PHYSICAL REVIEW VOLUME 89, NUMBER 4 FEBRUARY 15, 1953

Interpretation of High Energy $p-p$ Scattering*

DON R. SWANSON†
Radiation Laboratory, Department of Physics, University of California, Berkeley, California
(Received September 19, 1952)

A study is made of the scattering of high energy protons by protons. Several types of "cutoffs" are introduced into the singular tensor interaction proposed by Christian and Noyes; the triplet P state radial equations are then solved by essentially exact numerical integration methods. The resulting cross sections show a more pronounced disagreement with experiment than do the Born approximation cross sections of Christian and Noyes. Calculations were carried out in the vicinity of 350 Mev and 120 Mev.

INTRODUCTION

SEVERAL experiments have been carried out on the scattering of protons by protons at energies greater than 100 Mev.[1–4] The resulting differential cross sections are characterized by spherically symmetric angular distributions (in the center-of-mass system) and by a lack of dependence on energy. Between scattering angles of 20° and 160° and between energies of 120 Mev and 350 Mev the cross section is about four or five millibarns per steradian. The results have been interpreted by Christian and Noyes[5] (hereafter referred to as "CN"), by Jastrow,[6] and by Case and Pais.[7] In the CN analysis (350 Mev) a square well singlet interaction was used which gave almost no scattering at angles greater than 40°. The problem then was to find a triplet interaction yielding an essentially isotropic differential cross section. It was observed that any triplet central potential is undesirable since the cross section due to it would vanish at 90° (the wave function is antisymmetric), accordingly a tensor force model was chosen. (The wave function must of course still be antisymmetric; however, with a noncentral potential the antisymmetrization is not expressed in terms of the polar scattering angle θ alone, but by the azimuthal angle ϕ as well. The antisymmetric spin scattering matrix $S(\theta,\phi)-S(\pi-\theta,\phi+\pi)$ does not necessarily vanish at $\theta=\pi/2$ as it would if there were no ϕ dependence.)

In order to obtain the desired "flat" cross section, Christian and Noyes found it necessary to use a potential with a "highly singular" radial dependence $e^{-r/R}/r^2$. All triplet state calculations were carried out in Born approximation. Jastrow, on the other hand, attempted to obtain agreement with experiment by introducing a hard core into the singlet interaction, thus permitting greater momentum transfers and accordingly a substantial amount of large angle (90°) scattering. The triplet interaction was not then required to yield an isotropic cross section. Neither the CN nor the Jastrow interpretation was entirely successful in fitting the experimental data, the principal difficulty being too large a theoretical peak in the forward direction due mostly to scattering of the singlet D state. However, it was not in the spirit of the analyses to indulge in a detailed program of "curve fitting" but rather to illustrate the important features of the various interactions chosen. This philosophy applies as well to the present paper.

It is proposed here to examine more critically the triplet state calculations of Christian and Noyes, and, in particular, to investigate the validity of their use of the Born approximation. Singlet scattering will be ignored. There is reason to suspect that results of the Born approximation applied to a highly singular potential may not be even qualitatively correct. Consider, for example, the radial equation for the $^3P_0$ state [Appendix, Eq. (A18)]. In the vicinity of the origin this takes the form

$$d^2u/dy^2+\lambda_0 u/y^2=0,\quad y=kr. \tag{1}$$

(Choosing $\lambda_0>0$ implies that the nuclear potential is effectively attractive in this state, and sufficiently deep to dominate the centrifugal term as $r\rightarrow 0$.) The solution, for $\lambda_0>\frac{1}{4}$, is composed of spherical Bessel functions of imaginary order having an oscillatory singularity at the origin:[8]

$$u\underset{y\rightarrow 0}{\rightarrow} y^{\frac{1}{2}}\cos[(\lambda_0-\tfrac{1}{4})^{\frac{1}{2}}\log y+B].$$

An interaction of this nature can be treated in a physically meaningful way only if the singularity at the origin is in some arbitrary way "cut off." It is evident, however, that the region of the cutoff cannot be arbitrarily small since several oscillations of the wave function within the region would lead to bound states of the diproton. Consider the integral equation satisfied by the solution to Eq. (A18):

$$u=A_1g_1(y)+4\lambda g_{-1}(y)\int_0^y \frac{e^{-ay}}{y^2}ug_1dy +4\lambda g_1(y)\int_y^\infty \frac{e^{-ay}}{y^2}ug_{-1}dy. \tag{2}$$

* This work was performed under the auspices of the U. S. Atomic Energy Commission.

† Now at Hughes Aircraft Company Research and Development Laboratories, Culver City, California.

[1] Chamberlain, Segrè, and Wiegand, Phys. Rev. 83, 923 (1951).

[2] C. L. Oxley and R. D. Schamberger, Phys. Rev. 85, 416 (1952).

[3] Birge, Kruse, and Ramsey, Phys. Rev. 83, 274 (1951).

[4] Cassels, Staford, and Pickavance, Nature 168, 468 (1951).

[5] R. S. Christian and H. P. Noyes, Phys. Rev. 79, 85 (1950) (referred to as "CN" in the text).

[6] R. Jastrow, Phys. Rev. 81, 165 (1950).

[7] K. M. Case and A. Pais, Phys. Rev. 80, 203 (1950).

[8] K. M. Case, Phys. Rev. 80, 797 (1950).

The left-hand side of the equation becomes the wave function in Born approximation if the plane wave solution $u=g_1(y)$ is inserted as a trial function in the integrand. The Born approximation is valid if the exact solution does not deviate greatly from the free particle trial function. Near the origin the latter, $g_1(y)$, becomes just $\frac{1}{3}y^2$; the next zero occurs beyond the region in which the nuclear interaction is appreciable, even for energies as high as 350 Mev. It is therefore evident that for sufficiently short cutoffs the Born approximation is invalid since the exact (possibly oscillatory) solution does not resemble the trial function. Examination of the integral equation (2) shows, moreover, that the presence of a short range cutoff has a negligible influence on the Born calculation itself, simply because the singularity in the potential is masked by the $\frac{1}{3}y^2$ factor from the trial function. (For convenience a square well cutoff may be visualized here; that is, the potential e^{-ay}/y^2 for $y\geq y_0$ is placed equal to the constant e^{-ay_0}/y_0^2 for $y\leq y_0$.) It is evident that the larger the cut-off radius the more nearly valid becomes the first-order iteration procedure. On the other hand, a long range cutoff cannot be ignored in a Born calculation. It seems, then, that the CN procedure (Born approximation without explicit introduction of cutoff) can be taken seriously only if there exists some kind of cutoff of sufficiently long range to permit first-order perturbation methods to have real meaning, yet short enough so that the perturbation calculation itself is not appreciably influenced by its presence. It will be shown here that, strictly speaking, a cutoff fulfilling these two conditions does not exist.

