PHYSICAL REVIEW VOLUME 89, NUMBER 4 FEBRUARY 15, 1953

Deviations from *LS* Coupling in the Spheroidal Core Nuclear Model*

J. P. DAVIDSON,† *Washington University, St. Louis, Missouri*

AND

E. FEENBERG,‡ *Washington University, St. Louis, Missouri and Brookhaven National Laboratory, Upton, New York*

(Received November 4, 1952)

Deviations of measured magnetic moments from the Schmidt limits have been interpreted in terms of a failure of *LS* coupling. The model of an odd nucleon coupled to a nonspherical core provides an unforced theoretical basis for this interpretation. Projection operators for pure *LS* coupling states are introduced and applied to compute the statistical weights of the $L=I-\frac{1}{2}$ and $L'=I+\frac{1}{2}$ components in Bohr's strong coupling wave function. The orbital gyromagnetic ratio of each component is also determined.

Numerical results on the energy displacements produced by the coupling of the odd nucleon to the distorted core are generally unfavorable to the strong coupling description. A perturbation method appropriate to a weak coupling description is more likely to provide a satisfactory solution of the dynamical problem.

I. INTRODUCTION

THE theory of a nucleon moving in the field of a spherical core requires two interaction operators: $V_1(r)$, the radial potential function, and $V_2(r)\mathbf{l}\cdot\mathbf{s}$, the spin-orbit coupling potential. Some degree of arbitrariness is involved in any attempt to generalize these operators to a nonspherical core. A simple generalization, possibly inadequate, but suitable for exploratory studies, can be derived from the working hypothesis that V_1 and V_2 are functions of the particle densities $\rho_N(r)$ and $\rho_Z(r)$. To adapt this assumption to a nonspherical core, r is replaced by a new variable,

$$r'=[\lambda(x_1^2+x_2^2)+x_3^2/\lambda^2]^{\frac{1}{2}}, \tag{1a}$$

or

$$r'=r[1-\sum a_\mu^2+\sum a_\mu Y_{2\mu}(\theta'\varphi')]^{-1}, \tag{1b}$$

and the particle densities (and consequently also the potentials) are taken to be the same functions of r' as they were originally of r.

The first r' represents a spheroidal distortion of arbitrary magnitude leaving the volume of the core unchanged. The second is suitable for discussing small deviations from spherical symmetry. Here also the volume is held constant. The angles θ' and φ' are referred to a coordinate system in which the x_3 direction is a principal axis of the distorted core.[1] With this choice of polar axis $a_1=a_{-1}=0$, $a_{-2}=a_2$ and a_0, a_2 are real numbers.

Some information on the eigenvalues and solutions of the Schrödinger equation,

$$\left[-\frac{\hbar^2}{2M}\Delta+V_1(r_1)+\tfrac{1}{2}\mathbf{l}\cdot\mathbf{s}V_2(r')+\tfrac{1}{2}V_2(r')\mathbf{l}\cdot\mathbf{s}\right]\Psi=E\Psi, \tag{2}$$

can be derived from a study of the interaction operators

$${}_1H_{\text{int}}=V_1(r')-V_1(r), \tag{3}$$

$${}_2H_{\text{int}}=\tfrac{1}{2}[V_2(r')-V_2(r)]\mathbf{l}\cdot\mathbf{s}+\tfrac{1}{2}\mathbf{l}\cdot\mathbf{s}[V_2(r')-V_2(r)], \tag{4}$$

in the limit of small deviations from spherical symmetry. In first order

$${}_1H_{\text{int}}=-r(dV_1/dr)\sum a_\mu Y_{2\mu}, \tag{5}$$

$${}_2H_{\text{int}}=-\tfrac{1}{2}r(dV_2/dr)\sum a_\mu[\mathbf{l}\cdot\mathbf{s}Y_{2\mu}+Y_{2\mu}l\cdot s]. \tag{6}$$

With a rectangular potential well of constant depth D, Eq. (5) reduces to

$${}_1H_{\text{int}}=-\delta(r-R)DR\sum a_\mu Y_{2\mu}, \tag{7}$$

in which R is the nuclear radius and $\delta(r-R)$ is a Dirac delta-function. This is essentially the Rainwater[2–4] interaction operator with corrections for the finite depth of the potential well and the possible absence of symmetry about the body fixed polar axis. The recent study by Bohr[1] of the coupling between the odd nucleon and the surface oscillations of the nonspherical core is based on Eq. (7) and a quadratic approximation for the potential energy of the surface oscillations

$$V_{s0}=\tfrac{1}{2}C\sum a_\mu^2,\quad C=\frac{1}{2\pi}(2E_s-E_c) \tag{8}$$

derived from the theory of nuclear fision.[5] (E_s and E_c are, respectively, surface and Coulomb energies of the spherical core.)

The approximation represented by Eq. (7) is probably adequate on the range $|a_0|<0.2$. This statement is based on explicit numerical calculations for a well of infinite depth and a range of eccentricities[6] and also on comparisons with exact solutions for two analytically simple problems (harmonic oscillator well and well of

* Assisted by the joint program of the U. S. Office of Naval Research and the U. S. Atomic Energy Commission.

† Submitted by J. P. Davidson in partial fulfillment of the requirements for the degree of Doctor of Philosophy at Washington University, St. Louis, Missouri. Now at Pupin Physics Laboratory, Columbia University.

‡ Visiting physicist at Brookhaven National Laboratory, summer, 1952, and visiting lecturer at Princeton University, 1952–1953.

[1] A. Bohr, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **26**, No. 14 (1952).

[2] J. Rainwater, Phys. Rev. **79**, 432 (1950).

[3] E. Feenberg and K. C. Hammack, Phys. Rev. **81**, 285 (1951).

[4] S. Gallone and C. Salvetti, Phys. Rev. **84**, 1064 (1951).

[5] N. Bohr and J. A. Wheeler, Phys. Rev. **56**, 426 (1939).

[6] J. A. Wheeler and D. L. Hill (to be published).

infinite depth in a rectangular region). The present study shows that Eqs. (7) and (8) generally determine deviations from the spherical shape approaching the indicated limits of validity. In some applications a_0 is not small compared to 1, and the theory clearly breaks down. Nevertheless, some interesting conclusions can be derived from the theory based on Eqs. (7) and (8). The need for a critical approach to the theory is indicated by recent work[7] on the quadrupole moment of O^{17}.

The possibilities for the exact solution of Eq. (2) are illustrated by the example of a nonisotropic harmonic oscillator (omitting the spin-orbit interaction):

$$H=-(\hbar^2/2M)\Delta+\tfrac{1}{2}M\omega^2[\lambda(x_1^2+x_2^2)+x_3^2/\lambda^2].$$

The eigenvalues are

$$E=\hbar\omega[\lambda^{\frac{1}{2}}(n_1+n_2+1)+(n_3+\tfrac{1}{2})\lambda], \qquad (9)$$

in which the quantum numbers n_1, n_2, n_3 take on all integral values including zero. Minimum energy occurs for

$$\lambda=\left(\frac{2n_3+1}{n_1+n_2+1}\right)^{\frac{2}{3}}. \qquad (10)$$

Equations (9) and (10) yield

$$E=\tfrac{3}{2}\hbar\omega(2n_3+1)^{\frac{1}{3}}(n_1+n_2+1)^{\frac{2}{3}}. \qquad (11)$$

Numerical results are shown in Table I. The inclusion of the surface energy in the calculation would give qualitatively similar results with somewhat smaller deviations from spherical symmetry and smaller energy displacements.

