The Beta-Decay of the Triton*

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A general formula is derived for the beta-decay matrix elements in the decay of a nucleus into its mirror nucleus. This formula is specialized to the decay of H^3. Under the assumption of charge independence of nuclear forces, it is shown that the value $(\int\sigma)^2=3$ computed on the basis of a pure $^2S_{\frac{1}{2}}$ ground state is an upper limit for the true value of this matrix element, provided only that the $^2S_{\frac{1}{2}}$ state is present with a probability of at least 25 percent.

1. INTRODUCTION

MANY years ago Wigner[1] pointed out that the wave functions of mirror nuclei should be very similar to each other. Indeed, if neutron-neutron forces equal proton-proton forces, and if the Coulomb forces can be treated validly as a small perturbation which changes the energy of the state without appreciably changing the wave function, then the wave functions of mirror nuclei are identical. In that case, the matrix elements for the beta-decay of a nucleus into its mirror nucleus can be computed without detailed knowledge of the wave function.[2]

If we assume full charge independence of nuclear forces, i.e., equality of *nn*, *pp*, and *np* forces, then the total isotopic spin T is a good quantum number. It is then easy to show that the matrix element usually denoted[3] by $(\int 1)^2$ is given by

$$\left(\int 1\right)^2=T(T+1)-T_\xi T_\xi'. \tag{1.1}$$

This equals unity for mirror nuclei, for which $T=\frac{1}{2}$, $T_\xi=\frac{1}{2}$, $T_\xi'=-\frac{1}{2}$.

Unfortunately, no such simple result holds for the matrix element $(\int\sigma)^2$ which arises from the Gamow-Teller interaction.[4] If the forces are spin-independent in addition to being charge-independent, then the supermultiplet theory of Wigner[5] can be applied. In addition to the total angular momentum J and the total isotopic spin T, there are then the following additional good quantum numbers: the partition quantum numbers P, P', P'', the intrinsic spin of the nucleus S, and the orbital angular momentum of the nucleus L. The value of $(\int\sigma)^2$ under these assumptions has been calculated by Wigner.[2]

Unfortunately, the present evidence indicates that the nuclear forces are probably charge independent but are certainly not spin-independent. Trigg[6] has generalized Wigner's calculation to the extent that the orbital angular momentum L of the nucleus is no longer considered a good quantum number. However, Trigg assumes no appreciable admixtures of states belonging to partitions other than the dominant $(\frac{1}{2},\frac{1}{2},\pm\frac{1}{2})$. Since the tensor force leads to admixtures of other partitions,[7] and since the tensor force is probably the main spin-dependent force, we thought it worth while to calculate the effect of admixtures of states belonging to other partitions.

We first derive a general expression for $(\int\sigma)^2$ under the assumption of equal wave functions for the two mirror nuclei, i.e., equality of *nn* and *pp* forces, but no assumption about *np* forces. We then specialize the formula to the case of full charge independence by omitting all states with isotopic spin $T\neq\frac{1}{2}$. The application of these results to the decay of the triton is particularly simple because there are only three nucleons present. The permutation group on three variables has only three different irreducible representations (partitions); hence, all states which can be present in H^3 can be enumerated easily, and the relevant matrix elements can be calculated explicitly. When this is done, assuming charge independence of the nuclear forces, it turns out that there are only two states which would tend to make the value of $(\int\sigma)^2$ exceed 3. Since these states have to be present to at least 75 percent of the total wave function before $(\int\sigma)^2$ exceeds 3, we conclude that the true value of $(\int\sigma)^2$ for the decay of H^3 is less than or equal to 3.

The calculations reported here can also be used for the decay of other mirror nuclei, although they are not complete (i.e., not all possible partitions are enumerated) for nuclei of mass number greater than 3. In practice, however, the partitions $(\frac{1}{2},\frac{1}{2},\pm\frac{1}{2})$ and $(\frac{3}{2},\frac{1}{2},\mp\frac{1}{2})$ are surely the most important ones, and the results given here are sufficient for the computation of their effects in all mirror nuclei decays.

An appendix contains the explicit forms of the spin-isotopic-spin wave functions for all states of the triton.

2. A GENERAL EXPRESSION FOR $(\int\sigma)^2$

Consider a decay between two "corresponding" states of mirror nuclei, each with total angular momentum J. The initial nucleus has a neutron excess $T_\xi=\frac{1}{2}(N-Z)$

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[1] E. P. Wigner, as quoted in White, Delsasso, Fox, and Creutz, Phys. Rev. **56**, 512 (1939).
[2] E. P. Wigner, Phys. Rev. **56**, 519 (1939).
[3] E. J. Konopinski, Revs. Modern Phys. **15**, 209 (1943).
[4] G. Gamow and E. Teller, Phys. Rev. **49**, 895 (1936).
[5] E. P. Wigner, Phys. Rev. **51**, 106 (1937).
[6] G. L. Trigg, Phys. Rev. **86**, 506 (1952).
[7] A. M. Feingold, thesis, Princeton University (1952) (unpublished).