PROCEDURE

The procedure adopted here is to introduce specific cutoffs into the CN interaction and obtain an essentially exact solution to the scattering problem by a numerical integration procedure. The cutoffs considered will be of two types: tensor force "square wells," in which the potential is given by

$$V(r)=\mp 15.2S_{12}\frac{e^{-r/R}}{(r/R)^2}\text{ Mev}\quad\text{for}\quad r\geq r_0, \tag{3}$$

$$V(r)=\mp 15.2S_{12}\frac{e^{-r_0/R}}{(r_0/R)^2}\text{ Mev}=\text{constant for } r\leq r_0,$$

$$r^2S_{12}\equiv 3\boldsymbol{\sigma}_1\cdot\mathbf{r}\boldsymbol{\sigma}_2\cdot\mathbf{r}-r^2\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2,\quad R=1.6\times 10^{-13}\text{ cm},$$

and "hard cores," where

$$\begin{aligned}V(r)&=\infty\text{ for } r\leq r_0,\\ V(r)&=\text{Eq. (3) for } r\geq r_0.\end{aligned}\tag{4}$$

The $\mp$ sign refers to what will be called "attractive" and "repulsive" interactions, respectively. The Born cross section of course is the same for the two signs of the interaction.

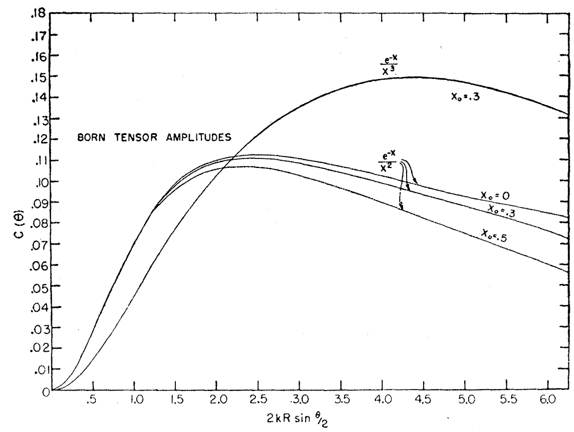

FIG. 1. Born tensor amplitudes, $C(\theta)$ [Eq. (A14)], for singular potentials with various ranges [$x_0=(r_0/R)$] of square well cutoffs. The radial dependence of the potentials is indicated on the plot. $x=r/R$.

In attempting to choose a more or less physically meaningful cut-off radius, r_0, the "nucleon Compton wavelength" $\hbar/Mc$ is a convenient guide. Part of the motivation for choosing a radial dependence of the form $e^{-r/R}/r^2$ is its similarity to terms in the phenomenological interactions predicted by meson theories. Such motivation hardly exists at distances as short as $\hbar/Mc$ where, for example, the nucleon structure, as well as relativistic effects, may be expected to play an important role. On the other hand, to introduce a cutoff as large as $3\hbar/Mc$ (about $\frac{1}{2}$ the meson Compton wavelength) more or less abandons the similarity to meson potentials. Essentially the same limits on r_0 are obtained by a few rough calculations which indicate that a cutoff somewhat smaller than $\hbar/Mc$ would lead to a bound di-proton, and a radius greater than $3\hbar/Mc$ tends to destroy the desired isotropy of the cross section even in Born approximation. (The latter point is illustrated by a plot of the Born tensor amplitude in Fig. 1.) The calculations were therefore carried out using a "short range cutoff," $r_0\approx\hbar/Mc$, and a "long range cutoff," $r_0\approx 2\hbar/Mc$, for both the square well and the hard core. The four cases considered will be denoted by the following abbreviations:

SRSW: short range square well cutoff;
Eq. (3) with $r_0=0.24\times 10^{-13}$ cm.
LRSW: long range square well cutoff;
Eq. (3) with $r_0=0.48\times 10^{-13}$ cm.
SRHC: short range hard core cutoff;
Eq. (4) with $r_0=0.24\times 10^{-13}$ cm.
LRHC: long range hard core cutoff;
Eq. (4) with $r_0=0.48\times 10^{-13}$ cm.

The ${}^3P_0$, ${}^3P_1$ and ${}^3P_2$, ${}^3F_2$ states for the SRSW case were solved by numerical integration and checked by iterating the resulting radial functions (using the integral equations) to produce the same phase shifts and amplitudes to within a few percent. All other states

TABLE I. Triplet $p-p$ phase shifts at 350 Mev for singular tensor potential with various cutoffs. (Cross sections plotted in Fig. 2.)

		Short range square well cutoff			Long range square well cutoff		Short range hard core		Long range hard core	
		Attractive	Repulsive	Born (Repulsive)	Attractive	Repulsive	Attractive	Repulsive	Attractive	Repulsive
$^3P_2$, $^3F_2$	B_1^2	-0.002 $+i.242$	-0.048 $+i.184$	$^B\delta_1^{20}=-0.033$	Same as SRSW		0.005 $+i.170$	-0.029 $+i.116$	-0.035 $-i.185$	-0.008 $-i.239$
	A_1^2	-0.022 $-i.100$	-0.152 $+i.567$	$^B\delta_1^{21}=\ 0.169$	Same as SRSW		-0.028 $-i.171$	-0.119 $+i.503$	-0.129 $-i.514$	-0.027 $+i.158$
	B_3^2	-0.017 $-i.016$	0.018 $-i.060$	$^B\delta_3^{20}=-0.014$	Same as SRSW		-0.013 $-i.016$	0.013 $-i.062$	0.012 $-i.020$	-0.016 $-i.066$
	A_3^2	-0.037 $-i.357$	-0.086 $+i.323$	$^B\delta_3^{21}=\ 0.188$	Same as SRSW		-0.045 $-i.356$	-0.078 $+i.325$	-0.082 $-i.349$	-0.035 $+i.331$
$^3P_0$	δ_1^0	-0.626	2.00	0.880	-0.616	1.24	-0.64	0.91	-0.66	0.32
$^3P_1$	δ_1^1	0.580	-0.360	-0.440	0.540	-0.351	0.38	-0.37	0.04	-0.47

($^3F_3$ $^3F_4$, $^3H_4$, etc.) were included in Born approximation, with cutoffs ignored. Some details of the procedure are given in the Appendix.