Of particular interest is the low state in the $n_1+n_2+n_3=2$ function space. This state has $m_l=0$ and, for small deviations from spherical symmetry, contains $l=0$ with statistical weight $\frac{1}{3}$ and $l=2$ with statistical weight $\frac{2}{3}$.

Another example possessing a simple exact solution has a well of infinite depth in a rectangular region with dimensions $L_1=L_2=L/\lambda^{\frac{1}{2}}$, $L_3=\lambda L$. The energy eigenvalue is

$$E=(\hbar^2\pi^2/2ML^2)[\lambda(n_1^2+n_2^2)+n_3^2/\lambda^2],$$

in which the quantum numbers n_1, n_2, n_3 take on all integral values excluding zero.

II. INTERPRETATION OF DEVIATIONS FROM THE SCHMIDT LIMITS

The nuclides $_{15}P_{16}$, $_{48}Cd_{63,65}$, $_{50}Sn_{65,67,69}$, and $_{81}Tl_{122,124}$ have $I=\frac{1}{2}$ and magnetic moments far removed from the Schmidt single particle limits. This group is assigned even parity by the shell model. Other groups with moments departing considerably from the appropriate Schmidt limits occur at $I=\frac{3}{2}$ (odd) and $I=5/2$ (even); $_{33}As_{42}$ is an extreme example in the first category and $_{53}I_{74}$ in the second.

[7] Geschwind, Gunther-Mohr, and Silvey, Phys. Rev. **85**, 474 (1952).

TABLE I. Nonisotropic harmonic oscillator.

n_1	n_2	n_3	m_l	$l(\lambda=1)$	λ	$\epsilon/\hbar\omega$
0	0	0	0	0	1	3/2
1	0	0	±1	1	0.630	2.38
0	1	0		1	0.630	2.38
0	0	1	0	1	2.08	2.16
2	0	0				
0	2	0	$\pm2, 0$	2 and 0	0.463	3.12
1	1	0				
1	0	1	±1	2	1.310	3.435
0	1	1				
0	0	2	0	0 and 2	2.93	2.565

These groups present no difficulties from the viewpoint of an extreme single particle model if it is supposed that the effective magnetic moment of the odd nucleon when bound to the core is substantially reduced from the free nucleon values.[8–10] However, as emphasized in recent publications,[11,12] the existence of many closed (±1) shell nuclides with magnetic moments close to the appropriate Schmidt limits[13] is difficult to reconcile with a large systematic modification in the magnetic properties of the bound nucleon.

An alternative working hypothesis puts the major share of the responsibility for the deviations from the Schmidt limits on the failure of *LS* coupling within the bounds of a pure doublet description.[11,14] The ground state is represented by a linear combination of pure *LS* coupling components, one with $L=I-\frac{1}{2}$ and the other with $L'=I+\frac{1}{2}$, both components having the same parity:

$$\Psi_M{}^I=\alpha\Psi_M{}^{LI}+(1-\alpha^2)^{\frac{1}{2}}\Psi_M{}^{L'I}. \qquad (12)$$

The model of an odd nucleon coupled to a nonspherical core provides an unforced theoretical basis for the failure of *LS* coupling. Bohr[1] has shown that this model is the dynamical equivalent of Rainwater's[2] spheroidal core considerations. In the present study we investigate the conditions under which the model is likely to produce an extreme failure of *LS* coupling after first determining the extent to which *LS* coupling fails when j is treated as a constant of motion.

III. MAGNETIC MOMENTS WHEN j IS A GOOD QUANTUM NUMBER

We consider the case of strong spin-orbit interaction and close coupling between the orbital motion of the odd nucleon and the motion of the deformed core.

[8] F. Bloch, Phys. Rev. **83**, 839 (1951).
[9] H. Mujazawa, Prog. Theoret. Phys. **6**, 263, 801 (1951).
[10] A. de Shalit, Helv. Phys. Acta **24**, 296 (1951).
[11] J. P. Davidson, Phys. Rev. **85**, 432 (1952).
[12] E. Feenberg, Ann. Rev. Nuc. Sci. **I**, 43 (1952).
[13] C^{13}, N^{15}, O^{17}, F^{19}, K^{41}, Y^{89}, and Pb^{207} are mentioned in reference 11. The list should include $_{56}Ba_{81}$, and $_{79}Au_{118}$ (closed $d_{\frac{3}{2}}$ shell at $Z=80$ followed by $s_{\frac{1}{2}}$ shell closing at 82). The magnitude and sign of the quadrupole moment of $_{79}Au_{118}$ [W. von Siemens, Naturwiss. **38**, 455 (1951)] support the single particle interpretation.
[14] G. L. Trigg, Phys. Rev. **86**, 506 (1952).

TABLE II. Magnetic moments for nucleon states with a definite value of j.

			Odd proton		Odd neutron	
l	j	I	μ_j	$\mu_{cj;I}$	μ_j	$\mu_{cj;I}$
0	1/2	1/2	2.79	2.79	−1.91	−1.91
1	3/2	3/2	3.79	2.52	−1.91	−0.90
2	5/2	5/2	4.79	3.71	−1.91	−1.07
3	7/2	7/2	5.79	4.82	−1.91	−1.17
4	9/2	9/2	6.79	5.89	−1.91	−1.23
1	1/2	1/2	−0.27	−0.27	0.63	0.63
2	3/2	3/2	0.12	0.31	1.14	0.93
3	5/2	5/2	0.87	0.91	1.37	1.27
4	7/2	7/2	1.71	1.65	1.47	1.46
5	9/2	9/2	2.62	2.46	1.56	1.60
2	3/2	1/2	0.12	0.37	1.14	0.03
3	5/2	3/2	0.87	0.56	1.37	0.74
4	7/2	5/2	1.71	1.16	1.47	1.04
5	9/2	7/2	2.62	1.90	1.56	1.25
1	3/2	1/2	3.79	−0.86	−1.91	1.05
2	5/2	3/2	4.79	1.96	−1.91	−0.45
3	7/2	5/2	5.79	3.24	−1.91	−0.68
4	9/2	7/2	6.79	4.42	−1.91	−0.84
5	11/2	9/2	7.79	5.53	−1.91	−0.95

Under these conditions there exist solutions for which the angular momentum components I_3 and j_3 are approximately constants of motion. The product function,

$$\Psi_M{}^I=\varphi[\chi_I{}^jD_{MI}{}^I+(-1)^{I-j}\chi_{-I}{}^jD_{M,-I}{}^I], \quad (13)$$

is a close approximation to such a solution.[1] Here $\chi_{\pm I}{}^j$ is a normalized nucleon wave function with $j_3=\pm I$, $D_{M,\pm I}{}^I$ describes a state of a symmetrical top with total angular momentum I, $I_3=\pm I$, $I_z=M$, and φ embodies the zero-point fluctuations of the surface waves.