$=+\frac{1}{2}$, the final nucleus has $T_\zeta'=-\frac{1}{2}$. The direction of the decay has no influence on the value of the matrix element. Let us define the "spherical components" of the Pauli spin vector $\boldsymbol{\sigma}^{(k)}$ for particle number k as follows:

$$\sigma_{+1}=-2^{-\frac{1}{2}}(\sigma_x+i\sigma_y),\quad \sigma_0=\sigma_z,\quad \sigma_{-1}=2^{-\frac{1}{2}}(\sigma_x-i\sigma_y). \tag{2.1}$$

The relevant operator for the allowed beta-decay with Gamow-Teller selection rules is the vector operator $\mathbf{Y}$ with the spherical components (A = mass number)

$$Y_\rho=\sum_{k=1}^{A}\sigma_\rho{}^{(k)}\tau_\xi{}^{(k)}, \tag{2.2}$$

where $\rho=1, 0, -1$, and τ_ξ is the conventional isotopic spin operator which changes a neutron into a proton and vice versa. Let the wave function of the initial state be Ψ_{JM}, the subscripts indicating the values of the total angular momentum and of its z component; the wave function of the final state is $\Phi_{JM'}$. Since the states are by assumption "corresponding" states, the value of J is the same, but M' may be any one of M, $M+1$, $M-1$. The quantity $(\int\boldsymbol{\sigma})^2$ is then defined as follows:

$$\left(\int\boldsymbol{\sigma}\right)^2=\sum_{\rho=-1}^{+1}\sum_{M'=-J}^{+J}|(\Phi_{JM'},\,Y_\rho\,\Psi_{JM})|^2. \tag{2.3}$$

The value of this double sum is, of course, independent of M, since a different choice of M corresponds to a different choice of the z direction in space, which cannot affect the beta-decay probability.

To simplify expression (2.3), we observe that the matrix elements of a vector operator such as $\mathbf{Y}$ depend upon the magnetic quantum numbers M, ρ, and M' in the same way as the Clebsch-Gordan (vector addition) coefficients $(1J\rho M\,|\,1JJM')$.[8] We use the notation of Condon and Shortley.[9] Thus we may restrict ourselves to particular values of M, ρ, and M'. We shall pick $M=M'=J$ and $\rho=0$ for ease of computation. Thus

$$(\Phi_{JM'},\,Y_\rho\,\Psi_{JM})=(1J_\rho M\,|\,1JJM')\frac{(\Phi_{JJ},\,Y_0\,\Psi_{JJ})}{(1J0J\,|\,1JJJ)}. \tag{2.4}$$

When we square this and sum over ρ and M', the sum of the squares of the Clebsch-Gordan coefficients is unity, so that we get

$$\left(\int\boldsymbol{\sigma}\right)^2=\left|\frac{(\Phi_{JJ},\,Y_0\,\Psi_{JJ})}{(1J0J\,|\,1JJJ)}\right|^2. \tag{2.5}$$

In order to get farther, we have to introduce a more detailed classification of the possible states. The states are linear superpositions of elementary states. Each elementary state has a definite orbital angular momentum L with z component m, a definite spin angular momentum S with z component μ, a definite isotopic spin quantum number T, and belongs to a definite row κ of some partition (P, P', P'') of the permutation group. For the sake of conciseness, we shall use the symbol P to denote all three quantities P, P', P''. We denote the space wave function belonging to row number κ of partition P and to an orbital angular momentum L, m by $\psi_{m\kappa}(L, P;\,\mathbf{r}_1\mathbf{r}_2,\,\cdots,\,\mathbf{r}_A)$. We denote the spin-isotopic-spin wave function belonging to row κ of the partition adjoint to P, to spin S with z component μ, to isotopic spin quantum number T, and neutron excess $T_\zeta=+\frac{1}{2}$, by $V_{\mu\kappa}(P, T, S)$. Explicit expressions for the functions V in the case of three particles are contained in the appendix, and the reader unfamiliar with the representations of the permutation group is urged to read the appendix before proceeding with the rest of this paper. The spin-isotopic-spin wave functions for neutron excess $T_\zeta'=-\frac{1}{2}$ will be denoted $U_{\mu\kappa}(P, T, S)$. They are obtained from the $V_{\mu\kappa}(P, T, S)$ by a systematic replacement of neutron wave functions by proton wave functions and vice versa (see the appendix).

We may now write the wave function Ψ_{JM} of the initial state as a linear superposition of wave functions $\Psi_{JM}(P, L, T, S)$ with definite values of the partition, orbital angular momentum, isotopic spin quantum number T, and spin quantum number S. We denote the probability amplitudes by α_{PLTS}:

$$\Psi_{JM}=\sum_{P,\,L,\,T,\,S}\alpha_{PLTS}\,\Psi_{JM}(P, L, T, S). \tag{2.6}$$

We shall normalize the functions $\Psi_{JM}(P, L, T, S)$ to unity; furthermore, they can be chosen in such a way that the coefficients α are all real numbers,[10] and we shall make such a choice. Thus, the quantities α are real and the sum of their squares is unity.

The functions $\Psi_{JM}(P, L, T, S)$ must be antisymmetric under the interchange of any two particles. Let f be the dimension of partition P. Then

$$\Psi_{JM}(P, L, T, S)=f^{-\frac{1}{2}}\sum_{\kappa=1}^{f}\sum_{m=-L}^{+L}\sum_{\mu=-S}^{+S}\times(LSm\mu\,|\,LSJM)\psi_{m\kappa}(L, P)V_{\mu\kappa}(P, T, S). \tag{2.7}$$

We now turn to the wave function of the final state, $\Phi_{JM'}$. At this point we assume explicitly that neutron-neutron forces equal proton-proton forces, and that the effect of the Coulomb forces on the wave function is negligible (even though there may be a measurable effect on the energy of the state). This means that the probability amplitudes α are the same as in Ψ_{JM} (these amplitudes are, of course, independent of M in any

[8] E. P. Wigner, *Gruppentheorie und ihre Anwendung auf die Quantenmechanik der Atomspektren* (Vieweg and Son, Braunschweig, 1931, reprinted by Edwards Brothers, Ann Arbor, Michigan, 1944).