The phase shifts for the LRSW cutoff were then obtained by a perturbation method using as trial functions in the integral Eqs. (A21) the radial functions for the SRSW case, except in the $^3P_0$ state of the "repulsive" interaction, which was integrated numerically. (The $^3P_0$ state is effectively attractive in the "repulsive" interaction and repulsive in the "attractive" because of a minus sign appearing in the corresponding matrix element of the tensor operator S_{12}.) Inspection of the differential Eqs. (A18) and (A19) shows that the effective well depth in the $^3P_0$ state is twice as great and of the opposite sign as that of the $^3P_1$ state. From the remarks following Eqs. (A23) in the Appendix, it is apparent that the most important quantity in the coupled system is the P-dominant P phase shift. Furthermore, in the P-dominant mode the term

$$(e^{-ay}/y^2)3(6)^{\frac{1}{2}}w$$

is asymptotically smaller than the term $(e^{-ay}/y^2)u$; from the power series expansion it is clear that it also starts out much smaller near the origin. Ignoring for the moment this coupling term, then, and comparing the size of the $^3P_2$ potential to the $^3P_1$ and $^3P_0$, it is seen that the latter are, in absolute value, five and ten times as large as the former. Accordingly it is reasonable to think that the $^3P_0$ phase shift in the "repulsive" case and the $^3P_1$ in the "attractive" will exhibit a great deal more sensitivity to the nature of the cutoff than will the coupled $^3P_2$, $^3F_2$ states. The perturbation calculations for the long range square well cutoffs indeed show just this sort of behavior. The coupled phase shifts in fact differ negligibly from those of the SRSW cutoff.

The foregoing arguments indicate that a fair approximation to the hard core cut-off cross sections should result from taking the core into consideration only in the $^3P_0$ and $^3P_1$ states and using the square well cut-off phase shifts in the coupled states. However, the following somewhat more refined procedure was used which still avoids the labor of repeating the coupled numerical integrations. Starting with the unperturbed SRSW solutions, the P-dominant P phase shift, δ_{11}^2, is added to the "hard sphere" P phase shift,

$$\tan\delta_p{}^{hs}=+J_{\frac{3}{2}}(y_0)/J_{-\frac{3}{2}}(y_0).$$

The nature of the approximation can be readily seen by considering a similar procedure for an uncoupled

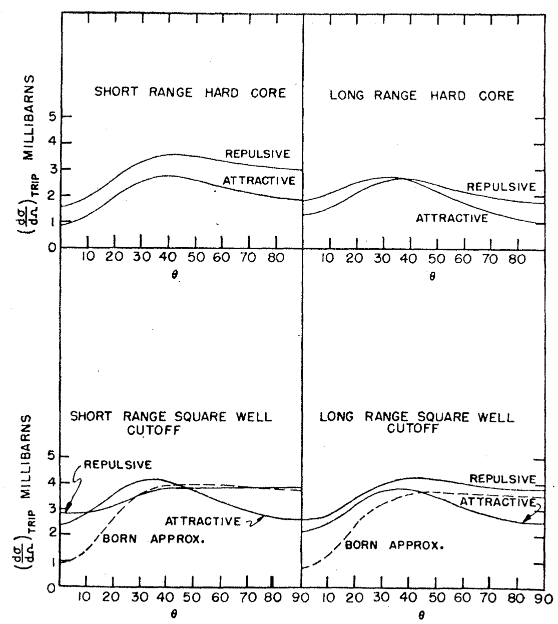

FIG. 2. Differential cross sections (center-of-mass system) for triplet $p-p$ scattering (neglecting Coulomb) at 350 Mev using a singular tensor potential $\mp 15.2S_{12}e^{-r/R}/(r/R)^2$ Mev with various cutoffs. Dotted curves shown Born cross sections, solid curves are "exact." Phase shifts are in Table I. "Short range" means $r_0=0.24\times10^{-13}$ cm; "long range" $=0.48\times10^{-13}$ cm.

integral equation (see Appendix for notation):

$$\sin\delta_{hc}\approx\int_0^{y_0}U_a u_a g_1 dy+\int_{y_0}^{\infty}U_b u_b g_1 dy$$

$$\approx\sin\delta_p{}^{hs}+\sin\delta_{sw},\quad \delta_{hc}\approx\delta_p{}^{hs}+\delta_{sw},\qquad (5)$$

where U_a= strong repulsion (approximates hard core); $U_b=e^{-ay}/y^2$ for $y\geq y_0$, $=e^{-ay_0}/y_0{}^2$= constant for $y\leq y_0$; trial function u_a= exact solution when $U_b=0$; trial function u_b= exact solution when $U_a=0$; δ_{sw}= phase shift for square well cutoff; δ_{hc}= phase shift for hard core cutoff; and $\delta_p{}^{hs}=P$ phase shift for hard core alone.

$$\int_0^{y_0}U_b u_b g_1 dy$$

is neglected. Analogous treatment of $a_{13}{}^2$, $a_{31}{}^2$, $a_{33}{}^2$ shows that the influence of the hard core on these quantities is negligible. The phase shifts $\delta_{13}{}^2$, $\delta_{31}{}^2$ (unperturbed SRSW value is $\pi/2$) feel the core somewhat more strongly; however, they may deviate as much as 20 percent from $\pi/2$ without changing the $\delta_l{}^{Jm_s}$ by more than a few percent [Eqs. (A8)].

All hard core coupled *PF* phase shifts were obtained in the manner just indicated; all uncoupled *P*-state equations were integrated numerically.

RESULTS AND CONCLUSIONS

Phase shifts and differential cross sections at 350 Mev are given in Table I and Fig. 2. The "attractive" interaction evidently leads to a greater anisotropy of the triplet cross section than does the "repulsive," regardless of the nature of the cutoff. The near agreement of the exact cross sections at 350 Mev with those calculated in Born approximation is surprising in view of the large discrepancies in the corresponding phase shifts. Similar discrepancies at 129 Mev lead to an exact cross section much larger than that obtained in Born approximation (Fig. 3); apparently, then, the

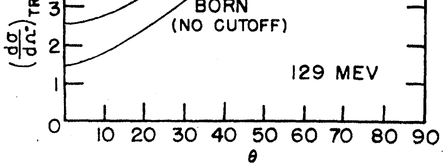

FIG. 3. Differential cross sections for triplet $p-p$ scattering (neglecting Coulomb) at 129 Mev using cutoff singular tensor potential. Curve *A*: SRSW cutoff. Curves *B*, *C* have cutoffs adjusted to give the $^3P_0$ phase shifts indicated on the plot.

