	0.41970	0.28620	0.15680
0.10	0.88828	0.69437	0.57889	0.41658	0.28500	0.15652
0.20	0.78806	0.67401	0.56678	0.41147	0.28303	0.15608
0.50	0.66884	0.61982	0.53345	0.39698	0.27731	0.15476
1.00	0.59639	0.54727	0.48618	0.37513	0.26836	0.15262
2.00	0.44530	0.44470	0.41374	0.33848	0.25239	0.14860
4.00	0.25949	0.32516	0.32010	0.28438	0.22631	0.14137
$F(\beta)$	1.0000	0.7957	0.7395	0.7030	0.7175	0.7849

[a] $J_1(\beta,\gamma)$ is the integral defined by Eq. (2.5).
[b] $F(\beta)$ is defined as the ratio of the sum of resistivity owing to lattice scattering only and the resistivity owing to impurity scattering only to the total resistivity when both types of scattering are present; the parameter β is defined as 6 times the ratio of resistivity owing to impurity scattering to resistivity owing to lattice scattering.
[c] γ is defined as $(9\pi/16)(\mu_L{}^0)^2H^2$, where H is the magnetic field strength in gauss and $\mu_L{}^0$ is the mobility associated with lattice scattering only if $H=0$, measured in cm²/abvolt-sec.

II. RESISTIVITY OF AN IMPURITY SEMICONDUCTOR

A sample may be termed an impurity semiconductor if conduction is practically due only to electrons excited to the conduction band from impurity donor levels (or to holes formed in the filled band by excitation of electrons to impurity acceptor levels); conduction owing to intrinsic electrons and holes is negligible in an impurity semiconductor. It is assumed that the conduction electron density does not exceed the limit of applicability of classical statistics.

Analysis[14,15] of measured resistivity data for impurity semiconducting samples, especially silicon and germanium, has shown that the temperature behavior can be explained only by the existence of another scattering process in addition to the scattering of conduction electrons by the lattice. Such an additional process, which has the appropriate temperature behavior to account for the experimental results, is scattering of conduction electrons by the impurity ions. A satisfactory model[16] for describing impurity ion scattering is based upon Rutherford scattering; the mean free path associated with this process is approximately proportional to ϵ^2. If an electron experiences collisions with both the lattice atoms and the impurity ions independently, the mean free path l is related to l_L, the mean free path if only lattice scattering exists, and l_I, the mean free path if only impurity scattering exists, by the relation:[17]

$$1/l=1/l_L+1/l_I. \tag{2.1}$$

This expression for l is now used to evaluate the quantities A and B.

The quantity l_L is approximately independent[18] of ϵ; it is also independent of the magnetic field strength. Equations (1.3) and (1.7) may be used to relate l_L to $\rho_L{}^0$, the resistivity for $H=0$ when only lattice scattering exists

$$l_L=3(2\pi mkT)^{\frac{1}{2}}(4ne^2\rho_L{}^0)^{-1}. \tag{2.2}$$

Let $l_I=\alpha_I\epsilon^2$; then Eqs. (1.3) and (1.7) relate α_I to $\rho_I{}^0$, the resistivity for $H=0$ when only impurity scattering exists

$$\alpha_I=(\pi m)^{1/2}(2^{5/2}ne^2\rho_I{}^0)^{-1}(kT)^{-3/2}. \tag{2.3}$$

A combination of Eqs. (2.1), (2.2), and (2.3) leads to the mean free path

$$l=l_Ll_I(l_L+l_I)^{-1}=l_L\epsilon^2(\epsilon^2+l_L/\alpha_I)^{-1}$$
$$=l_L\epsilon^2\{\epsilon^2+6\rho_I{}^0(kT)^2/\rho_L{}^0\}^{-1}. \tag{2.4}$$

Use Eq. (2.4) for l, replace ϵ/kT by x, and express l_L by Eq. (2.2) to convert the quantity A of Eq. (1.3) into

$$A=\frac{1}{\rho_L{}^0}\int_0^\infty\frac{x^3e^{-x}(x^2+\beta)dx}{(x^2+\beta)^2+\gamma x^3}, \tag{2.5}$$

where

$$\beta=6\rho_I{}^0/\rho_L{}^0, \tag{2.6}$$

and

$$\gamma=(9\pi/16)(\mu_L{}^0)^2H^2, \tag{2.7}$$

in which $\mu_L{}^0$, the mobility associated with lattice scattering only, if $H=0$, is given by

$$\mu_L{}^0=4el_L(18\pi mkT)^{-\frac{1}{2}}. \tag{2.8}$$

Denote the integral on the right side of Eq. (2.5) by $J_1(\beta,\gamma)$ so that

$$J_1(\beta,\gamma)=\rho_L{}^0A. \tag{2.9}$$

In a similar manner, one obtains

$$B=\frac{\gamma^{\frac{1}{2}}}{\rho_L{}^0}\int_0^\infty\frac{x^{9/2}e^{-x}dx}{(x^2+\beta)^2+\gamma x^3}\equiv\frac{\gamma^{\frac{1}{2}}}{\rho_L{}^0}J_2(\beta,\gamma). \tag{2.10}$$