[9] E. U. Condon and G. H. Shortley, *Theory of Atomic Spectra* (Cambridge University Press, Cambridge, 1935). Unlike Condon and Shortley, the Clebsch-Gordan coefficients $(jj'mm'\,|\,jj'JM)$ used here are *defined* to vanish identically unless $m+m'=M$. Thus, a double summation over m and m', say, really reduces to a single sum.

[10] E. P. Wigner, Gött Nachr. **31**, 546 (1932).

case), and furthermore, that the space wave functions $\psi_{m\kappa}(L, P; \mathbf{r}_1, \mathbf{r}_2, \cdots, \mathbf{r}_A)$ are the same as the ones occurring in the initial state. Thus, the integrations over the space coordinates can all be reduced to the orthogonality and normalization integrals for the functions $\psi_{m\kappa}(L, P; \mathbf{r}_1, \mathbf{r}_2, \cdots, \mathbf{r}_A)$, which form an orthonormal set. Hence, we can compute the beta-decay matrix element without any detailed knowledge of the wave functions; indeed, all we shall need to know are the probability amplitudes α_{PLTS}.

We are interested in the matrix element $(\Phi_{JJ}, Y_0 \Psi_{JJ})$ which occurs in (2.5). The operator Y_0 does not involve any operations on space coordinates. Hence, this matrix element vanishes unless L and m are the same for the initial and final states. Furthermore, Y_0 is invariant under permutations of the particles [see (2.2)]. Thus, the matrix element vanishes unless the partition P and the row index κ are the same for the initial and final state. Since Y_0 is a component of a vector operator as far as the spin S is concerned, S' may differ from S by at most one unit. Similarly T' may differ from T by at most one unit. We, therefore, obtain

$$(\Phi_{JJ}, Y_0 \Psi_{JJ}) = \sum_P \sum_L \sum_{T'S'} \sum_{TS} \alpha_{PLT'S'} \alpha_{PLTS} \times (\Phi_{JJ}(PLT'S'), Y_0 \Psi_{JJ}(PLTS)), \quad (2.8)$$

where

$$(\Phi_{JJ}(PLT'S'), Y_0 \Psi_{JJ}(PLTS)) = f^{-1} \sum_{\kappa=1}^{f} \sum_{m=-L}^{+L} \sum_{\mu'=-S'}^{+S'} \sum_{\mu=-S}^{+S} (LS'm\mu' | LS'JJ) \times (LSm\mu | LSJJ)\, (U_{\mu'\kappa}(PT'S'), Y_0 V_{\mu\kappa}(PTS)). \quad (2.9)$$

One of the sums over μ and μ' is spurious, since μ must equal μ'.

Next we use the fact that the matrix element of a symmetric operator such as Y_0 is independent of the row index κ. Hence, we may omit the sum over κ provided we also omit the factor f^{-1} in front. Furthermore, Y_0 is the component with $\rho=0$ of a vector operator as far as the spin S is concerned. Thus, the dependence of the matrix element on the right side of (2.9) upon μ and μ' is given by the Clebsch-Gordan coefficients $(1S0\mu | 1SS'\mu')$. All these matrix elements may, therefore, be expressed in terms of the one with $\mu=\mu'=\frac{1}{2}$ (we recall that mirror nuclei beta-decays always involve odd mass number nuclei), as follows

$$(U_{\mu'\kappa}(PT'S'), Y_0 V_{\mu\kappa}(PTS)) = (1S0\mu | 1SS'\mu') \frac{(U_{\frac{1}{2},\kappa}(PT'S'), Y_0 V_{\frac{1}{2},\kappa}(PTS))}{(1S0\tfrac{1}{2} | 1SS'\tfrac{1}{2})}. \quad (2.10)$$

When we substitute (2.10) into (2.9), the sums over m, μ, and μ' can be evaluated explicitly by methods due to Racah.[11] We obtain

$$\sum_{m,\mu,\mu'} (LS'm\mu' | LS'JJ)\, (LSm\mu | LSJJ)\, (1S0\mu | 1SS'\mu') = (-1)^{J+S+L+1} (2J+1)^{\frac{1}{2}} (2S'+1)^{\frac{1}{2}} \times W(SJS'J, L1)\, (1J0J | 1JJJ), \quad (2.11)$$

where the W are the Racah coefficients defined in reference 11, and the last factor just cancels the Clebsch-Gordan coefficient which appears in Eq. (2.5). We define the following quantity:

$$Q(P, T'S', TS) = \frac{(2S'+1)^{\frac{1}{2}} (U_{\frac{1}{2},\kappa}(PT'S'), Y_0 V_{\frac{1}{2},\kappa}(PTS))}{(1S0\tfrac{1}{2} | 1SS'\tfrac{1}{2})}, \quad (2.12)$$

with the symmetry property

$$Q(P, TS, T'S') = (-1)^{S'-S} Q(P, T'S', TS). \quad (2.13)$$

$Q(P, T'S', TS)$ may be considered a matrix with rows labeled by $T'S'$ and columns labeled by T, S. There is one such matrix for every possible partition $P=(P, P', P'')$.