TABLE II. Phase shifts for triplet $p-p$ scattering at 129 Mev, using repulsive singular tensor interaction with short range square well (SRSW) cutoff.

	Exact	Born
$^3P_0$	$\delta_1{}^0=\ 1.81$	${}^B\delta_1{}^0=\ 0.590$
$^3P_1$	$\delta_1{}^1=-0.240$	${}^B\delta_1{}^1=-0.295$
$^3P_2$, $^3F_2$	$B_1{}^2=-0.008+i.012$	${}^B\delta_1{}^{20}=-0.021$
	$A_1{}^2=-0.037+i.228$	${}^B\delta_1{}^{21}=\ 0.112$
	$B_3{}^2=0.002-i.082$	${}^B\delta_3{}^{20}=-0.024$
	$A_3{}^2=-0.027+i.195$	${}^B\delta_3{}^{21}=\ 0.109$

close agreement at 350 Mev is accidental. Figure 2 also indicates that the greater the "volume" of potential removed by the cutoff the greater is the angular variation of the cross section; Fig. 1 illustrates the same point in Born approximation.

The SRSW cutoff for the "repulsive" case was calculated in detail at 129 Mev. The results, Fig. 3 (curve *A*) and Table II, show that the predicted scattering is much too great. The trouble comes almost entirely from the large $^3P_0$ phase shift. To investigate the effect (at 129 Mev) of modifying the cutoff, attention will be restricted to the $^3P_0$ state. (The arguments of the preceding section indicate that the coupled phase shifts are only slightly influenced by the nature of the potential at short range; the $^3P_1$ state is repulsive and so obviously insensitive to the cutoff.) A $^3P_0$ phase shift of 0.80 (instead of the 1.8 of Table II) yields roughly the desired cross section (Fig. 3, curve *B*). The required phase shift can be produced, for example, by the combination of a square well cutoff at 0.48×10^{-13} cm and a hard core of radius 0.24×10^{-13} cm, (or, of course, by a hard core alone of radius somewhat larger than 0.24×10^{-13} cm). The cross section at 350 Mev will then in any case lie between that of the SRHC and the LRHC cutoffs shown in Fig. 2.

It is concluded, therefore, that, within the framework of the singlet and triplet models adopted by Christian and Noyes, something similar to the following triplet potential seems to yield the closest approach to the experimental cross sections (Fig. 4) at 120 Mev and 350 Mev:

$$V(r)=15.2S_{12}\frac{e^{-r/R}}{(r/R)^2}\text{ Mev},\quad r\geq r_1=0.48\times10^{-13}\text{ cm},$$

$$V(r)=15.2S_{12}\frac{e^{-r_1/R}}{(r_1/R)^2}\text{ Mev}=\text{constant},\quad r_0\leq r\leq r_1,\qquad (6)$$

$$V(r)=\infty,\quad r\leq r_0=0.24\times10^{-13}\text{ cm}.$$

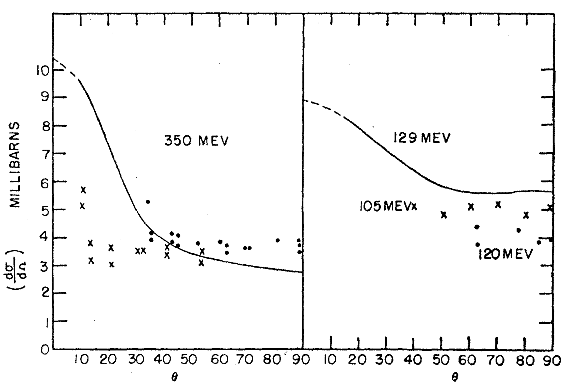

FIG. 4. $P-p$ scattering at 350 Mev, 129 Mev using the cut-off singular tensor potential given by Eq. (6) and a square well singlet interaction (Christian and Noyes). Coulomb scattering neglected. The experimental points at 350 Mev and those at 120 Mev denoted by ● are taken from Chamberlain, Segrè, and Wiegand. (See reference 1.) The points, x, at 105 Mev are from Birge, Kruse, and Ramsey (see reference 3).

It is to be emphasized that significance should be attached only to the necessary degree of "severeness" (i.e., volume of potential affected) of the cutoff and not to its precise nature.

It should be mentioned that a cutoff sufficiently short to increase the ${}^3P_0$ phase shift at 129 Mev to 2.2 is not obviously less desirable than (6) (see Fig. 3, curve C); the 350-Mev scattering would be changed, but not drastically. The effect of so short a cutoff would be more pronounced at some energy less than 120 Mev where the ${}^3P_0$ phase shift will have decreased to $\pi/2$.

By using the potential given by (6), the discrepancy with the experimental forward scattering is considerably greater than originally was indicated by the CN calculations. The disagreement seems sufficiently conclusive to justify ruling out a large class of static potentials for the $p-p$ interaction. The class of inacceptable potentials is by no means exhaustive, however. Whenever a strong short range component (e.g., hard core) is included in the singlet interaction (thus permitting large angle scattering), the triplet potential acquires several more degrees of freedom since the requirement of isotropy may be dropped. In particular, triplet central potentials then merit consideration.

APPENDIX

The nucleon-nucleon scattering problem for a noncentral static potential will be formulated and discussed. The notation and method of treatment adapts conveniently to a description of polarization effects carried out in a concurrent paper.[9]

The asymptotic form of the triplet state wave function can be written[10]

$$\psi \sim e^{ikz}\chi_{\mathrm{inc}} + (e^{ikr}/r)S\chi_{\mathrm{inc}}. \qquad \text{(A1)}$$

[9] Don R. Swanson, Phys. Rev. **89**, 749 (1953).
[10] J. Ashkin and T. Wu, Phys. Rev. **73**, 973 (1948).