The detailed numerical evaluation of the energy matrix in the space of the strong coupling wave functions indicates that strong coupling is an extreme case rather far removed from the actual situation. Nevertheless, the simplicity of the strong coupling model is sufficient justification for a careful examination of all its consequences.[15]

Consider first a nucleon state with a definite value of j. The magnetic moment operator of the compound system has the form

$$\mu_{cj}=g_j\mathbf{j}+g_c(\mathbf{I}-\mathbf{j}) =(g_j-g_c)\mathbf{j}+g_c\mathbf{I}, \quad (14)$$

in which $\mu_j=g_jj$ is the magnetic moment of the odd nucleon on a spherical core and $g_c\sim Z/A$ is the gyromagnetic ratio of the core.[16] The magnetic moment computed from Eq. (14) is

$$\mu_{cj;I}=g_cI+(g_j-g_c)\frac{(IM|\mathbf{I}\cdot\mathbf{j}|IM)}{I+1}, \quad (15)$$

in which $(IM|\mathbf{I}\cdot\mathbf{j}|IM)$ is the diagonal matrix element of $\mathbf{I}\cdot\mathbf{j}$ with respect to the wave function of Eq. (13). Straightforward calculations yield

$$I_3j_3\Psi_M{}^I=I^2\Psi_M{}^I,$$
$$(I_1j_1+I_2j_2)\Psi_M{}^I=\tfrac{1}{2}[2I(j-I+1)(j+I)]^{\frac{1}{2}}\varphi \cdot[\chi_{I-1}D^I{}_{M,I-1}+(-1)^{I-j}\chi_{-I+1}D^I{}_{M,-I+1}]. \quad (16)$$

In deriving Eq. (16) it must be remembered that the components of $\mathbf{I}$ and $\mathbf{j}$ with respect to the principal axes of the core commute and also that $\mathbf{j}\times\mathbf{j}=i\mathbf{j}$, the normal commutation relations, for both space and body fixed axes, but $\mathbf{I}\times\mathbf{I}=-i\mathbf{I}$ for body fixed axes. Equation (16) implies

$$(IM|\mathbf{I}\cdot\mathbf{j}|IM)=I^2,\quad I>\tfrac{1}{2},$$
$$=\tfrac{1}{4}[1+(-1)^{I-j}(2j+1)],\quad I=\tfrac{1}{2}, \quad (17)$$

and, consequently,

$$\mu_{cj;I}=\frac{I}{I+1}[g_jI+g_c],\quad I>\tfrac{1}{2}, \quad (18)$$

$$\mu_{cj;\frac{1}{2}}=\tfrac{1}{6}[3g_c+(g_j-g_c)\{1+(-1)^{I-j}(2j+1)\}]. \quad (19)$$

In particular,

$$\mu_{c\frac{1}{2};\frac{1}{2}}=\mu_{\frac{1}{2}},\quad \mu_{c\frac{3}{2};\frac{1}{2}}=g_c-\tfrac{1}{3}\mu_{\frac{3}{2}}. \quad (20)$$

Equation (18) with $I=j$ is not new,[17] but the important restriction to $I>\frac{1}{2}$ has not been observed in previous applications. Numerical results are listed in Table II.

Excellent agreement of $\mu_{cI;I}$ with the general trend of the experimental moments is found[17] for $I=l+\frac{1}{2}>\frac{1}{2}$. Two examples of extreme departure from the appropriate Schmidt limits may be interpreted in terms of a failure of l and j as good quantum numbers; the moment of $_{33}A_{42}$ suggests approximately equal parts of $\mu_{c\frac{3}{2};\frac{3}{2}}$ and $\mu_{c\frac{5}{2};\frac{3}{2}}$ (odd parity) while that of $_{53}I_{74}$ can be fitted with two-thirds $\mu_{c\frac{5}{2};\frac{5}{2}}$ and one-third $\mu_{c7/2;\frac{5}{2}}$ (even parity). Close coupling of $f_{\frac{5}{2}}$ and $p_{\frac{3}{2}}$ orbitals through the mediation of the distorted core is a reasonable possibility on the range $29\leqq Z\leqq 38$; similarly an example showing close coupling between $g_{7/2}$ and $d_{\frac{5}{2}}$ is not unexpected on the range $51\leqq Z\leqq 58$.

For $I=l-\frac{1}{2}$, $\mu_{cI;I}$ is generally only slightly better than μ_j leaving a substantial discrepancy between the theory and the general trend of the experimental moments. The possibility of a partial decoupling of $\mathbf{l}$ and $\mathbf{s}$ must be considered here since a state with $I=j=l-\frac{1}{2}$ is coupled through the distorted core to $j'=l+\frac{1}{2}$. One needs the nondiagonal matrix element of the magnetic moment operator

$$(l-\tfrac{1}{2},l-\tfrac{1}{2}|g_ss_3+g_ll_3|l+\tfrac{1}{2},l-\tfrac{1}{2}) =(g_s-g_l)(2l)^{\frac{1}{2}}/(2l+1) =4.58(2l)^{\frac{1}{2}}/(2l+1),\ \text{odd } Z =-3.82(2l)^{\frac{1}{2}}/(2l+1),\ \text{odd } N. \quad (21)$$

[15] We are indebted to Dr. N. Kroll and Dr. L. Foldy for valuable discussions on the subject of the strong coupling approximation.

[16] H. Margenau and E. P. Wigner, Phys. Rev. 58, 103 (1939).

[17] A. Bohr, Phys. Rev. 81, 134 (1951); all the B curves in Figs. 1 and 2 of this reference should coincide with the A curves at $I=\frac{1}{2}$.

If now

$$\Psi_I=(1-\beta^2)^{\frac{1}{2}}\Psi_{cI;I}+\beta\Psi_{cI+1;I}, \quad (22)$$

the correction to the magnetic moment is

$$\mu_I-\mu_{cI;I}\sim\beta(g_s-g_l)(2I+1)^{\frac{1}{2}}/(I+1) \quad (23)$$

for $\beta^2\ll 1$. Rather small values of β^2 are able to account for most of the observed deviations. Thus, at $I=7/2$ and odd Z a correction of 0.8 nuclear magneton requires $\beta^2\sim 0.06$. Both odd N and odd Z require negligible corrections at $I=\frac{1}{2}$ and $\frac{3}{2}$ and the positive sign for β at $I>\frac{3}{2}$.