Algebraic combination of Eqs. (1.7), (2.9), and (2.10) yields the result

$$\frac{\rho(H)}{\rho_T{}^0}=\frac{J_1}{J_1{}^2+\gamma J_2{}^2}\cdot\frac{\rho_L{}^0}{\rho_T{}^0}, \tag{2.11}$$

TABLE II. Values of $J_2(\beta,\gamma)$, the integral defined by Eq. (2.10).

γ = \ β =	0	2/3	3/2	4	9	24
0.00	0.88623	0.39704	0.27416	0.14674	0.07334	0.02484
0.01	0.87132	0.39569	0.27348	0.14649	0.07323	0.02481
0.04	0.82537	0.39190	0.27164	0.14591	0.07306	0.02479
0.10	0.74423	0.38454	0.26804	0.14475	0.07272	0.02473
0.20	0.63769	0.37289	0.26225	0.14286	0.07215	0.02464
0.50	0.52255	0.34197	0.24633	0.13751	0.07051	0.02438
1.00	0.45711	0.30076	0.22382	0.12946	0.06796	0.02396
2.00	0.32871	0.24289	0.18947	0.11602	0.06342	0.02318
4.00	0.18613	0.17606	0.14537	0.09633	0.05608	0.02178

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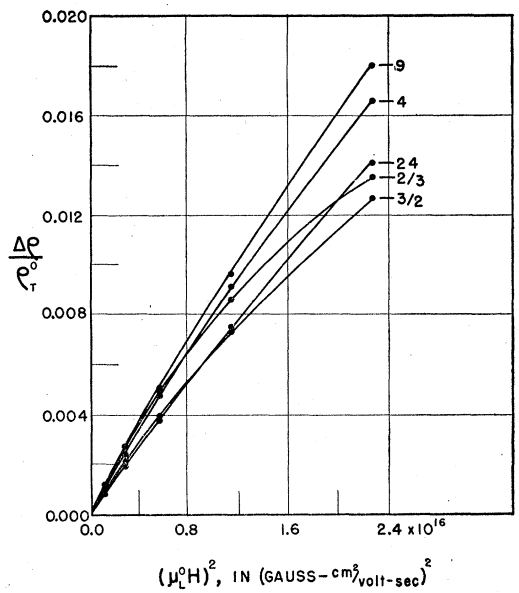

FIG. 1. Fractional increase in resistivity with rising magnetic field for an impurity semiconductor. The numbers at the ends of the curves are the values of the parameter β, defined as six times the ratio of resistivity resulting from impurity scattering to resistivity owing to lattice scattering. The abscissae, given as squares of the product of magnetic field strength and carrier mobility in the absence of magnetic field and scattering by impurities, may be converted into H^2T^{-3} values by taking $\mu_L{}^0=1.7\times10^7T^{-\frac{3}{2}}$ for germanium, or $5\times10^5T^{-\frac{3}{2}}$ for silicion, or $2.6\times10^6T^{-\frac{3}{2}}$ for tellurium, where T is the temperature in °K.

where $\rho_T{}^0$ is the total resistivity at zero magnetic field. The ratio $\rho_L{}^0/\rho_T{}^0$ can be replaced by $F(\beta)(1+\beta/6)^{-1}$, where the quantity $F(\beta)$, defined as $(\rho_L{}^0+\rho_I{}^0)/\rho_T{}^0$, has been previously evaluated[17] as a function of the ratio of impurity scattering to lattice scattering. Finally, the magnetoresistive effect is calculable from the relation

$$\frac{\Delta\rho}{\rho_T{}^0}=\frac{\rho(H)-\rho_T{}^0}{\rho_T{}^0}=\frac{J_1}{J_1{}^2+\gamma J_2{}^2}\cdot\frac{F}{1+\beta/6}-1. \quad (2.12)$$

Tables I and II give the values, as found by numerical integration, of the integrals J_1 and J_2 for selected values β and γ. The β values are chosen to correspond to $\rho_I{}^0/(\rho_I{}^0+\rho_L{}^0)$ values of 0.0, 0.1, 0.2, 0.4, 0.6, and 0.8. Table I also includes the values of $F(\beta)$ corresponding to the chosen β values. Figure 1 shows the behavior of the ratio $\Delta\rho/\rho_T{}^0$ as a function of the quantity $(\mu_L{}^0)^2H^2$. The quantity γ, or $(\mu_L{}^0H)^2$, is used because the theoretical plot of $\Delta\rho/\rho_T{}^0$ against γ, for a given value of β, is the same for all materials. From Fig. 1 one can get a plot of $\Delta\rho/\rho_T{}^0$ against H^2 for a given material at a chosen temperature T by using the appropriate value of $\mu_L{}^0$. One can also consider Fig. 1 to be a plot of magnetoresistive change against H^2T^{-3} if $\mu_L{}^0$ is proportional to $T^{-\frac{3}{2}}$, as is expected[18] for a classical semiconductor.