We now combine (2.5), (2.8), (2.9), (2.10), (2.11), and (2.12) to obtain the *final result:*

$$\left(\int \sigma\right)^2 = \left[\sum_P \sum_L \sum_{T'S'} \sum_{TS} (-1)^{J+S+L+1} \times (2J+1)^{\frac{1}{2}} \alpha_{PLT'S'} \alpha_{PLTS} \times W(SJS'J, L1) Q(P, T'S', TS)\right]^2. \quad (2.14)$$

We first show that this expression reduces to the one derived by Trigg[6] if we assume that there is no admixture of states belonging to partitions other than the dominant $P=(P, P', P'')=(\frac{1}{2}, \frac{1}{2}, \pm\frac{1}{2})$. That is, we assume that the probability coefficients α_{PLTS} are zero unless P is this particular partition. Since this partition contains only one multiplet, namely, $S=T=\frac{1}{2}$, the sums over P, T', S', T, and S in (2.14) all reduce to one term. Furthermore, for $S=\frac{1}{2}$ there are only two possible values of L, namely, $L=J+\frac{1}{2}$ and $L=J-\frac{1}{2}$. It is shown in the appendix that $Q=-\sqrt{6}$ for this partition, so that we get, with the notation $\alpha_{PL\frac{1}{2}\frac{1}{2}}^2 = p_L$ (= the probability of finding orbital angular momentum L),

$$\left(\int \sigma\right)^2 = 6(2J+1)\left[\sum_{L=J-\frac{1}{2}}^{J+\frac{1}{2}} (-1)^L W(\tfrac{1}{2}, J, \tfrac{1}{2}J; L, 1) p_L\right]^2 = [J(J+1)]^{-1} \{\sum_{L=J-\frac{1}{2}}^{J+\frac{1}{2}} [J(J+1) + \tfrac{3}{4} - L(L+1)] p_L\}^2. \quad (2.15)$$

This result is the same as the one given by Trigg. As was

[11] G. Racah, Phys. Rev. **62**, 438 (1942).

mentioned in the introduction, the applicability of this formula is somewhat questionable. As long as the forces are both spin-independent and charge-independent, the orbital angular momentum L is a good quantum number and Wigner's calculation is applicable. It is believed at present that the forces are charge-independent to a good approximation, and that the main deviation from spin-independence is produced by the tensor force. Now, the tensor force has the property that its matrix elements are zero between states belonging to the partition $(\frac{1}{2}, \frac{1}{2}, \pm\frac{1}{2})$. Thus, the one additional state considered by Trigg is *not* admixed in first order by the tensor force. Rather, the tensor force admixes in first order a term belonging to partition $(\frac{3}{2}, \frac{1}{2}, \mp\frac{1}{2})$ with $T=\frac{1}{2}$ and $S=\frac{3}{2}$. For example, the main admixture to the dominant[12] $(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$ $^{22}S_{\frac{1}{2}}$ state of H^3 is of type $(\frac{3}{2}, \frac{1}{2}, \frac{1}{2})$ $^{24}D_{\frac{1}{2}}$ (this is the state considered by Gerjuoy and Schwinger),[13] rather than of the type $(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$ $^{22}P_{\frac{1}{2}}$ implied by the use of Trigg's expression (2.15). In this respect the triton is typical of all the mirror nuclei. Thus, Trigg's analysis of the data, which is based on Eq. (2.15), is open to some doubt.

We now proceed to give the values of Q, (2.12), for the partitions occurring in the wave function of the triton. The computation of these values of Q is outlined in the appendix. The dominant partition is $(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$ corresponding to a completely symmetric space wave function and a completely antisymmetric spin-isotopic-spin wave function. This partition gives rise to only one multiplet, namely, $T=S=\frac{1}{2}$, and

$$Q(P, \tfrac{1}{2}\tfrac{1}{2}, \tfrac{1}{2}\tfrac{1}{2}) = -\sqrt{6} \quad \text{for} \quad P=(\tfrac{1}{2}, \tfrac{1}{2}, -\tfrac{1}{2}). \tag{2.16}$$

The next partition in order of importance is $(\frac{3}{2}, \frac{1}{2}, \frac{1}{2})$ with the three multiplets $T=S=\frac{1}{2}$; $T=\frac{1}{2}$, $S=\frac{3}{2}$; and $T=\frac{3}{2}$, $S=\frac{1}{2}$. We write Q as a matrix with rows labeled by $T'S'$ and columns labeled by TS:

Partition $(\frac{3}{2}, \frac{1}{2}, \frac{1}{2})$

$T'S'$ \ TS	$\frac{1}{2}\ \frac{1}{2}$	$\frac{1}{2}\ \frac{3}{2}$	$\frac{3}{2}\ \frac{1}{2}$
$\frac{1}{2}\ \frac{1}{2}$	$\sqrt{2/3}$	$4\sqrt{2/3}$	$-2\sqrt{2/3}$
$\frac{1}{2}\ \frac{3}{2}$	$-4\sqrt{2/3}$	$2\sqrt{5/3}$	$2\sqrt{2/3}$
$\frac{3}{2}\ \frac{1}{2}$	$-2\sqrt{2/3}$	$-2\sqrt{2/3}$	$-2\sqrt{2/3}$

(2.17)

If the forces are completely charge-independent (so far we have assumed only the equality of *nn* and *pp* forces), the total isotopic spin T is a good quantum number. In that case $T=T'=\frac{1}{2}$ and the last row and last column of the matrix (2.17) should be omitted.