The remaining important special case is $I=\frac{1}{2}$ (even parity). Here the coupling between ${}^2s_{\frac{1}{2}}$ and ${}^2d_{\frac{3}{2}}$ through the distorted core provides a possible explanation of magnetic moments far removed from the ${}^2s_{\frac{1}{2}}$ Schmidt limit. The calculations are described in a later section.

IV. RESOLUTION INTO *LS* COUPLING COMPONENTS WHEN *j* IS A GOOD QUANTUM NUMBER

We return to the strong coupling wave function of Eq. (13) under the restriction that l and j are good quantum numbers. The resolution of $\Psi_M{}^I$ into LS coupling components is accomplished by the introduction of projection operators

$$\begin{aligned}P_{ljI}{}^{I\pm\frac{1}{2}}&=0, \quad L=I\mp\tfrac{1}{2}\\&=1, \quad L=I\pm\tfrac{1}{2}.\end{aligned} \quad (24)$$

Explicitly,

$$\begin{aligned}P_{ljI}{}^{I+\frac{1}{2}}&=[(\mathbf{I}-\mathbf{S})^2-(I-\tfrac{1}{2})(I+\tfrac{1}{2})]/(2I+1)\\&=(I+1-2\mathbf{I}\cdot\mathbf{S})/(2I+1),\end{aligned} \quad (25)$$

$$\begin{aligned}P_{ljI}{}^{I-\frac{1}{2}}&=[(I+\tfrac{1}{2})(I+\tfrac{3}{2})-(\mathbf{I}-\mathbf{S})^2]/(2I+1)\\&=(I+2\mathbf{I}\cdot\mathbf{S})/(2I+1).\end{aligned} \quad (26)$$

These relations and Eq. (12) yield

$$\alpha^2=(IM|P_{ljI}{}^{I-\frac{1}{2}}|IM), \quad (27)$$

independent of M.

In the body fixed reference frame, the components of $\mathbf{I}$ and $\mathbf{S}$ commute; this fact and the relations $\mathbf{I}\times\mathbf{I}=-i\mathbf{I}$ and $\mathbf{S}\times\mathbf{S}=i\mathbf{S}$ yield

$$4\mathbf{I}\cdot\mathbf{S}\mathbf{I}\cdot\mathbf{S}=I(I+1)-2\mathbf{I}\cdot\mathbf{S}, \quad (28)$$

in agreement with the idempotence property of the projection operators.

The substitution

$$\begin{aligned}\mathbf{I}\cdot\mathbf{S}&\sim(\mathbf{I}\cdot\mathbf{j})(\mathbf{j}\cdot\mathbf{S})/[j(j+1)]\\&=(\mathbf{I}\cdot\mathbf{j})(-1)^{l+\frac{1}{2}-j}/(2l+1)\end{aligned} \quad (29)$$

leaves unchanged all matrix elements of $\mathbf{I}\cdot\mathbf{S}$ in the function space defined by $\Psi_{-I}{}^I\cdots\Psi_I{}^I$. Consequently the projection operators can be expressed in the convenient forms:

$$P_{ljI}{}^{I+\frac{1}{2}}=\frac{(I+1)(2l+1)-2\mathbf{I}\cdot\mathbf{j}(-1)^{l+\frac{1}{2}-j}}{(2I+1)(2l+1)}, \quad (30)$$

$$P_{ljI}{}^{I-\frac{1}{2}}=\frac{I(2l+1)+2\mathbf{I}\cdot\mathbf{j}(-1)^{l+\frac{1}{2}-j}}{(2I+1)(2l+1)}. \quad (31)$$

For $I=\frac{1}{2}$ the statistical weight of the ${}^2S_{\frac{1}{2}}$ component in $\Psi_M{}^{\frac{1}{2}}$ of Eqs. (9) and (13) is

$$\begin{aligned}\alpha^2&=\frac{2l+1+[1+(-1)^{j-\frac{1}{2}}(2j+1)](-1)^{l+\frac{1}{2}-j}}{4(2l+1)}\\&=1, \quad l=0, \quad j=I=\tfrac{1}{2}\\&=0, \quad l=1, \quad j=I=\tfrac{1}{2}\\&=0.4, \quad l=2, \quad j=I+1=\tfrac{3}{2}.\end{aligned} \quad (32)$$

In the same way, the statistical weight of the LS coupling component with $L=I-\frac{1}{2}$ in $\Psi_M{}^I$ for $I>\frac{1}{2}$ is

$$\begin{aligned}\alpha^2&=\frac{I(2l+1)+2I^2(-1)^{l+\frac{1}{2}-j}}{(2I+1)(2l+I)}\\&=\frac{2I}{(2I+1)}, \qquad j=l+\tfrac{1}{2}=I\\&=\frac{I}{(I+1)(2I+1)}, \quad j=l-\tfrac{1}{2}=I.\end{aligned} \quad (33)$$

Maximum mixing occurs at $I=\frac{3}{2}$ with 75 percent $P_{\frac{3}{2}}$ and 25 percent $D_{\frac{3}{2}}$ from a nucleon in a $p_{\frac{3}{2}}$ state.

The orbital gyromagnetic ratios are quantities of interest in connection with the resolution of $\Psi_M{}^I$ into pure LS coupling components. To compute these quantities we start from the operator for the orbital magnetic moment:

$$\begin{aligned}\boldsymbol{\mu}_{\text{orb}}&=g_l\mathbf{l}+g_c(\mathbf{I}-\mathbf{j})\\&=(g_l-g_c)\mathbf{j}-g_l\mathbf{S}+g_c\mathbf{I}.\end{aligned} \quad (34)$$

The easily verified relations,

$$\begin{aligned}[\mathbf{j},\mathbf{I}\cdot\mathbf{S}]&=[\mathbf{S},\mathbf{I}\cdot\mathbf{S}]=[\mathbf{I},\mathbf{I}\cdot\mathbf{S}]\\&=i\mathbf{I}\times\mathbf{S},\end{aligned} \quad (35)$$

have the consequence that $\boldsymbol{\mu}_{\text{orb}}$ commutes with $\mathbf{I}\cdot\mathbf{S}$ and therefore also with the orbital angular momentum projection operators. This property enables us to write

$$\begin{aligned}&\mu_{\text{orb}}{}^{I\pm\frac{1}{2}}(II|P_{ljI}{}^{I\pm\frac{1}{2}}|II)\\&\quad=(II|\mathbf{I}\cdot\boldsymbol{\mu}_{\text{orb}}P_{ljI}{}^{I\pm\frac{1}{2}}|II)/(I+1).\end{aligned} \quad (36)$$