It is observable that (a) the change in resistivity is directly proportional to the square of the magnetic flux density at weak magnetic fields, (b) the predicted magnetoresistive effects are quite small, under 2 percent for the mobility-field combinations likely to be encountered except at very low temperatures, and (c) the effect of impurity scattering is to decrease the calculated resistivity change, thus increasing the gap between measured and calculated values.

Figure 2 gives a comparison between theory and experiment for germanium sample 33E, an n-type polycrystalline sample of rather low resistivity, prepared by doping with antimony. This sample was chosen because its magnetoresistive effect had been measured[6] at liquid nitrogen and liquid hydrogen temperatures as well as at room temperature, and also because a previous study[19] of resistivity and Hall coefficient for this sample gave information about the ratio of impurity to lattice scattering at various temperatures. As may be seen from Fig. 2, there is a great discrepancy between theory and experiment at 295°K, a much smaller discrepancy at 77°K, and relatively little discrepancy at 20°K.

III. HALL COEFFICIENT OF AN IMPURITY SEMICONDUCTOR

The dependence of the Hall coefficient of an impurity semiconductor upon magnetic field strength is computed from Eq. (1.8) with the use of the mean free path

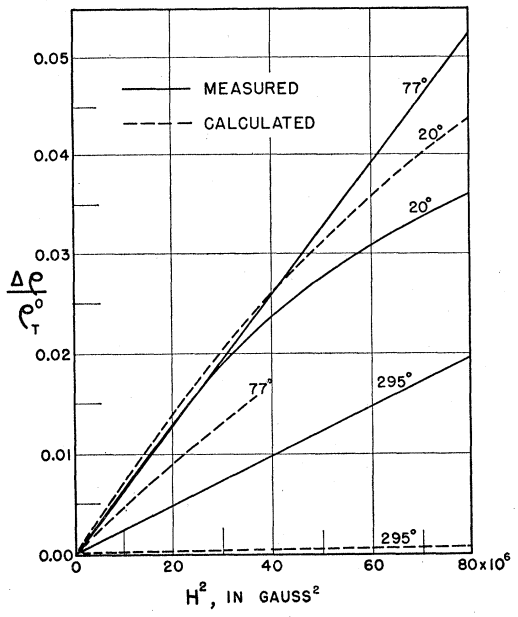

FIG. 2. Comparison between measured and calculated values of the fractional increase in resistivity with magnetic field for germanium sample 33E, an n-type polycrystalline sample, prepared by doping with antimony, having a room temperature resistivity of 0.05 ohm-cm, and found to be quite homogeneous. The measured curves are from Estermann and Foner.[6]

[19] K. Lark-Horovitz, National Defense Research Committee Report NDRC 14-585, 1945, unpublished, Fig. 7.

defined by Eq. (2.4). Simplification with the use of Eqs. (2.7), (2.9), and (2.10) yields

$$R=-\frac{\gamma^{\frac{1}{2}}J_2}{\rho_L{}^0H}\left\{\frac{J_1{}^2}{(\rho_L{}^0)^2}+\frac{\gamma J_2{}^2}{(\rho_L{}^0)^2}\right\}^{-1}=-\frac{3\pi^{\frac{1}{2}}}{4}\frac{\mu_L{}^0\rho_L{}^0J_2}{J_1{}^2+\gamma J_2{}^2}. \quad (3.1)$$

Since $\mu_L{}^0=(ne\rho_L{}^0)^{-1}$, one can write

$$R=-\frac{3\pi^{\frac{1}{2}}}{4ne}\frac{J_2}{J_1{}^2+\gamma J_2{}^2}. \quad (3.2)$$

If one defines a ratio r by

$$|R|=r/(ne), \quad (3.3)$$

one can also write

$$r(\beta,\gamma)=\frac{3\pi^{\frac{1}{2}}}{4}\frac{J_2}{J_1{}^2+\gamma J_2{}^2}. \quad (3.4)$$