χ_{inc} is the triplet spin function of the initial state, where $S(\theta,\phi)$ is the triplet spin scattering operator, the matrix for which is given explicitly in terms of the complex phase shifts, $\delta_l^{Jm_s}$, by

$$S=\frac{1}{2ik}\sum_{J,m_s,l}(2l+1)[\exp(2i\delta_l^{Jm_s})-1]\prod\nolimits_l^{Jm_s}P_l(\cos\theta)$$

$$=\frac{1}{4ik}\begin{array}{c|ccc|c} & 1 & 0 & -1 & m_s'/m_s \\ & A & \sqrt{2}Be^{-i\phi} & Ce^{-2i\phi} & 1 \\ & -\sqrt{2}De^{i\phi} & 2E & \sqrt{2}De^{-i\phi} & 0. \\ & Ce^{2i\phi} & -\sqrt{2}Be^{i\phi} & A & -1 \end{array} \qquad \text{(A2)}$$

$$A=\sum_l[(l+2)A_l^{l+1}+(2l+1)A_l^l+(l-1)A_l^{l-1}]P_l,$$

$$B=\sum_l[B_l^{l+1}-B_l^{l-1}]P_l^1,$$

$$C=\sum_l\left[\frac{1}{l+1}A_l^{l+1}-\frac{2l+1}{l(l+1)}A_l^l+\frac{1}{l}A_l^{l-1}\right]P_l^2,$$

$$D=\sum_l\left[\frac{l+2}{l+1}A_l^{l+1}-\frac{2l+1}{l(l+1)}A_l^l-\frac{l-1}{l}A_l^{l-1}\right]P_l^1,$$

$$E=\sum\nolimits_l[(l+1)B_l^{l+1}+lB_l^{l-1}]P_l,$$

$$A_l^J=\exp(2i\delta_l^{J,\pm1})-1$$

$$B_l^J=\exp(2i\delta_l^{J,0})-1. \qquad \text{(A3)}$$

$\prod_l^{Jm_s}P_l$ is an operator in triplet spin space defined by Eq. (A6). Coulomb scattering is neglected. The boundary conditions of the scattering problem yield also the relationship between the complex phase shifts and the asymptotic form of the radial wave functions. To obtain this relationship, first expand the wave function of the system in eigenfunctions, ψ^{Jm}, of total angular momentum J^2 and J_z. Separate the radial from the spin-angular dependence by means of the expansion

$$\psi^{Jm}=\sum\nolimits_l(u_l^J(r)/r)\psi_l^{Jm},$$

where the ψ_l^{Jm} are eigenfunctions of J^2, J_z and the orbital angular momentum L^2. The Schrödinger equation for the radial functions becomes

$$\frac{d^2u_l^J}{dr^2}-\frac{l(l+1)}{r^2}u_l^J+k^2u_l^J-\frac{M}{\hbar^2}\sum_{l'=J-1}^{J+1}V_{ll'}^J(r)u_{l'}^J=0, \qquad \text{(A4)}$$

where $V_{ll'}^J(r)=(\psi_l^{Jm}, V(\mathbf{r},\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2)\psi_{l'}^{Jm})$ is independent of m. The scalar product denotes an integration over the surface of a sphere and summation over spin variables.

For a tensor interaction, the orbital angular momentum is not a constant of the motion and $V(\mathbf{r},\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2)$ contains off-diagonal elements between states of the same parity.[11]

$$S_{12}=\frac{1}{2J+1}\left|\begin{array}{ccc|c} J-1 & J & J+1 & l'/l \\ -2(J-1) & 0 & 6[J(J+1)]^{\frac{1}{2}} & J-1 \\ 0 & 2(2J+1) & 0 & J \\ 6[J(J+1)]^{\frac{1}{2}} & 0 & -2(J+2) & J+1 \end{array}\right|. \tag{A5}$$

The orthonormal set of spin-angular functions, ψ_l^{Jm}, can be expressed in terms of spherical harmonics and spin functions by means of the Clebsch-Gordon expansion:

$$\psi_l^{Jm}=\sum_{m_l,m_s}(lsJm|lsm_lm_s)Y_{m_l}^l(\theta,\phi)\chi^{sm_s}$$

and

$$Y_{m_l}^l\chi^{sm_s}=\sum_J(lsJm|lsm_lm_s)\psi_l^{J,m_l+m_s},$$

where $m=m_l+m_s$. Defining the projection operator, $\prod_l^{Jm}$, by

$$\Pi_l^{Jm}Y_{m-m_s}^l\chi^{sm_s}=(lsJm|ls,\,m-m_s,\,m_s)\psi_l^{Jm},$$

so that

$$(l1Jm_s|l10m_s)\psi_l^{Jm_s}=\prod_l^{Jm_s}Y_0^l\chi^{1m_s}, \tag{A6}$$

the general expansion for the wave function of the system takes the form

$$\bar{\psi}=\sum_{Jm}\sum_{l=J-1}^{J+1}\sum_{i=1}^{2}\frac{C_i^{Jm}}{(l1Jm|l1,\,m-m_s,\,m_s)}\frac{u_{li}^J(r)}{r}\times\prod_l^{Jm}Y_{m-m_s}^l\chi^{1m_s}, \tag{A7}$$

$u_{li}^J\sim a_{li}^J\sin(kr-l\pi/2+\delta_{li}^J)$. The subscript i is summed over the two regular solutions to the coupled equation in (A4) [see discussion following Eqs. (A20)]. For uncoupled states, put $C_2^{Jm}=0$. The asymptotic form of (A7) is the same as (A1) with S defined by (A2) provided that[5]

$$\begin{aligned} D_J\exp(2i\delta_{J-1}^{J,\pm1})&=E_J+2i[J/(J+1)]^{\frac{1}{2}}W \\ &=[D_J\exp(2i\delta_{J+1}^{J,0})]^*, \\ D_J\exp(2i\delta_{J-1}^{J,0})&=E_J-2i[(J+1)/J]^{\frac{1}{2}}W \\ &=[D_J\exp(2i\delta_{J+1}^{J,\pm1})]^*, \end{aligned} \tag{A8}$$

where

$$\begin{aligned} D_J\equiv{}&\exp[-i(\delta^J{}_{J-1,\,J-1}+\delta^J{}_{J+1,\,J+1})] \\ &-a^J{}_{J-1,\,J+1}a^J{}_{J+1,\,J-1} \\ &\times\exp[-i(\delta^J{}_{J-1,\,J+1}+\delta^J{}_{J+1,\,J-1})], \end{aligned}$$

$$\begin{aligned} E_J\equiv{}&\exp[i(\delta^J{}_{J-1,\,J-1}-\delta^J{}_{J+1,\,J+1})] \\ &-a^J{}_{J-1,\,J+1}a^J{}_{J+1,\,J-1} \\ &\times\exp[i(\delta^J{}_{J-1,\,J+1}-\delta^J{}_{J+1,\,J-1})]. \end{aligned}$$

$$\begin{aligned} W&=a^J{}_{J-1,\,J+1}\sin(\delta^J{}_{J-1,\,J-1}-\delta^J{}_{J-1,\,J+1}) \\ &=a^J{}_{J+1,\,J-1}\sin(\delta^J{}_{J+1,\,J+1}-\delta^J{}_{J+1,\,J-1}), \end{aligned} \tag{A9}$$

$a^J{}_{J-1,\,J-1}=a^J{}_{J+1,\,J+1}=1.$

$\delta_l^{Jm_s}=\delta_l^J$ for all uncoupled states.