From Eqs. (25), (26), and (34),

$$\begin{aligned}\mathbf{I}\cdot\boldsymbol{\mu}_{\text{orb}}P_{ljI}{}^{I+\frac{1}{2}}=&\frac{g_l-g_c}{2I+1}[(I+1)\mathbf{I}\cdot\mathbf{j}-2\mathbf{I}\cdot\mathbf{j}\mathbf{I}\cdot\mathbf{S}]\\&+I[g_c(I+1)+\tfrac{1}{2}g_l]P_{ljI}{}^{I+\frac{1}{2}},\end{aligned} \quad (37)$$

$$\begin{aligned}\mathbf{I}\cdot\boldsymbol{\mu}_{\text{orb}}P_{ljI}{}^{I-\frac{1}{2}}=&\frac{g_l-g_c}{2I+1}[I(\mathbf{I}\cdot\mathbf{j})+2\mathbf{I}\cdot\mathbf{j}\mathbf{I}\cdot\mathbf{S}]\\&+(I+1)[g_cI-\tfrac{1}{2}g_l]P_{ljI}{}^{I-\frac{1}{2}}.\end{aligned} \quad (38)$$

Since the operator $\mathbf{I}\cdot\mathbf{j}$ does not mix the two values of j associated with the given l, the substitution

$$\mathbf{I}\cdot\mathbf{j}\mathbf{I}\cdot\mathbf{S}\sim(\mathbf{I}\cdot\mathbf{j})^2(-1)^{l+\frac{1}{2}-j}/(2l+1) \quad (39)$$

TABLE III. Values of g_{orb}^L.

$I=j$	l	L	g_{orb}^L (odd Z)	g_{orb}^L (odd N)
1/2	0	0	0	0
		1	—	—
	1	0	—	—
		1	1	0
3/2	1	1	0.760	0.160
		2	0.600	0.267
	2	1	0.280	0.480
		2	0.835	0.110
5/2	2	2	0.829	0.114
		3	0.700	0.200
	3	2	0.571	0.286
		3	0.870	0.086
7/2	3	3	0.867	0.089
		4	0.760	0.160
	4	3	0.689	0.208
		4	0.893	0.071
9/2	4	4	0.891	0.073
		5	0.800	0.133
	5	4	0.755	0.164
		5	0.909	0.061

is permitted in Eqs. (37) and (38). Equations (16) and (17) suggest the procedure followed in computing the diagonal matrix elements of $(\mathbf{I}\cdot\mathbf{j})^2$; these are found to have the values

$$(\tfrac{1}{2}\tfrac{1}{2}|(\mathbf{I}\cdot\mathbf{j})^2|\tfrac{1}{2}\tfrac{1}{2})=\tfrac{1}{16}[1+(-1)^{I-j}(2j+1)]^2,$$
$$(II|(\mathbf{I}\cdot\mathbf{j})^2|II)=I^2(I^2+1),\quad I=j>\tfrac{1}{2}. \tag{40}$$

Orbital gyromagnetic ratios g_L for the odd nucleon-distorted core system are defined by the equations

$$\mu_{orb}^{I-\frac{1}{2}}=(I-\tfrac{1}{2})g_{orb}^{I-\frac{1}{2}},\quad L=I-\tfrac{1}{2};$$
$$\mu_{orb}^{I+\frac{1}{2}}=\frac{I(I+\frac{3}{2})}{I+1}g_{orb}^{I+\frac{1}{2}},\quad L=I+\tfrac{1}{2}. \tag{41}$$

For $I=j=\frac{1}{2}$ and $l=0$, both left- and right-hand members of Eq. (36) vanish. In this case no calculations are needed to establish the absence of an orbital contribution to the magnetic moment. A more interesting check is supplied by $I=j=\frac{1}{2}$, $l=1$ (a pure $^2P_{\frac{1}{2}}$ state). Now

$$\mu_{orb}^{I+\frac{1}{2}}=\tfrac{2}{3}g_l \quad\text{and}\quad g_{orb}^{I+\frac{1}{2}}=g_l, \tag{42}$$

in agreement with expectations.

General formulas for g_L are listed below

j	L	g_{orb}^L
$l+\frac{1}{2}$	l	$g_l+(g_c-g_l)/(I+1)$
$l+\frac{1}{2}$	$l+1$	$g_l+2(g_c-g_l)/(I+\frac{3}{2})$
$l-\frac{1}{2}$	l	$g_l+(g_c-g_l)(2I^2+I+1)/(I+\frac{3}{2})(2I^2+2I+1)$
$l-\frac{1}{2}$	$l-1$	$g_l+(g_c-g_l)2I/(I+1)(I-\frac{1}{2})$.

(43)

These formulas with $g_l=1$ (odd proton), $g_l=0$ (odd neutron), and $g_c=0.4$ are used to compute the values of g_{orb}^L listed in Table III.

There is little resemblance between these numerical results for g_{orb}^L and the values postulated in the $S-MW$ interpolation procedure employed by Trigg[14] and Davidson.[11] Possibly these differences reveal the crudeness of the $S-MW$ interpolation procedure; however, the inadequacy of the distorted core model in regard to magnetic moments for $I=j=l-\frac{1}{2}$ suggests that judgement should be reserved on this question.

Equations (33) and (43) can be used for a simple alternative derivation of the magnetic moment formula of Eq. (18). This procedure is useful in providing a partial check on the formulas for α^2 and g_{orb}^L.

V. MATRIX ELEMENTS OF $_1H_{int}$

We compute matrix elements of $_1H_{int}$ in the linear approximation of Eq. (7). In the jm_j representation with the axis of quantization along the symmetry axis of the core the diagonal matrix elements are[1]

$$(jm_j|_1H_{int}|jm_j)=k_{nl}\left(\frac{5}{4\pi}\right)^{\frac{1}{2}}a_0\frac{3m_j^2-j(j+1)}{4j(j+1)}, \tag{44}$$

in which

$$k_{nl}=DR^3R_{nl}^2(R), \tag{45}$$

and R_{nl} is the radial function of the state occupied by the odd nucleon. Since this matrix element vanishes at $j=\frac{1}{2}$, there is no direct coupling between a nucleon in a $p_{\frac{1}{2}}$ or $s_{\frac{1}{2}}$ orbital and the surface oscillations of the core.

For each value of $|m_j|$ there exists a low state of the compound system with total angular momentum $I=|m_j|$. The ground state is associated with maximum absolute value of $(jm_j|_1H_{int}|jm_j)$, i.e., with

$$(j,\pm j|_1H_{int}|j,\pm j)=k_{nl}\left(\frac{5}{4\pi}\right)^{\frac{1}{2}}a_0\frac{2j-1}{4(j+1)}, \tag{46}$$

and $I=j$; the next largest absolute value of the matrix element occurs at $I=|m_j|=\frac{1}{2}$ with

$$(j,\pm\tfrac{1}{2}|_1H_{int}|j,\pm\tfrac{1}{2})=-k_{nl}\left(\frac{5}{4\pi}\right)^{\frac{1}{2}}a_0\frac{(2j-1)(2j+3)}{16j(j+1)}. \tag{47}$$

One exception must be noted; at $j=\frac{3}{2}$ the absolute value of the matrix element is independent of m_j,

$$(\tfrac{3}{2},\pm\tfrac{3}{2}|_1H_{int}|\tfrac{3}{2},\pm\tfrac{3}{2})=k_{nl}\frac{1}{5}\left(\frac{5}{4\pi}\right)^{\frac{1}{2}}a_0,$$
$$(\tfrac{3}{2},\pm\tfrac{3}{2}|_1H_{int}|\tfrac{3}{2},\pm\tfrac{1}{2})=-k_{nl}\frac{1}{5}\left(\frac{5}{4\pi}\right)^{\frac{1}{2}}a_0. \tag{48}$$

In this case the dynamical theory of the coupling between the odd nucleon and the surface oscillations should yield two closely spaced low levels, one with $I=\frac{3}{2}$ and the other with $I=\frac{1}{2}$, both having the same parity as the odd nucleon state.