If β is taken equal to zero, corresponding to the absence of impurity scattering, Eq. (3.2) yields the magnetic field dependence of Hall coefficient calculated by Harding.[9] If β is set equal to infinity, one finds the magnetic field dependence of Hall coefficient for a sample in which impurity scattering completely dominates; this case was worked out earlier by Johnson and Lark-Horovitz.[20] The values of the ratio $r(\beta,\gamma)$ for these limiting cases and for β values corresponding to $\rho_I{}^0/(\rho_I{}^0+\rho_L{}^0)=0.1$, 0.2, 0.4, 0.6, and 0.8 are given in Table III. Because $\mu_L{}^0$ no longer has significance when only impurity scattering exists ($\beta\to\infty$), the r values for this limiting case are given as a function of a new parameter γ^*, defined by

$$\gamma^*=(\pi/64)(\mu_I{}^0)^2H^2. \quad (3.5)$$

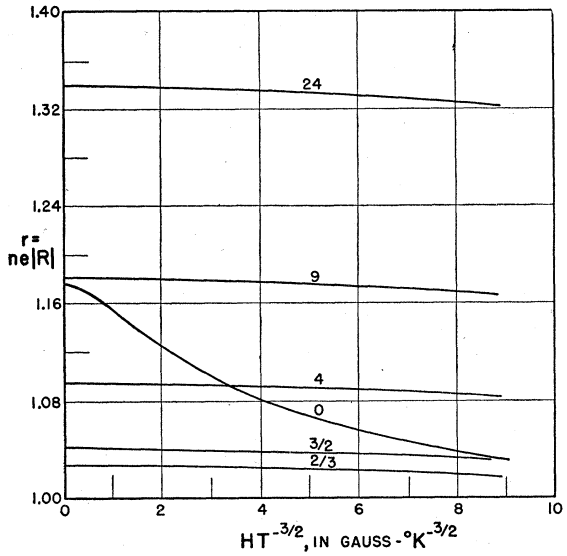

FIG. 3. Dependence of Hall effect upon magnetic field strength for germanium. The ordinate r is defined as the dimensionless product of the Hall coefficient, electronic charge, and carrier density. The numbers on the various curves are the values of β, defined as six times the ratio of resistivity owing to impurity scattering to resistivity owing to lattice scattering.

[20] V. A. Johnson and K. Lark-Horovitz, Phys. Rev. **79**, 176 (1950).

TABLE III. Values of the Hall ratio $r=ne|R|$ as a function of β and γ.

$\gamma=$ \ $\beta=$	0	2/3	3/2	4	9	24	γ^*[a]	$\beta=\infty$
0.00	1.1781	1.0289	1.0411	1.0962	1.1833	1.3399	0.0000	1.9328
0.01	1.1718	1.0289	1.0411	1.0959	1.1828	1.3390	0.0005	1.8616
0.04	1.1591	1.0289	1.0411	1.0958	1.1826	1.3389	0.002	1.7561
0.10	1.1441	1.0286	1.0410	1.0956	1.1824	1.3386	0.005	1.6558
0.20	1.1264	1.0282	1.0407	1.0953	1.1820	1.3381	0.025	1.4565
0.50	1.0996	1.0270	1.0399	1.0942	1.1808	1.3369	0.050	1.3750
1.00	1.0762	1.0253	1.0386	1.0928	1.1788	1.3346	0.250	1.2183
2.00	1.0545	1.0226	1.0366	1.0900	1.1751	1.3306	1.000	1.1251
4.00	1.0356	1.0188	1.0334	1.0853	1.1685	1.3230		

[a] The parameter $\gamma=(9\pi/16)(\mu_L{}^0)^2H^2$, where $\mu_L{}^0$ is the mobility when only lattice scattering is present, becomes meaningless for $\beta=\infty$, which corresponds to the absence of lattice scattering. Hence r for $\beta=\infty$ is given in terms of a parameter γ^*, defined as $(\pi/64)(\mu_I{}^0)^2H^2$, where $\mu_I{}^0$ is the mobility when only impurity scattering is present.

Figure 3 shows a plot of r *vs* $HT^{-\frac{3}{2}}$ for six different β values. The γ-parameter has been converted into terms of $HT^{-\frac{3}{2}}$, by assuming that $\mu_L{}^0$ is proportional to $T^{-\frac{3}{2}}$, the behavior expected[18] for lattice scattering in any classical semiconductor; the constant of proportionality is found by setting $\mu_L{}^0$ at 300°K equal to 3300 cm²/volt-sec, a value characteristic of germanium.[21] It is apparent that the introduction of impurity scattering lessens the variation of r with magnetic field strength. Scanlon[3] and Cleland[4] have observed that pure germanium samples show more variation in Hall coefficient with changing magnetic field than do relatively impure samples. However, a quantitative comparison between the behavior of Hall coefficients in a magnetic field as predicted by Eq. (3.2) and as measured[2–5] shows that the theoretical values are almost always too low, usually by a factor between 2 and 10.