[11] W. Rarita and J. Schwinger, Phys. Rev. **59**, 556 (1941).

The subscript i in Eqs. (A7) here takes on the values $J-1$, $J+1$ instead of 1, 2. The Wronskian conditions (A9) follow immediately from the differential equations (A4) or (A20):

$$u_1u_2'-u_2u_1'+w_1w_2'-w_2w_1'=\text{constant}. \tag{A10}$$

Boundary conditions at the origin require the constant to be zero; the asymptotic form of (A10) is (A9).

In Born approximation, Eqs. (A2) and (A8) become

$$^BS=\frac{1}{k}\sum_{J,l,m_s}(2l+1)\,{}^B\delta_l^{Jm_s}\prod_l^{Jm_s}P_l, \tag{A11}$$

$$^B\delta_l^{Jm_s}={}^B\delta_{ll}^J-\epsilon_l^{Jm_s}\,{}^Ba_{lL}^J,\quad l=J-1,\,J+1;\; L=2J-l.$$

$$\begin{aligned} \epsilon_{J-1}^{J\pm}&=-\epsilon_{J+1}^{J,0}=1/\epsilon_{J+1}^{J\pm} \\ &=-1/\epsilon_{J-1}^{J,0}=[J/(J+1)]^{\frac{1}{2}}. \end{aligned} \tag{A12}$$

For any linear combination of central and tensor potentials, with arbitrary exchange dependence, $V(\mathbf{r},\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2)=[-J(r)S_{12}-J_c(r)][a+bP_x]$, Eq. (A11) can be written in the closed form[10]

$$^BS=F1+\begin{vmatrix} C_1 & C_2e^{-i\phi} & C_3e^{-2i\phi} \\ C_2e^{i\phi} & -2C_1 & -C_2e^{-i\phi} \\ C_3e^{2i\phi} & -C_2e^{i\phi} & C_1 \end{vmatrix}, \tag{A13}$$

where

$$\begin{aligned} F&=aF(\theta)+bF(\pi-\theta)\equiv aF_K+bF_L; \\ F(\theta)&=\frac{M}{\hbar^2}\int r^2J_C(r)\frac{\sin Kr}{Kr}dr, \\ C_1&=-\tfrac{1}{2}C_++\tfrac{3}{2}\cos\theta C_-, \\ C_2&=(3/\sqrt{2})\sin\theta C_-, \\ C_3&=-\tfrac{3}{2}[C_++\cos\theta C_-], \\ C_\pm&=aC_K\pm bC_L\equiv aC(\theta)\pm bC(\pi-\theta), \\ C(\theta)&=\frac{M}{\hbar^2}\int r^2J(r)\frac{g_2(Kr)}{Kr}dr. \end{aligned} \tag{A14}$$

The procedure for calculating S will be to remove from the Born scattering matrix (A13) the first few terms of its partial wave expansion (A11), and to replace them by the corresponding terms in the exact scattering matrix. The result will then correspond to a scattering matrix containing explicitly the phase shifts of the few lowest angular momentum states and implicitly the Born approximation on all higher states.

$$S={}^BS+S'$$

$$\begin{aligned} A_l'^J&=\exp(2i\delta_l^{J,\pm1})-1-2i\,{}^B\delta_l^{J,\pm1}; \\ B_l'^J&=\exp(2i\delta_l^{J,0})-1-2i\,{}^B\delta_l^{J,0}, \end{aligned} \tag{A15}$$

where S' is defined analogously to Eqs. (A2), (A3), but with $A_l'^J$, $B_l'^J$ replacing A_l^J, B_l^J.

For $p-p$ scattering, replace $S(\theta,\phi)$ with $S(\theta,\phi)-S(\pi-\theta,\phi+\pi)$.

The triplet contribution to the differential scattering cross section, reduced to terms containing just Legendre polynomials, is

$$(d\sigma/d\Omega)_{\mathrm{triplet}}=(d\sigma/d\Omega)_{\mathrm{trip}}{}^{\mathrm{Born}}+(d\sigma/d\Omega)_{\mathrm{TI}}+(d\sigma/d\Omega)_{\mathrm{CI}}+(d\sigma/d\Omega)'=\tfrac{1}{4}\,\mathrm{Tr}(S^{\dagger}S),$$

$$(d\sigma/d\Omega)_{\mathrm{trip}}{}^{\mathrm{Born}}=\tfrac{1}{4}\,\mathrm{Tr}({}^{B}S^{\dagger}\,{}^{B}S)=\tfrac{3}{4}|F|^{2}+6[a^{2}C_K{}^{2}+b^{2}C_L{}^{2}-abC_KC_L],$$

$$(d\sigma/d\Omega)_{\mathrm{TI}}=-(1/4k)\{[aC_K+bC_L]\times[\epsilon_0(\Delta_0+\Delta_2P_2)+\epsilon_2(\Delta_1P_1+\Delta_3P_3)]+[aC_K-bC_L][\epsilon_2(\nu_0+\nu_2P_2)+\epsilon_0\nu_1P_1]\},$$

$$(d\sigma/d\Omega)_{\mathrm{CI}}=(1/4k)[aF_K+bF_L]\times[\epsilon_0(\Delta_0{}^{C}+\Delta_2{}^{C}P_2)+\epsilon_2(\Delta_1{}^{C}P_1+\Delta_3{}^{C}P_3)],$$

$$(d\sigma/d\Omega)'=(\epsilon_4/4k^{2})[\Lambda_0+\Lambda_1P_1+\Lambda_2P_2+\Lambda_3P_3+\Lambda_4P_4]=\tfrac{1}{4}\,\mathrm{Tr}(S'^{\dagger}S'). \quad \text{(A16)}$$