Equation (2) clearly implies that l and j are not exact quantum numbers. A number of nondiagonal matrix elements are actually quite large, large enough in some cases to invalidate the usual implicit assumption that l and j are good approximate constants of motion. Elementary calculations yield the following results:

$$({}^2X_{l+\frac{1}{2},\,l-\frac{1}{2}}|\,{}_1H_{\rm int}|\,{}^2X_{l-\frac{1}{2},\,l-\frac{1}{2}})=-ka_03\left(\frac{5}{4\pi}\right)^{\frac{1}{2}}\frac{(2l)^{\frac{1}{2}}}{(2l+1)(2l+3)},\qquad(49)$$

connecting two states with the same l; and

$$({}^2X_{l+\frac{1}{2},\,l+\frac{1}{2}}|\,{}_1H_{\rm int}|\,{}^2X_{l'-\frac{1}{2},\,l+\frac{1}{2}})=-k'a_03\left(\frac{5}{4\pi}\right)^{\frac{1}{2}}\frac{2^{\frac{1}{2}}(l+1)^{\frac{1}{2}}}{(2l+3)(2l+5)},\qquad(50)$$

$$({}^2X_{l+\frac{1}{2},\,l+\frac{1}{2}}|\,{}_1H_{\rm int}|\,{}^2X_{l'+\frac{1}{2},\,l+\frac{1}{2}})=-k'a_03\left(\frac{5}{4\pi}\right)^{\frac{1}{2}}\frac{(l+1)^{\frac{1}{2}}}{(2l+3)^{\frac{1}{2}}(2l+5)},\qquad(51)$$

connecting l and $l'=l+2$. Reversing the sign of $m_I=l\pm\frac{1}{2}$ multiplies the matrix element by the factor $(-1)^{j-j'}$. Here k' denotes the geometric mean of the k's associated with the nl and $n'l'$ states. Extensive calculations[18] of particle densities in oscillator potentials and rectangular wells show that k' and k generally differ by less than 20 percent. In the following we ignore the difference and replace k' by k.

Special cases under Eq. (50) of particular interest are

$$\begin{aligned}({}^2s_{\frac{1}{2}},\pm\tfrac{1}{2}|\,{}_1H_{\rm int}|\,{}^2d_{\frac{3}{2}},\pm\tfrac{1}{2})&=\mp k'a_0/(10\pi)^{\frac{1}{2}},\\({}^2p_{\frac{3}{2}},\pm\tfrac{3}{2}|\,{}_1H_{\rm int}|\,{}^2f_{\frac{5}{2}},\pm\tfrac{3}{2})&=\mp k'a_03/7(5\pi)^{\frac{1}{2}},\\({}^2d_{\frac{5}{2}},\pm5/2|\,{}_1H_{\rm int}|\,{}^2g_{7/2},\pm5/2)&=\mp k'a_0\frac{1}{5}\left(\frac{5}{6\pi}\right)^{\frac{1}{2}}.\end{aligned}\qquad(52)$$

Comparing Eqs. (48) and (52) one sees that the coupling energy between ${}^2s_{\frac{1}{2}}$ and ${}^2d_{\frac{3}{2}}$ through the mediation of the distorted core is indeed quite large; in fact, larger by a factor $2^{\frac{1}{2}}$ than the diagonal matrix elements of $H_{\rm int}$ for the ${}^2d_{\frac{3}{2}}$ state.

VI. COUPLING OF ${}^2s_{\frac{1}{2}}$ AND ${}^2d_{\frac{3}{2}}$ NUCLEON STATES

Consider the group of nuclides with $I=\frac{1}{2}$ and even parity. Experimental evidence locating a state with $I=\frac{3}{2}$ and even parity close to the ground state exists for many of these; for the others the existence of such a low-lying level can be inferred from the shell model. We write

$$\chi_{\pm\frac{1}{2}}=\beta\chi({}^2s_{\frac{1}{2}},\pm\tfrac{1}{2})\pm(1-\beta^2)^{\frac{1}{2}}\chi({}^2d_{\frac{3}{2}},\pm\tfrac{1}{2})\qquad(53)$$

in the static approximation for the core, and proceed to compute the energy and the amplitude β from the relations§

$$\begin{aligned}&\beta[E-\tfrac{1}{2}CA_0^2]+(1-\beta^2)^{\frac{1}{2}}ka_0/(10\pi)^{\frac{1}{2}}=0,\\&\beta ka_0/(10\pi)^{\frac{1}{2}}+(1-\beta^2)^{\frac{1}{2}}[E-\Delta E+ka_0/(20\pi)^{\frac{1}{2}}-\tfrac{1}{2}Ca_0^2]=0.\end{aligned}\qquad(54)$$

Here ΔE denotes the excitation energy of $d_{\frac{3}{2}}$ relative to $s_{\frac{1}{2}}$ when the core is spherical; negative ΔE means $d_{\frac{3}{2}}$ below $s_{\frac{1}{2}}$.

With the notation

$$\Delta E=nk^2/20\pi C,\quad y=(20\pi)^{\frac{1}{2}}Ca_0/k,\qquad(55)$$

the lower of the two eigenvalues determined by Eq. (54) is given by the formula

$$E20\pi C/k^2=\tfrac{1}{2}[n-y+y^2-\{8y^2+(n-y)^2\}^{\frac{1}{2}}].\qquad(56)$$

The detailed analysis of Eq. (56) yields the result that $\beta^2/(1-\beta^2)$ lies between 2.0 and 0.5 and y between 1.67 and 2.00 on the range $0\leqq n\leqq3.33$. The corresponding values of μ range from $\frac{2}{3}$ to $\frac{1}{3}$ (approximately) of the Schmidt value.