IV. RESISTIVITY OF AN INTRINSIC SEMICONDUCTOR

At high temperatures, thermal energy is sufficient to excite appreciable numbers of electrons from the filled band to the conduction band of a semiconductor; for each electron excited to the conduction band, there exists a conducting hole in the filled band. Conduction by such electron-hole pairs is termed intrinsic.

Since both holes and electrons act as current carriers, the electrical current density equations must include terms characteristic of each carrier. Thus one replaces Eqs. (1.1) and (1.2) by

$$j_x=(A_1+A_2)E_x-(B_1-B_2)E_y, \quad (4.1)$$

$$j_y=(B_1-B_2)E_x+(A_1+A_2)E_y, \quad (4.2)$$

where subscript 1 applies to electrons and subscript 2 to holes. The coefficients A_1 and A_2 are defined by Eq. (1.3) with the electron mean free path l_1, the effective electron mass m_1, and the electron distribution function f_1 used in A_1 and the corresponding quantities for holes,

[21] Pearson, Haynes, and Shockley, Phys. Rev. **78**, 295 (1950).

TABLE IV. Values of $K_1(\gamma)$,[a] and $K_2(\gamma)$.[b]

γ	$K_1(\gamma)$	$K_2(\gamma)$
0.0000	1.00000	0.88623
0.0025	0.99760	0.88217
0.01	0.99041	0.87132
0.05	0.95649	0.82537
0.15	0.88828	0.74423
0.35	0.78806	0.63769
0.70	0.66884	0.52255
1.00	0.59639	0.45711
2.00	0.44530	0.32871
5.00	0.25945	0.18613

[a] $K_1(\gamma)=\int_0^\infty x^2e^{-x}(x+\gamma)^{-1}dx$.
[b] $K_2(\gamma)=\int_0^\infty x^{\frac{1}{2}}e^{-x}(x+\gamma)^{-1}dx$.

l_2, m_2, and f_2, used in A_2. The symbol e represents the magnitude of the electron charge; the proper signs for electrons and holes have been considered in writing Eqs. (4.1) and (4.2). The terms B_1 and B_2 are found from Eq. (1.4) by the appropriate insertions of l_1, m_1, and f_1, for B_1 and of l_2, m_2, and f_2 for B_2.

Equations (4.1) and (4.2) lead to the relations

$$\sigma=\frac{j_x}{E_x}=\frac{(A_1+A_2)^2+(B_1-B_2)^2}{A_1+A_2} \tag{4.3}$$

and

$$R=\frac{E_y}{j_xH}=\frac{-(B_1-B_2)}{H\{(A_1+A_2)^2+(B_1-B_2)^2\}}. \tag{4.4}$$

Resistivity due to scattering by impurity ions decreases rapidly as temperature rises,[16] whereas resistivity because of scattering by the lattice rises as the $\frac{3}{2}$ power of the temperature;[18] hence impurity scattering is treated as negligible in an intrinsic semiconductor. Further simplification comes from assuming that the temperature of the sample is high enough that the concentration of impurity electrons or holes is negligible in comparison to the concentration of intrinsic electrons and holes; in this case, one takes n_1, the electron concentration, equal to n_2, the hole concentration.

Since l_1 and l_2 are independent of the velocity v the terms A_1 and A_2 reduce to

$$A_1=n_1e\mu_1{}^0\int_0^\infty x^2e^{-x}(x+\gamma_1)^{-1}dx \tag{4.5}$$

and

$$A_2=n_2e\mu_2{}^0\int_0^\infty x^2e^{-x}(x+\gamma_2)^{-1}dx, \tag{4.6}$$

where $\mu_1{}^0$ and $\mu_2{}^0$ represent the electron and hole mobilities, respectively, in zero magnetic field, and

$$\gamma_{1,2}=(9\pi/16)(\mu_{1,2}{}^0H)^2. \tag{4.7}$$

Define

$$K_1(\gamma)\equiv\int_0^\infty x^2e^{-x}(x+\gamma)^{-1}dx$$
$$=1-\gamma+\gamma^2e^\gamma\{-\mathrm{Ei}(-\gamma)\}. \tag{4.8}$$

The terms B_1 and B_2 reduce to

$$B_{1,2}=n_{1,2}e\mu_{1,2}{}^0\gamma_{1,2}{}^{\frac{1}{2}}\int_0^\infty x^{\frac{1}{2}}e^{-x}(x+\gamma_{1,2})^{-1}dx. \tag{4.9}$$

Define

$$K_2(\gamma)\equiv\int_0^\infty x^{\frac{1}{2}}e^{-x}(x+\gamma)^{-1}dx$$
$$=\pi^{\frac{1}{2}}/2-\gamma\int_0^\infty x^{\frac{1}{2}}e^{-x}(x+\gamma)^{-1}dx. \tag{4.10}$$

The integral in $K_2(\gamma)$ has been evaluated numerically for a series of γ-values; the results are presented in Table IV along with the corresponding values of $K_1(\gamma)$.