The last partition which can occur in the triton is $(\frac{3}{2}, \frac{3}{2}, \frac{3}{2})$ which corresponds to a completely antisymmetric space function and a completely symmetric spin-isotopic-spin function. This partition gives rise to two multiplets, $T=S=\frac{1}{2}$ and $T=S=\frac{3}{2}$. Only the first of these occurs if the forces are charge-independent. The matrix Q is given by

Partition $(\frac{3}{2}, \frac{3}{2}, \frac{3}{2})$

$T'S'$ \ TS	$\frac{1}{2}\ \frac{1}{2}$	$\frac{3}{2}\ \frac{3}{2}$
$\frac{1}{2}\ \frac{1}{2}$	$5\sqrt{2/3}$	$-4\sqrt{1/3}$
$\frac{3}{2}\ \frac{3}{2}$	$4\sqrt{1/3}$	$-4\sqrt{(5/3)}$

(2.18)

The rest of this paper will be devoted to the triton decay. We would like to point out, however, that the partition $(\frac{3}{2}, \frac{1}{2}, \mp\frac{1}{2})$ is the dominant admixture produced by the tensor force in *all* mirror nuclei, so that (2.16) and (2.17) together with (2.14) give probably a more adequate description of mirror nuclei decays than Trigg's expression (2.15). Unfortunately, there are in general two or more states admixed in first order by the tensor force (the triton is an exception to this rule), making the analysis much more difficult in practice.

3. THE DECAY OF THE TRITON

We shall assume from the outset that the forces are charge-independent so that states with isotopic spin $T=\frac{3}{2}$ can be omitted. There are then the following states possible:[14]

Partition $(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$:	$^{22}S_{\frac{1}{2}}$	$^{22}P_{\frac{1}{2}}$,			
Partition $(\frac{3}{2}, \frac{1}{2}, \frac{1}{2})$:	$^{22}S_{\frac{1}{2}}$	$^{22}P_{\frac{1}{2}}$	$^{24}P_{\frac{1}{2}}$	$^{24}D_{\frac{1}{2}}$,	(3.1)
Partition $(\frac{3}{2}, \frac{3}{2}, \frac{3}{2})$:	$^{22}S_{\frac{1}{2}}$	$^{22}P_{\frac{1}{2}}$.			

The first state listed is the dominant state, and is the only one present if the forces are spin- and charge-independent. The tensor force admixes in first order only the $(\frac{3}{2}, \frac{1}{2}, \frac{1}{2})$ $^{24}D_{\frac{1}{2}}$ state. In the next order all states are admixed, but states belonging to the completely antisymmetric partition $(\frac{3}{2}, \frac{3}{2}, \frac{3}{2})$ are high up in energy and consequently are not admixed to any appreciable extent.

For convenience of notation, we shall number the partitions by Roman numerals I, II, III. Since all states considered here have $T=\frac{1}{2}$ and $J=\frac{1}{2}$, we shall omit those symbols. Hence, the first state in (3.1) will be denoted by I^2S, the last state in the second line of (3.1) by II^4D, and so on. Their probability amplitudes will be denoted by $\alpha(\mathrm{I}^2S)$ and $\alpha(\mathrm{II}^4D)$, respectively, and their squares, the probabilities themselves, by $p(\mathrm{I}^2S)$ and $p(\mathrm{II}^4D)$. With this notation the matrix element $(\int\boldsymbol{\sigma})^2$ for the triton decay is

$$\begin{aligned}\left(\int\boldsymbol{\sigma}\right)^2 = 3\{&p(\mathrm{I}^2S) - \tfrac{1}{3}p(\mathrm{I}^2P) - \tfrac{1}{3}p(\mathrm{II}^2S) \\ &+\tfrac{1}{9}[p(\mathrm{II}^2P) - 5p(\mathrm{II}^4P) - 16\alpha(\mathrm{I}^2P)\alpha(\mathrm{II}^4P)] \\ &+\tfrac{1}{3}p(\mathrm{II}^4D) \ -(5/3)p(\mathrm{III}^2S) \\ &+(5/9)p(\mathrm{III}^2P)\}^2.\end{aligned} \tag{3.2}$$

In the special case that the II^4D state is the *only* appreciable admixture, so that $p(\mathrm{I}^2S)=1-p(\mathrm{II}^4D)$, ex-

[12] The two superscripts on the term symbols denote the multiplicities $2T+1$ and $2S+1$, in this order.

[13] E. Gerjuoy and J. S. Schwinger, Phys. Rev. **61**, 138 (1942).

[14] The two superscripts on the left side of each state stand for the multiplicities $2T+1$ and $2S+1$, in that order.

pression (3.2) reduces to[15]

$$\left(\int\sigma\right)^2=3[1-\tfrac{2}{3}p(\mathrm{II}^4D)]^2. \tag{3.3}$$

This quantity is less than or equal to 3, no matter what the amount of admixture is. A D-state admixture of 4 percent follows from the measured magnetic moments of H^3 and He^3, if the exchange magnetic moments are assumed to be equal and opposite in these two nuclei, and if relativistic effects on the magnetic moments are ignored, the latter assumption being a rather doubtful one. If we assume a 4 percent D-state admixture for the sake of argument, the value of $(\int\sigma)^2$ is changed by roughly 5 percent, becoming 2.84 instead of 3.00. Such a 5 percent change is not much smaller than the experimental uncertainty in the ft value of the triton decay.[16] Since the amount of D-state admixture is by no means well known, and since other states may be admixed appreciably also, the value of $(\int\sigma)^2$ is quite uncertain.

We would like to point out that admixtures of $(\frac{3}{2},\frac{1}{2},\mp\frac{1}{2})$ states of the same order of magnitude, or even larger, may be expected in all mirror nuclei. It is quite likely that these are the dominant admixtures to the usually considered $(\frac{1}{2},\frac{1}{2},\pm\frac{1}{2})P$ states in the odd nuclei with mass numbers between 7 and 15. If this conjecture is correct, one should not attempt to make too detailed an analysis of the ft values of these mirror nuclei without taking this correction into account.[17] It is not apparent that the admixture correction is negligible even for those mirror nuclei which have one more or one less nucleon than some closed shell. Indeed, the triton satisfies this condition, yet the admixture correction to the matrix element is probably of the order of 5 percent, and may be a lot more than that if the magnetic moments of H^3 and He^3 are corrected for relativistic effects, which would change the D-state probability. Unfortunately, we do not know at this time how to make this correction, not even the direction of the relativistic effect; all we know is that the relativistic effect on the magnetic moment is *not* negligible compared to the admixture effect, which is not much consolation.