VII. ROTATIONAL AND ZERO-POINT ENERGIES

Equation (54) represents a static approximation in which the rotational kinetic energy and the zero point vibrational energies are neglected. Bohr's equations (50) and (98) in reference 1 yield the following operator for the rotational kinetic energy:

$$T_{\rm rot}=\frac{\hbar^2}{6Ba_0^2}[I(I+1)+j(j+1)-I_3^2-j_3^2-2(I_1j_1+I_2j_2)],\qquad(57)$$

subject to the restriction $j_3=I_3=I$. The diagonal matrix elements of $T_{\rm rot}$ can be computed with the help of Eq. (17) and have the values

$$\begin{aligned}&\frac{\hbar^2}{2Ba_0^2}\quad\text{for}\quad I=j=\tfrac{3}{2},\\&\frac{\hbar^2}{Ba_0^2}\quad\text{for}\quad I=\tfrac{1}{2},\quad j=\tfrac{3}{2}.\end{aligned}\qquad(58)$$

Here $B=\frac{1}{2}\rho_0R^5$, ρ_0 denoting the mean density of nuclear matter. In the absence of coupling to a ${}^2s_{\frac{1}{2}}$ state the energies are given by the minimum values of

$$\begin{aligned}(20\pi C/k^2)E_{c\frac{3}{2};\frac{3}{2}}&=y+\tfrac{1}{2}y^2+\Gamma/2y^2,\\(20\pi C/k^2)E_{c\frac{3}{2};\frac{1}{2}}&=-y+\tfrac{1}{2}y^2+\Gamma/y^2,\end{aligned}\qquad(59)$$

with

$$\Gamma=\frac{(20\pi\hbar)^2}{B}\frac{C^3}{k^4}.\qquad(60)$$

---

[18] K. C. Hammack, Doctor's thesis, Washington University (1951) (unpublished).

§ We are indebted to Dr. A. Bohr for calling our attention to an error in a first statement of this problem.

TABLE IV. Values of the physical parameters.

A	C*	k*	$k^2/20\pi C$*	$k/(20\pi)^{\frac{1}{2}}C$	Γ	$\hbar^2/B$*
31	41.5	40	0.613	0.123	53	0.50
203	63.8	40	0.400	0.080	9.0	0.023

* Energy in Mev units.

Numerical values of the physical parameters are listed in Table IV.

Equation (59) is a special case of the general form

$$(20\pi C/k^2)E_{cj;I} = -\eta y + \tfrac{1}{2}y^2 + \eta'\Gamma/2y^2 \quad (61)$$

in which η and η' are constants determined by j and I. In the extreme case of $\eta'\Gamma \gg 1$ the minimum value of $E_{cj;I}$ occurs near

$$\begin{aligned} y &= \tfrac{1}{4}\eta + (\eta'\Gamma)^{\frac{1}{4}}, \quad \eta > 0, \\ &= \tfrac{1}{4}\eta - (\eta'\Gamma)^{\frac{1}{4}}, \quad \eta < 0, \end{aligned} \quad (62)$$

and has the value

$$\frac{20\pi C}{k^2}E_{cj;I} = (\eta'\Gamma)^{\frac{1}{2}} - |\eta|(\eta'\Gamma)^{\frac{1}{4}} - \tfrac{1}{8}\eta^2 + \cdots. \quad (63)$$

Equation (62) yields $a_0 \cong -0.16$ for $I = j = \frac{3}{2}$, $A = 203$. The zero point energies E_β and E_γ of the distorted core [Bohr's equations (108) and (113)] are given by

$$\begin{aligned} (20\pi C/k^2)E_\beta &= 2\Gamma/y^2 + \tfrac{1}{2}(\Gamma p/C)^{\frac{1}{2}} \\ &= \Gamma^{\frac{1}{2}}\left[1 + \frac{2}{\eta'^{\frac{1}{2}}} - \left(\frac{3}{8} + \frac{1}{\eta'^{\frac{1}{2}}}\right)\frac{|y|}{(\eta'\Gamma)^{\frac{1}{4}}} \right. \\ &\quad \left. + \left(\frac{3}{8\eta'^{\frac{1}{2}}} + \frac{21}{128}\right)\frac{\eta^2}{(\eta'\Gamma)^{\frac{1}{2}}} + \cdots\right], \end{aligned} \quad (64)$$

$$\begin{aligned} (20\pi C/k^2)E_\gamma &= (2\eta')^{\frac{1}{2}}\Gamma/y^2[1 + |\eta y^3|/2\eta'\Gamma]^{\frac{1}{2}} \\ &= (2\Gamma)^{\frac{1}{2}}[1 - |\eta|/4(\eta'\Gamma)^{\frac{1}{4}} \\ &\quad + (\tfrac{3}{16} - 7\eta^2/64)/(\eta'\Gamma)^{\frac{1}{2}} + \cdots]. \end{aligned} \quad (65)$$

The zero-point energy of the spherical core [Bohr's equation (9)] is

$$\begin{aligned} \frac{20\pi C}{k^2}E_\omega &= \frac{20\pi C}{k^2}\frac{5}{2}\left(\frac{\hbar^2 C}{B}\right)^{\frac{1}{2}} \\ &= (5/2)\Gamma^{\frac{1}{2}}. \end{aligned} \quad (66)$$

A necessary condition for the validity of strong coupling is that the diagonal matrix elements of the energy are lowered by the introduction of the Rainwater interaction. This condition takes the form

$$E_{cj;I} + E_\beta + E_\gamma - E_\omega < 0. \quad (67)$$

It is easily seen that Eq. (67) fails for $\Gamma \gg 1$ and $\eta'\Gamma \gg 1$. A numerical example will make this clear. At $A = 203$, $j = I = \frac{3}{2}$, $\eta = -1$, $\eta' = 1$, $\Gamma = 9$,

$$\begin{aligned} (20\pi C/k^2)E_{c\frac{3}{2};\frac{3}{2}} &= 1.14, \\ (20\pi C/k^2)E_\beta &= 7.16, \\ (20\pi C/k^2)E_\gamma &= 3.74, \\ (20\pi C/k^2)E_\omega &= 7.5, \end{aligned}$$

or

$$(E_{c\frac{3}{2};\frac{3}{2}} + E_\beta + E_\gamma - E_\omega) = 1.8 \text{ Mev.}$$

These results show that the numerical values of the physical parameters are inconsistent with the assumption of strong coupling embodied in Eq. (13).[19] An alternative treatment employing the occupation numbers of the surface oscillation states as diagonal variables is indicated.[20]

We turn now to estimate the influence of the rotational kinetic energy on the location of the $I = \frac{1}{2}$ state when $^2s_{\frac{1}{2}}$ and $^2d_{\frac{3}{2}}$ nucleon states are coupled through the distorted core. The energy formula Eq. (56) is replaced by

$$\begin{aligned} (20\pi C/k^2)E_{\frac{1}{2}} = \tfrac{1}{2}[&n - y + y^2 + \Gamma/y^2 \\ &- \{8y^2 + (n - y + \Gamma/y^2)^2\}^{\frac{1}{2}}]. \end{aligned} \quad (68)$$

Numerical results appear in Table V for a particular value of $\beta/(1-\beta^2)^{\frac{1}{2}}$ corresponding to a moment approximately midway between the Schmidt limits. The example at $A = 203$ shows the substantial influence of the rotational kinetic energy on the location and magnitude of the energy minimum under relatively favorable conditions for the validity of the strong coupling approximation. It appears that a perturbation method appropriate to a weak coupling description is more likely to provide a satisfactory solution of the dynamical problem, particularly for small values of the mass number.