Let c denote the ratio of electron to hole mobility, $\mu_1{}^0/\mu_2{}^0$, and, in addition, take $n_1=n_2=n$. Then it follows that

$$\gamma_1=c^2\gamma_2. \tag{4.11}$$

The resistivity at zero magnetic field is

$$\rho^0=(ne\mu_1{}^0+ne\mu_2{}^0)^{-1}. \tag{4.12}$$

The algebraic combination of Eqs. (4.3), (4.5), through (4.12) yields an expression for the magnetoresistive effect

$$\frac{\Delta\rho}{\rho^0}=\frac{\rho(H)-\rho^0}{\rho^0}=(c+1)\left[cK_1(\gamma_1)+K_1(\gamma_2)+\frac{\gamma_2\{c^2K_2(\gamma_1)-K_2(\gamma_2)\}^2}{cK_1(\gamma_1)+K_1(\gamma_2)}\right]^{-1}-1. \tag{4.13}$$

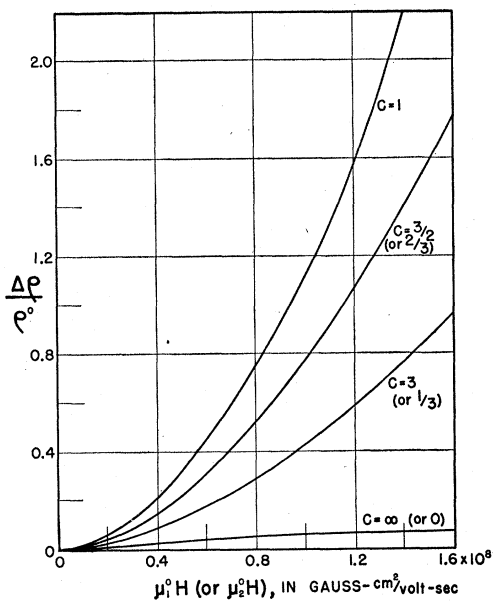

FIG. 4. Fractional increase in resistivity with magnetic field strength for an intrinsic semiconductor. The ratio of electron mobility to hole mobility is denoted by c. The abscissa is the product of magnetic field strength and electron mobility when $c=1$, $\frac{3}{2}$, 3, or ∞, or the product of magnetic field strength and hole mobility when $c=1$, $\frac{2}{3}$, $\frac{1}{3}$, or 0.

TABLE V. Relation between γ_1 and H for intrinsic germanium.[a]

γ_1	$\mu_1{}^0H$ gauss-cm²/volt-sec	H (gauss) at 295°K	H (gauss) at 625°K
0.01	0.07522×10^8	2250	6900
0.05	0.16821×10^8	5030	15 420
0.15	0.29135×10^8	8710	26 700
0.35	0.44504×10^8	13 300	40 800
0.70	0.62938×10^8	18 800	57 700
1.00	0.75225×10^8	22 500	69 000
2.00	1.06385×10^8	31 800	97 500
5.00	1.68209×10^8	50 300	154 300

[a] Values based upon taking $\mu_1{}^0=1.7\times10^6T^{-\frac{3}{2}}$ cm²/volt-sec, corresponding to $\mu_1{}^0=3300$ cm²/volt-sec at 300°K.

Inspection of Eq. (4.13) shows that $\Delta\rho/\rho^0$ depends upon the ratio c as well as upon the quantity γ_1 or $\mu_1{}^0H$. For this reason, the dependence of $\Delta\rho/\rho^0$ upon $\mu_1{}^0H$ is shown for four c values in Fig. 4. Because of the symmetry of terms arising from electron conduction and from hole conduction, one finds that $\Delta\rho/\rho^0$ for given c and $\mu_1{}^0H$ values is equal to $\Delta\rho/\rho^0$ for $1/c$ and $\mu_2{}^0H$. The curve in Fig. 4 for $c=\infty$ or 0 corresponds to the curve of Fig. 1 with $\beta=0$ (i.e., to Harding's curve). Thus, Fig. 4 indicates that the introduction of a second type of carrier causes an increase in the magnetoresistive effect, which becomes greatest when the two carriers have the same mobility.