In view of the uncertainty in the actual value of $(\int\sigma)^2$, we think it may be useful to point out that, under very reasonable assumptions about the wave function of H^3, the true value of $(\int\sigma)^2$ is no larger than the elementary value 3. A look at expression (3.2) shows that there are only two states which can lead to larger matrix elements. One of them is the III^2S state, which would lead to $(\int\sigma)^2=25/3$ if it were the only state present. The other state is a linear combination of II^2P and II^4P obtained by diagonalizing the quadratic form in the brackets of (3.2). The eigenvalues of this quadratic form are (including the factor $\frac{1}{9}$ in front) $+0.727$ and -1.172. Only the latter of these is larger than unity. We clearly get the most unfavorable estimate if we assume that the two bad states are the *only* admixtures. Let p_1 stand for the probability of the particular mixture of II^2P and II^4P involved here, and denote $p(\mathrm{III}^2S)$ by p_2. Then $p(\mathrm{I}^2S)=1-p_1-p_2$ by assumption, and we get from (3.2)

$$\left(\int\sigma\right)^2=3(1-2.172p_1-2.667p_2)^2. \tag{3.4}$$

This can exceed 3 only if the following inequality is satisfied:

$$1.086p_1+1.333p_2\geqslant 1. \tag{3.5}$$

If p_1 is the only admixture (i.e., $p_2=0$), the state involved has to be present to the extent of at least 92 percent. The situation is a little more favorable for the III^2S state; it would have to be present to only (!) 75 percent. Conversely, *if the dominant* I^2S *state is present to at least 25 percent probability, the true value of* $(\int\sigma)^2$ *does not exceed 3.*

Because of this theorem it is very much worth while to measure the comparative half-life (ft value) of the triton decay with high accuracy, even though the true value of $(\int\sigma)^2$ is not known theoretically to anything like comparable accuracy. For such a measurement will allow us to put a very accurate *upper limit* on the product $(\int\sigma)^2ft$, which would help considerably in narrowing down the beta-decay interaction. This point is discussed in more detail in the preceding paper.

APPENDIX. SPIN- AND ISOTOPIC-SPIN FUNCTIONS FOR THE TRITON

Let α be the spin function for a particle with spin up, β the spin function for a particle with spin down. In accordance with expression (2.10) we may restrict ourselves to spin functions with $S_z=\mu=\frac{1}{2}$. There are three linearly independent functions, namely, $\alpha(1)\alpha(2)\beta(3)$, $\alpha(1)\beta(2)\alpha(3)$, and $\beta(1)\alpha(2)\alpha(3)$. These three functions form an orthonormal set. We now construct another orthonormal set with the property that members of the new set transform according to irreducible representations of the permutation group. These new functions are

$$\begin{aligned}
q_1&=6^{-\frac{1}{2}}[\beta(1)\alpha(2)+\alpha(1)\beta(2)]\alpha(3)\\
&\qquad -(2/3)^{\frac{1}{2}}\alpha(1)\alpha(2)\beta(3),\\
q_2&=2^{-\frac{1}{2}}[\beta(1)\alpha(2)-\alpha(1)\beta(2)]\alpha(3),\\
q_3&=3^{-\frac{1}{2}}[\alpha(1)\alpha(2)\beta(3)+\alpha(1)\beta(2)\alpha(3)\\
&\qquad +\beta(1)\alpha(2)\alpha(3)].
\end{aligned} \tag{A.1}$$

[15] This expression was first calculated by E. Feenberg (quoted in reference 6); a trivial mistake in sign in the formula quoted in reference 6 was corrected by Professor Feenberg in a private communication to the author.

[16] For references regarding the end-point energy and lifetime of the triton, see Hornyak, Lauritsen, Morrison, and Fowler, Revs. Modern Phys. **22**, 291 (1950).

[17] R. Nataf and R. Bouchez, Phys. Rev. **87**, 155 (1952); R. Bouchez and R. Nataf, Compt. rend. **234**, 86 (1952); O. Kofoed-Hansen and A. Winther, Phys. Rev. **86**, 428 (1952); and unpublished work by the same authors.

The function q_3 is completely symmetric under permutations of 1, 2, 3. It is the $S_z=\frac{1}{2}$ function belonging to the multiplet $S=\frac{3}{2}$ (quartet state). Since the completely symmetric representation of the permutation group is one-dimensional, there is only one such function, which transforms into itself under all permutations.

The functions q_1 and q_2 are necessary to describe the doublet ($S=\frac{1}{2}$) spin state with $S_z=\frac{1}{2}$. It should be noted that the complete doublet state for three particles cannot be written as the product of a space wave function and a spin wave function, but rather it is the sum of two such products. This should not be too surprising since, in the usual way of writing such states by means of Slater determinants, the wave function appears as a sum of even more products (the products are obtained by writing out the determinant explicitly). Under the permutation which exchanges coordinates 1 and 2, q_1 transforms into itself whereas q_2 transforms into its negative. But under all other permutations (except the identity, of course) q_1 and q_2 transform into linear combinations of each other. The matrices corresponding to the various permutations are given in Wigner's book[8] on page 63, with the notation given at the bottom of page 71.