One of the authors (JPD) wishes to thank the Central Administration of Washington University for the grant of a University Fellowship under whose tenure this work was done.

TABLE V. Coupling of $^2s_{\frac{1}{2}}$ and $^2d_{\frac{3}{2}}$, $\beta/(1-\beta^2)^{\frac{1}{2}} = 1$.

A	Γ	n	y	a_0	$(20\pi C/k^2)E_{\frac{1}{2}}$
90	20.7	0.24	2.83	0.28	0.00
203	9.0	1.06	2.50	0.20	−0.41
∞	0.0	1.91	1.91	...	−0.87

[19] The strong coupling approximation appears to better advantage when two or more equivalent nucleons are coupled to the core (private communication from K. W. Ford).

[20] L. L. Foldy and F. J. Milford, Phys. Rev. 80, 751 (1950).

APPENDIX A

We evaluate the allowed beta-decay matrix element $|\int\boldsymbol{\sigma}|^2$ for image transitions. Trigg[14] has shown that

$$|\int\boldsymbol{\sigma}|^2=[(I+1)\alpha^2-I(1-\alpha^2)]^2/[I(I+1)]. \quad \text{(A1)}$$

Introducing α^2 from Eq. (33), we get

$$|\int\boldsymbol{\sigma}|^2=I/(I+1), \qquad I=j=l+\tfrac{1}{2}>\tfrac{1}{2},$$
$$=[I/(I+1)]^3, \quad I=j=l-\tfrac{1}{2}>\tfrac{1}{2}. \quad \text{(A2)}$$

Thus, the coupling to the core leaves the ratio for $l+\frac{1}{2}$ and $l-\frac{1}{2}$ unchanged relative to the values for pure LS coupling, but reduces both by the factor $I^2/(I+1)^2$. An alternative derivation starts from the equivalence relation,

$$\boldsymbol{\sigma}\sim\mathbf{I}\frac{2I}{I+1}\frac{(-1)^{l+\frac{1}{2}-j}}{2l+1}. \quad \text{(A3)}$$

Then

$$\left|\int\boldsymbol{\sigma}\right|^2=\frac{I^3}{I+1}\frac{1}{(l+\frac{1}{2})^2}, \quad \text{(A4)}$$

in agreement with Eq. (A2).

APPENDIX B

The diagonal matrix elements of $(\mathbf{I}-\mathbf{j})^2$ and $(\mathbf{I}-\mathbf{j})^4$ can be evaluated with the aid of Eqs. (16), (17), and (40). Results are

$$(\tfrac{1}{2}M|(\mathbf{I}-\mathbf{j})^2|\tfrac{1}{2}M)=(j+\tfrac{1}{2})[j+\tfrac{1}{2}-(-1)^{j-\frac{1}{2}}];$$
$$(IM|(\mathbf{I}-\mathbf{j})^2|IM)=2I, \quad I>\tfrac{1}{2};$$
$$(IM|(\mathbf{I}-\mathbf{j})^4|IM)=2(2I)^2, \quad I>\tfrac{1}{2}.$$

For $I=\frac{1}{2}$, Eq. (16) shows that $(I-j)^2$ is a constant of motion.

PHYSICAL REVIEW VOLUME 89, NUMBER 4 FEBRUARY 15, 1953

A Third Rydberg Series of N_2

R. EDWIN WORLEY
Department of Physics, University of Nevada, Reno, Nevada
(Received October 31, 1952)

A third series of Rydberg bands, converging to the $^2\Pi$ state of N_2^+, has been identified in the far ultraviolet absorption spectrum of nitrogen. Its interpretation substantiates an assignment of vibrational numbers for the $^2\Pi$ state, and indicates that this state is derived from the o state of N_2.

TWO series of Rydberg terms have previously been identified for N_2, one[1,2] converging to the $X\,^2\Sigma$ state of N_2^+, the other[3,4] to the excited $B\,^2\Sigma$ state. I shall refer to these as the $X-X$ and the $B-X$ series, and discuss here a third, or $A-X$, series that converges to the $A\,^2\Pi$ state, about 1 ev above the ground state. Some years ago Professor Mulliken suggested I examine my spectrograms of the far ultraviolet absorption of N_2, with the object of identifying the corresponding bands and thus locating the $^2\Pi$ state. However, due to overlying absorption on the best plates (nitrogen pressure too high), this proved unfeasible.

The recent identification of $^2\Pi-{}^2\Sigma$ bands of N_2^+ by Meinel,[5] in auroral spectra, affords an approximate value for the limit of the anticipated Rydberg series. I have accordingly re-examined my plates, and have found five distinct bands[6] which fit the following Rydberg formula with residuals of +1, −2, +2, +2, −2 cm^{-1}.

$\nu_m=136\,607-R/(m-0.0441-0.018/m)^2$; $m=2$ to 6, $R=109\,735$ cm^{-1}. The bands are rather narrow, are shaded to longer wavelengths, and show but one head. Starting with the first member, intensities progressively decrease in a normal manner, the values being comparable to those of corresponding $X-X$ bands. The upper term for the first band is the $v=1$ level of state o.

Dalby and Douglas[7] have photographed the $^2\Pi-{}^2\Sigma$ bands of N_2^+ at large dispersion, using a laboratory source. From preliminary results of the analysis, kindly supplied by Dr. Douglas, the empirical series limit, above, is found to correspond to the $v=1$ level of the $^2\Pi$ state, and apparently to its upper component $^2\Pi_{\frac{1}{2}}$. The position of this component, as found by adding $\nu_{00}+\Delta G(\frac{1}{2})+\frac{1}{2}A$ to the limit of the $X-X$ Rydberg series,[2] is 136 597 cm^{-1} above the ground state of N_2. In fact, if the heads of the higher members of the $X-X$ series represent origins—that is, if they are of Q-form as is suggested by the extreme narrowness of the bands—and if a computed origin-to-head interval is added

[1] R. E. Worley and F. A. Jenkins, Phys. Rev. 54, 305 (1938).
[2] R. E. Worley, Phys. Rev. 64, 207 (1943).
[3] J. J. Hopfield, Phys. Rev. 36, 789 (1930). See also Takamine, Suga, and Tanaka, Sci. Pap. Inst. Phys. Chem. Res. Tokyo 34, 854 (1938).
[4] R. S. Mulliken, Phys. Rev. 46, 144 (1934).
[5] A. B. Meinel, Astrophys. J. 114, 431 (1951); 112, 562 (1950).
[6] Most bands referred to herein are listed in Table I, reference 2. Omitted were the following (cm^{-1}): 123 995* ($m=3$); 124 069* (footnote 8). Recently measured for use in Table I herein were 122 068; 133 995; 135 361 cm^{-1}.
[7] F. W. Dalby and A. E. Douglas, Phys. Rev. 84, 843 (1951). See also R. Herman, Compt. rend. 233, 926 (1951); N. D. Sayers, Proc. Phys. Soc. (London) 65, 152 (1952).