Table V indicates the magnetic field intensities corresponding to various γ_1 values for intrinsic germanium at temperatures of 295°K and 625°K. Thus, relatively large magnetoresistive effects should be observed in pure samples which become intrinsic at comparatively low temperatures, whereas magnetic field strengths several times stronger would be required to produce the same $\Delta\rho/\rho^0$ value in an impure sample that becomes intrinsic only at very high temperatures.

V. HALL COEFFICIENT OF AN INTRINSIC SEMICONDUCTOR

The Hall coefficient of an intrinsic semiconductor is given by Eq. (4.4). By employing Eqs. (4.5) through (4.10) and taking $n_1=n_2=n$, one obtains an expression for the Hall coefficient

$$R=-\frac{3\pi^{\frac{1}{2}}}{4ne}\frac{c^2K_2(\gamma_1)-K_2(\gamma_2)}{\{cK_1(\gamma_1)+K_1(\gamma_2)\}^2+\gamma_2\{c^2K_2(\gamma_1)-K_2(\gamma_2)\}^2}. \tag{5.1}$$

For very weak magnetic fields, this equation reduces to

$$R_0=-\frac{3\pi}{8ne}\frac{c-1}{c+1}. \tag{5.2}$$

Thus the ratio R/R_0 becomes

$$\frac{R}{R_0}=\frac{2}{\pi^{\frac{1}{2}}}\frac{(c+1)}{(c-1)}\times\frac{c^2K_2(\gamma_1)-K_2(\gamma_2)}{\{cK_1(\gamma_1)+K_1(\gamma_2)\}^2+\gamma_2\{c^2K_2(\gamma_1)-K_2(\gamma_2)\}^2}. \tag{5.3}$$

The change in Hall coefficient with magnetic field may be described by

$$\Delta R/R_0=(R/R_0)-1. \tag{5.4}$$

This quantity depends upon the mobility ratio c as well as upon $\mu_1{}^0H$; in fact, $\Delta R/R_0$ for given c and $\mu_1{}^0H$ equals $\Delta R/R_0$ for $1/c$ and $\mu_2{}^0H$, in analog to the resistivity behavior. Figure 5 shows the dependence of $\Delta R/R_0$ upon $\mu_1{}^0H$ for several mobility ratios. As $\mu_1{}^0H$ increases, R decreases to a value about 85 percent of R_0. When $c=\infty$ (or 0), R approaches $0.85R_0$ asymptotically, as was found by Harding.[9] For other c values, the $\Delta R/R_0$ curve shows a minimum at about -0.15 with the position of the minimum moving to smaller $\mu_1{}^0H$ values as c goes toward unity. Although R_0 becomes zero as c approaches unity, the ratio $\Delta R/R_0$ has a determinate limit;

$$\lim_{c\to1}\left(\frac{\Delta R}{R_0}\right)=\frac{2}{\pi^{\frac{1}{2}}}\frac{K_2(\gamma_1)}{\{K_1(\gamma_1)\}^2}, \tag{5.5}$$

a positive value, which is indicated on Fig. 5.

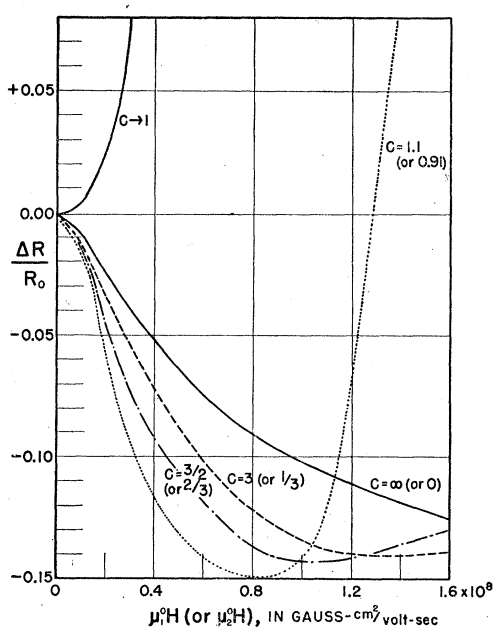

FIG. 5. Fractional change in Hall coefficient with magnetic field strength for an intrinsic semiconductor, given for several values of the ratio of electron mobility to hole mobility.

Thus, it is found that the calculated variation of resistivity and Hall coefficient with magnetic field intensity is very much greater for an intrinsic semiconductor than for an impurity semiconductor. At the present time experimental investigations of the magnetic field dependence of resistivity and Hall coefficient have not been published, and hence theory and experiment cannot be compared for such conditions.

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Tables for Second Born Approximation Scattering from Various Potential Fields

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By means of a variational approach, the scattering amplitude for electron scattering from various potentials is calculated. Numerical values for the functions involved are tabulated as a function of energy and scattering angle.