There is no completely antisymmetric spin function among the set (A.1), corresponding to the fact that it is impossible to put three neutrons into the same space state (which would mean a completely symmetric space wave function).

If we were considering a nucleus containing three neutrons, say, the spin function q_3 would have to be multiplied by a completely antisymmetric space wave function to give an acceptable total wave function for the quartet spin state. To get a wave function for the doublet state of three neutrons, we would have to construct two space wave functions ψ_1 and ψ_2 which transform analogously to q_1 and q_2, and use them to construct a sum of two products of type $2^{-\frac{1}{2}}(\psi_1 q_2-\psi_2 q_1)$ which would then be completely antisymmetric also.

The situation is appreciably more complicated for the triton with its two neutrons and one proton. Let ν be the isotopic spin wave function for the neutron state, π be the isotopic spin wave function for the proton state. ν corresponds to $T_z=+\frac{1}{2}$, π to $T_z=-\frac{1}{2}$. We now construct three orthonormal base functions for the triton analogous to (A.1):

$$\begin{aligned} v_1&=6^{-\frac{1}{2}}[\pi(1)\nu(2)+\nu(1)\pi(2)]\nu(3)\\ &\qquad -(2/3)^{\frac{1}{2}}\nu(1)\nu(2)\pi(3),\\ v_2&=2^{-\frac{1}{2}}[\pi(1)\nu(2)-\nu(1)\pi(2)]\nu(3),\\ v_3&=3^{-\frac{1}{2}}[\nu(1)\nu(2)\pi(3)+\nu(1)\pi(2)\nu(3)\\ &\qquad +\pi(1)\nu(2)\nu(3)]. \end{aligned} \tag{A.2}$$

We must now find linear combinations of products of q's and v's which transform under permutations of the spin and isotopic spin coordinates *jointly* according to the three possible irreducible representations of the permutation group (two of these three representations are exemplified by the set (v_1, v_2) and by the one function v_3, respectively; the third representation is exemplified by a completely antisymmetric function).

We start with the completely symmetric spin-isotopic-spin functions which go along with completely antisymmetric space wave functions. Clearly one of these symmetric spin-isotopic-spin functions is the product v_3q_3 which corresponds to isotopic spin $T=\frac{3}{2}$ (because of v_3) and mechanical spin $S=\frac{3}{2}$ (because of q_3). The only other symmetric function is $2^{-\frac{1}{2}}(v_1q_1+v_2q_2)$, corresponding to isotopic spin $T=\frac{1}{2}$ (because of the occurrence of v_1 and v_2) and to mechanical spin $S=\frac{1}{2}$ (because of the occurrence of q_1 and q_2). Thus, the supermultiplet $(P, P', P'')=(\frac{3}{2}, \frac{3}{2}, \frac{3}{2})$ (meaning a completely antisymmetric space wave function) gives rise to two multiplets, namely, $T=S=\frac{3}{2}$ and $T=S=\frac{1}{2}$, with the spin-isotopic-spin wave functions $V_{\frac{1}{2},\kappa}(P, T, S)$:[18]

$$V(\mathrm{III};\tfrac{1}{2},\tfrac{1}{2})=2^{-\frac{1}{2}}(v_1q_1+v_2q_2), \tag{A.3}$$

$$V(\mathrm{III};\tfrac{3}{2},\tfrac{3}{2})=v_3q_3. \tag{A.4}$$

There is only one completely antisymmetric spin-isotopic-spin wave function, namely [we use the Roman numeral I for the partition $(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$],

$$V(\mathrm{I};\tfrac{1}{2},\tfrac{1}{2})=2^{-\frac{1}{2}}(v_1q_2-v_2q_1). \tag{A.5}$$

It is easy to verify that this is indeed the properly normalized Slater determinant corresponding to three nucleons in the states $\nu\alpha$, $\pi\alpha$, and $\nu\beta$, respectively.

Finally, we write down the functions belonging to partition $(\frac{3}{2}, \frac{1}{2}, \frac{1}{2})$, which we shall denote by the Roman numeral II. There are two functions for each multiplet, which will be distinguished by the subscript $\kappa=1, 2$. The functions V_κ for the multiplet $T=S=\frac{1}{2}$ are

$$\begin{aligned} V_1(\mathrm{II};\tfrac{1}{2},\tfrac{1}{2})&=-2^{-\frac{1}{2}}(v_1q_1-v_2q_2),\\ V_2(\mathrm{II};\tfrac{1}{2},\tfrac{1}{2})&=2^{-\frac{1}{2}}(v_1q_2+v_2q_1). \end{aligned} \tag{A.6}$$

The signs are chosen in such a way that V_1 and V_2 transform into each other under permutations precisely in the same way as q_1 and q_2 of (A.1), or as v_1 and v_2 of (A.2).

The second multiplet in this partition is $T=\frac{1}{2}$, $S=\frac{3}{2}$ with the spin-isotopic-spin functions

$$V_1(\mathrm{II};\tfrac{1}{2},\tfrac{3}{2})=v_1q_3, \quad V_2(\mathrm{II};\tfrac{1}{2},\tfrac{3}{2})=v_2q_3. \tag{A.7}$$

Finally, we have the multiplet $T=\frac{3}{2}$, $S=\frac{1}{2}$ with the functions

$$V_1(\mathrm{II};\tfrac{3}{2},\tfrac{1}{2})=v_3q_1, \quad V_2(\mathrm{II};\tfrac{3}{2},\tfrac{1}{2})=v_3q_2. \tag{A.8}$$

Equations (A.3) through (A.8) contain nine different and mutually orthogonal functions. Since there are only nine linearly independent products v_iq_j we have exhausted all possibilities.