(For $n-p$ scattering, $\epsilon_0\equiv\epsilon_2\equiv\epsilon_4\equiv 1$.) (For $p-p$ scattering, $\epsilon_0\equiv 0$, $\epsilon_2\equiv 2$, $\epsilon_4\equiv 4$ and $a\equiv 1$, $b\equiv -1$.)

$$\Delta_0=-\Delta_0{}^{10}+\Delta_0{}^{11}+\tfrac{3}{2}\Delta_2{}^{11}-(5/2)\Delta_2{}^{21}+\Delta_2{}^{31},$$
$$\Delta_2=-2\Delta_2{}^{10}-\Delta_2{}^{11}+5\Delta_2{}^{21}+\Delta_2{}^{31}-3\Delta_2{}^{30},$$
$$\Delta_1=-\Delta_1{}^{00}+\tfrac{3}{2}\Delta_1{}^{11}+\tfrac{3}{2}\Delta_1{}^{21}-2\Delta_1{}^{20}+3\Delta_3{}^{21},$$
$$\Delta_3=-2\Delta_3{}^{21}-3\Delta_3{}^{20},$$
$$\Delta_0{}^{C}=\Delta_0{}^{10}+2\Delta_0{}^{11},$$
$$\Delta_2{}^{C}=2\Delta_2{}^{10}+\Delta_2{}^{11}+5\Delta_2{}^{21}+3\Delta_2{}^{30}+4\Delta_2{}^{31},$$
$$\nu_0=3\Delta_1{}^{00}+\tfrac{3}{2}\Delta_1{}^{21}+\Delta_3{}^{21}-(9/2)\Delta_1{}^{11},$$
$$\nu_2=6\Delta_1{}^{20}-6\Delta_1{}^{21}+9\Delta_3{}^{20}-4\Delta_3{}^{21},$$
$$\nu_1=3\Delta_0{}^{10}-3\Delta_0{}^{11}+6\Delta_2{}^{10}-\tfrac{3}{2}\Delta_2{}^{11}-(15/2)\Delta_2{}^{21}+3\Delta_2{}^{31},$$
$$\Delta_1{}^{C}=\Delta_1{}^{00}+3\Delta_1{}^{11}+2\Delta_1{}^{20}+3\Delta_1{}^{21},$$
$$\Delta_3{}^{C}=3\Delta_3{}^{20}+2\Delta_3{}^{21}.$$

$$\Lambda_0=\Delta_{11}{}^{000}+3\Delta_{11}{}^{111}+2\Delta_{11}{}^{220}+3\Delta_{11}{}^{221}+3\Delta_{33}{}^{220}+2\Delta_{33}{}^{221}+\Delta_{00}{}^{110}+2\Delta_{00}{}^{111}+2\Delta_{22}{}^{110}+\Delta_{22}{}^{111}+5\Delta_{22}{}^{221}+3\Delta_{22}{}^{330}+4\Delta_{22}{}^{331}.$$

$$\Lambda_2=4\Delta_{11}{}^{020}+\tfrac{3}{2}\Delta_{11}{}^{111}+9\Delta_{11}{}^{121}+2\Delta_{11}{}^{220}+\tfrac{3}{2}\Delta_{11}{}^{221}+6\Delta_{13}{}^{020}+6\Delta_{13}{}^{121}+(12/7)\Delta_{13}{}^{220}-(6/7)\Delta_{13}{}^{221}+(24/7)\Delta_{33}{}^{220}+(8/7)\Delta_{33}{}^{221}+4\Delta_{02}{}^{110}+2\Delta_{02}{}^{111}+10\Delta_{02}{}^{121}+6\Delta_{02}{}^{130}+8\Delta_{02}{}^{131}+2\Delta_{22}{}^{110}-\tfrac{1}{2}\Delta_{22}{}^{111}+(25/14)\Delta_{22}{}^{221}+(24/7)\Delta_{22}{}^{331}+(24/7)\Delta_{22}{}^{330}+5\Delta_{22}{}^{121}+(40/7)\Delta_{22}{}^{231}-(8/7)\Delta_{22}{}^{131}+(12/7)\Delta_{22}{}^{130}.$$

$$\Lambda_4=(72/7)\Delta_{13}{}^{220}+(48/7)\Delta_{13}{}^{221}+(18/7)\Delta_{33}{}^{220}-(8/7)\Delta_{33}{}^{221}+(40/7)\Delta_{22}{}^{221}+(100/7)\Delta_{22}{}^{231}+(4/7)\Delta_{22}{}^{331}+(36/7)\Delta_{22}{}^{131}+(72/7)\Delta_{22}{}^{130}+(18/7)\Delta_{22}{}^{330}.$$

$$\Lambda_1=2\Delta_{10}{}^{010}+6\Delta_{01}{}^{111}+6\Delta_{01}{}^{121}+4\Delta_{01}{}^{120}+4\Delta_{12}{}^{010}+3\Delta_{12}{}^{111}+9\Delta_{12}{}^{121}-\tfrac{3}{5}\Delta_{21}{}^{121}+\tfrac{4}{5}\Delta_{21}{}^{120}+3\Delta_{12}{}^{221}+(48/5)\Delta_{12}{}^{231}+(36/5)\Delta_{12}{}^{230}.$$

$$\Lambda_3=6\Delta_{12}{}^{121}+(18/5)\Delta_{21}{}^{121}+(36/5)\Delta_{21}{}^{120}+12\Delta_{12}{}^{221}+6\Delta_{12}{}^{030}+12\Delta_{12}{}^{131}+(12/5)\Delta_{12}{}^{231}+(24/5)\Delta_{12}{}^{230}+4\Delta_{03}{}^{121}+6\Delta_{03}{}^{120}, \quad \text{(A17)}$$

where

$$\Delta_l{}^{J1}=\Delta_l{}^{J,-1}=\mathrm{Im}A_l{}'^{J},$$
$$\Delta_{ll'}{}^{JJ',-1}=\Delta_{ll'}{}^{JJ'1}=\tfrac{1}{4}\Re[A_l{}'^{J}(A_{l'}{}'^{J'})^{*}],$$
$$\Delta_l{}^{J0}=\mathrm{Im}B_l{}'^{J},\qquad \Delta_{ll'}{}^{JJ'0}=\tfrac{1}{4}\Re[B_l{}'^{J}(B_{l'}{}'^{J'})^{*}]. \quad \text{(A17b)}$$

The identity $A_{J-1}{}^{J}-A_{J+1}{}^{J}=B_{J-1}{}^{J}-B_{J+1}{}^{J}$ follows from (A8), (A9). The summation has been carried out explicitly over the ${}^3S_1$, ${}^3P_0$, ${}^3P_1$, ${}^3P_2$, ${}^3D_1$, ${}^3D_2$, ${}^3D_3$, and ${}^3F_2$ states. In $(d\sigma/d\Omega)'$, DF interference has been omitted. For $p-p$ scattering, terms which contain an even subscript do not appear and (A17) simplifies considerably.