RECENTLY variational methods have been applied to scattering problems in nuclear physics; it was of interest to see with what success these methods would meet in problems on the atomic scale. At the same time it was felt that it was important to make available in tabular form the various functions involved. Here, scattering of a particle from various static potentials is considered. The variational principle used is essentially that as introduced by Schwinger and is in the form suited for determining scattering amplitudes as presented by Lax[1] and by Morse and Feshbach.[2] It is as follows:

$$J(\theta)=\frac{-\int e^{i\mathbf{k}_i\cdot\mathbf{r}}U(r)\psi_s^*(r)dv\cdot\int e^{-i\mathbf{k}_s\cdot\mathbf{r}_1}U(r_1)\psi_i(r_1)dv_1}{4\pi\int\psi_s^*(r)U(r)\psi_i(r)dv+\int\int\psi_s^*(r)U(r)(e^{ik|r-r_1|}/|r-r_1|)U(r_1)\psi_i(r_1)dv,\,dv_1}.$$

This quantity $J(\theta)$, the scattering amplitude which we wish to minimize, was calculated for various potentials $U(r)$; namely, an exponential, Yukawa, and a potential of the form $-2(U_0/a_0)(1/r+\lambda/2)\exp(-r/\lambda)$. The trial wave function which was used in the above expression for $J(\theta)$ was $\psi_i=\exp(i\mathbf{k}_i\cdot\mathbf{r})$ and $\psi_s^*=\exp(-i\mathbf{k}_s\cdot\mathbf{r})$, where $\mathbf{k}_i$ is the momentum vector in the direction of the incident wave, $\mathbf{k}_s$ is the momentum vector in the direction of the scattered wave, $|\mathbf{k}_s|^2=|\mathbf{k}_i|^2$, and θ is the angle between the two vectors.

Many of the integrals involved have been calculated previously; the evaluation of those integrals not readily

TABLE I. Algebraic values for $S_i(\theta)$, where $S_i(\theta)$ is essentially the ratio of the second to the first Born approximation for the indicated potentials.

Potential	$S_i(\theta)$
Yukawa $U(r)=U_0(\lambda/r)\exp(-\lambda r)$	$S_y=-U_0(1+x^2c^2)I_y$ [a]
Exponential $U(r)=U_0\lambda^2\exp(-\lambda r)$	$S_e=U_0(1+x^2c^2)^2I_e$
Mixed $U(r)=-U_0(2/a_0)(1/r+\lambda/2)\times\exp(-\lambda r)$	$S_m=U_0(2/\lambda a_0)(1+x^2c^2)^2I_m/(2+x^2c^2)$

where

$$I_y=(2/xcA)\left[\arctan\left(\frac{xc}{A}\right)+\frac{i}{2}\ln\left(\frac{A+x^2c}{A-x^2c}\right)\right],$$

$$I_e=a+ib+dI_y,$$

$$a=4/[c^2A^2(A^2+x^2c^2)]-(2+x^2)^2(4-8x^2-2x^4c^2)/A^4(A^2+x^2c^2)-2/(x^2c^2+4)^2,$$

$$b=x/A^2c^2(1+x^2)-3x(2+x^2)^2/A^4(1+x^2),$$

$$d=-2/A^2c^2+4/A^2-6(2+x^2)^2/A^4,$$

$$I_m=[1+2(x^2+2)/A^2]I_y+(4-x^4c^2)/A^2(A^2+x^2c^2)+ix(2+x^2)/A^2(1+x^2)-I_e/2,$$

and where we have also used the notation:

$$x=2k/\lambda,\qquad A^2=4+4x^2+x^4c^2,$$
$$c=\sin\theta/2,\qquad \cos\theta=\mathbf{k}_i\cdot\mathbf{k}_s/k^2,$$
$$k=|\mathbf{k}_i|=|\mathbf{k}_s|=(mc/\hbar)(2E/mc^2)^{\frac{1}{2}}.$$

E denotes the energy of the incident electrons, m their mass, and $\hbar$ Planck's constant.

[a] See reference 3.

TABLE II. Algebraic values for $S_i(\theta)$ for $\theta=0$ and for the potentials indicated.

Potential	$S_i(\theta)_{\theta=0}$
Yukawa	$S_y=-U_0(1+ix)/2(1+x^2)$
Exponential	$S_e=-2U_0[(15+10x^2+3x^4)/24(1+x^2)^3+ix(3+3x^2+x^4)/3(1+x^2)^3]$
Mixed	$S_m=(U_0/2a_0\lambda)[(75+106x^2+39x^4)/24(1+x^2)^3+ix(12+18x^2+7x^4)/3(1+x^2)^3]$

[1] M. Lax, Phys. Rev. 78, 306 (1950).
[2] P. M. Morse and H. Feshbach, *Methods of Theoretical Physics*, to be published.