[18] We omit the common index $\mu=S_z=\frac{1}{2}$, and denote the partition $(\frac{3}{2}, \frac{3}{2}, \frac{3}{2})$ by the Roman numeral III for convenience as was also done in Sec. 3 of the paper. Since the representation is one-dimensional, we can omit the index κ.

We have written down these functions without indicating any method of constructing them systematically. The easiest way is by means of the Young symmetry operators (see Van der Waerden,[19] Sec. 129, or Weyl,[20] Chapter 5C). However, a detailed exposition of this method is beyond the scope of this appendix.

The spin-isotopic-spin functions U for the nucleus He^3 are obtained simply by replacing v_1, v_2, v_3 everywhere by u_1, u_2, u_3 which are defined analogously:

$$\begin{aligned} u_1&=6^{-\frac{1}{2}}[\nu(1)\pi(2)+\pi(1)\nu(2)]\pi(3)\\ &\qquad -(2/3)^{\frac{1}{2}}\pi(1)\pi(2)\nu(3),\\ u_2&=2^{-\frac{1}{2}}[\nu(1)\pi(2)-\pi(1)\nu(2)]\pi(3), \qquad (A.9)\\ u_3&=3^{-\frac{1}{2}}[\pi(1)\pi(2)\nu(3)+\pi(1)\nu(2)\pi(3)\\ &\qquad +\nu(1)\pi(2)\pi(3)]. \end{aligned}$$

In order to calculate the matrix elements (U, Y_0V) which appear in the numerator of (2.12), we note first of all that Y_0 is a symmetric operator under permutations and, therefore, does not have matrix elements connecting different partitions. Within the same partition it connects only functions with the same value of κ, and furthermore (U_κ, Y_0V_κ) is independent of κ. As far as the z component of the mechanical spin is concerned, Y_0 connects only states with the same $S_z=\mu$, which we have chosen to be $+\frac{1}{2}$. From here on the calculation is trivial though slightly tedious, leading to the following matrices for (U, Y_0V):

Partition I: $(U, Y_0V)=1$

Partition II:

$T'S'$ \ TS	$\frac{1}{2}\,\frac{1}{2}$	$\frac{1}{2}\,\frac{3}{2}$	$\frac{3}{2}\,\frac{1}{2}$
$\frac{1}{2}\,\frac{1}{2}$	$-1/3$	$-4/3$	$2/3$
$\frac{1}{2}\,\frac{3}{2}$	$-4/3$	$-1/3$	$2/3$
$\frac{3}{2}\,\frac{1}{2}$	$2/3$	$2/3$	$2/3$

Partition III:

$T'S'$ \ TS	$\frac{1}{2}\,\frac{1}{2}$	$\frac{3}{2}\,\frac{3}{2}$
$\frac{1}{2}\,\frac{1}{2}$	$-5/3$	$8\frac{1}{2}/3$
$\frac{3}{2}\,\frac{3}{2}$	$8\frac{1}{2}/3$	$2/3$

(A.10)

Combination of (A.10) and (2.12) then gives the matrices $Q(P, T'S', TS)$ listed in Sec. 2 of the paper.

[19] B. L. Van der Waerden, *Moderne Algebra* (Julius Springer, Berlin, 1940); reprinted by F. Ungar Publishing Company, New York (1943).

[20] H. Weyl, *The Theory of Groups and Quantum Mechanics* (Methuen and Company, London, 1931).

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The Lamb Shift for Hydrogen and Deuterium

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Previous calculations of all second- and fourth-order radiative corrections, which contribute to the Lamb shift, are discussed. Further corrections to the Lamb shift in hydrogen and deuterium, resulting from the finite mass and internal structure of the nuclei, are calculated. The final theoretical results for the Lamb shift, excluding all sixth-order corrections, are (1057.19±0.16) Mc/sec for hydrogen and (1058.49±0.16) Mc/sec for deuterium. These values are about half a megacycle smaller than the corresponding experimental results.

1. INTRODUCTION

THE accuracy of the experimental determination of the Lamb shift has improved considerably over the last few years. A detailed account of the experimental techniques and calculations involved in obtaining a precision measurement of the Lamb shift will be found in a series of papers entitled "Fine Structure of the Hydrogen Atom."[1-3] The total Lamb shift (denoted by S and expressed in megacycles per second) is the total energy difference between any $n\,^2S_{\frac{1}{2}}$ and $n\,^2P_{\frac{1}{2}}$ levels for a hydrogen-like atom, which coincide in the elementary Dirac theory for an electron in a Coulomb field. In the present paper we shall be mainly concerned with the case of $n=2$ for hydrogen and deuterium. For this case S is now known experimentally to within one-tenth of a megacycle.

The contributions of a large number of effects to the Lamb shift have by now been calculated by different authors. The main aim of the present paper is to summarize the results of these previous calculations and to calculate the contributions of a few more terms. Corrections to the Lamb shift of relative order α^2, $(\alpha m/M)$ and $(\alpha^{-1}m^2/M^2)$ and smaller corrections will not be treated in this paper.

The electromagnetic displacement of the energy levels of an electron bound in a fixed Coulomb potential involves an expansion in powers of α, the fine structure constant. All the terms of the lowest two orders in α

[1] W. E. Lamb and R. C. Retherford, Phys. Rev. **79**, 549 (1950); **81**, 222 (1951); **86**, 1014 (1952).

[2] W. E. Lamb, Phys. Rev. **85**, 259 (1952).

[3] Triebwasser, Dayhoff, and Lamb, Phys. Rev. **89**, 98 (1953).