The radial differential equations for which "exact" solutions were obtained in the present paper will be considered now in more detail.

Let

$$U=e^{-ay}/y^{2}\ \text{for}\ y\geq y_0,$$
$$=e^{-ay_0}/y_0{}^{2}\ \text{for}\ y\leq y_0=\text{constant}.$$

$${}^3P_0:\quad \frac{d^2u^0}{dy^2}-\frac{2}{y^2}u^0+u^0=4\lambda Uu^0; \quad \text{(A18)}$$

$${}^3P_1:\quad \frac{d^2u^1}{dy^2}-\frac{2}{y^2}u^1+u^1=-2\lambda Uu^1; \quad \text{(A19)}$$

$$\left.\begin{aligned} {}^3P_2:&\quad \frac{d^2u}{dy^2}-\frac{2}{y^2}u+u=\tfrac{2}{5}\lambda U(u-3(6)^{\frac{1}{2}}w);\\ {}^3F_2:&\quad \frac{d^2w}{dy^2}-\frac{12}{y^2}w+w=\tfrac{2}{5}\lambda U(4w-3(6)^{\frac{1}{2}}u);\end{aligned}\right\} \quad \text{(A20)}$$

$$a=1/kR;\ \lambda=MV_0R^2/\hbar^2;\ k^2=ME/\hbar^2;\ y=kr.$$

The potential $V(r)$ is given in Eq. (3). There are four independent sets of solutions,

$$\begin{Bmatrix}u_1\\ w_1\end{Bmatrix},\quad \begin{Bmatrix}u_2\\ w_2\end{Bmatrix},\quad \begin{Bmatrix}u_3\\ w_3\end{Bmatrix},\quad \begin{Bmatrix}u_4\\ w_4\end{Bmatrix}$$

to Eq. (A20). Examination of the power series representation in the neighborhood of the origin shows that two of the solutions always vanish at the origin, and the other two are irregular and must be discarded because of the usual arguments on quadratic integrability and conservation of current. The two regular solutions,

$$\begin{Bmatrix}u_1\\ w_1\end{Bmatrix},\quad \begin{Bmatrix}u_2\\ w_2\end{Bmatrix},$$

will be called a "fundamental set." Any set of solutions arising from a linear transformation of the fundamental set will also satisfy all of the boundary conditions of the scattering problem and hence may be used to calculate the complex phase shifts $\delta_l{}^{Jm_s}$. It is not difficult to give a plausible argument showing that there ought to be one pair of solutions,

$$\left\{\begin{matrix} u_1 \rightarrow u_{11}{}^2 \\ w_1 \rightarrow u_{31}{}^2 \end{matrix}\right\},$$

in which the P-state is dominant, at least asymptotically, and another pair,

$$\left\{\begin{matrix} u_2 \rightarrow u_{13}{}^2 \\ w_2 \rightarrow u_{33}{}^2 \end{matrix}\right\},$$

in which the F-state is dominant. Consider the integral equations corresponding to the coupled differential equations (A20) and their boundary conditions:

$$u_{1\alpha}{}^2(y) = A_{1\alpha}{}^2 g_1(y) + \tfrac{2}{5}\lambda g_{-1}(y)\int_0^y U(u_{1\alpha}{}^2 - 3\sqrt{6}u_{3\alpha}{}^2)g_1 dy$$
$$+\tfrac{2}{5}\lambda g_1(y)\int_y^\infty U(u_{1\alpha}{}^2 - 3\sqrt{6}u_{3\alpha}{}^2)g_{-1}dy;$$
$$u_{3\alpha}{}^2(y) = A_{3\alpha}{}^2 g_3(y) + \tfrac{2}{5}\lambda g_{-3}(y)\int_0^y U(4u_{3\alpha}{}^2 - 3\sqrt{6}u_{1\alpha}{}^2)g_3 dy$$
$$+\tfrac{2}{5}\lambda g_3(y)\int_y^\infty U(4u_{3\alpha}{}^2 - 3\sqrt{6}u_{1\alpha}{}^2)g_{-3}dy;$$
$$g_{\pm l}(y) = (\tfrac{1}{2}\pi y)^{\frac{1}{2}} J_{\pm(l+\frac{1}{2})}(y);\quad g_1 \underset{y\to 0}{\longrightarrow} \frac{y^2}{3};\quad g_3 \underset{y\to 0}{\longrightarrow} \frac{y^4}{105}. \tag{A21}$$

The constants $A_{1\alpha}{}^2$, $A_{3\alpha}{}^2$ are arbitrary; the subscripts α denote the duplicity of regular solutions and take on the values 1, 3. The asymptotic form of (A21) yields integral expressions for the amplitudes and phase shifts:

$$A_{1\alpha}{}^2 = a_{1\alpha}{}^2 \cos\delta_{1\alpha}{}^2,$$
$$a_{1\alpha}{}^2 \sin\delta_{1\alpha}{}^2 = -\tfrac{2}{5}\lambda\int_0^\infty U(u_{1\alpha}{}^2 - 3\sqrt{6}u_{3\alpha}{}^2)g_1 dy;$$
$$A_{3\alpha}{}^2 = a_{3\alpha}{}^2 \cos\delta_{3\alpha}{}^2,$$
$$a_{3\alpha}{}^2 \sin\delta_{3\alpha}{}^2 = -\tfrac{2}{5}\lambda\int_0^\infty U(4u_{3\alpha}{}^2 - 3\sqrt{6}u_{1\alpha}{}^2)g_3 dy, \tag{A22}$$

where

$$u_{1\alpha}{}^2 \sim a_{1\alpha}{}^2 \sin(y - \pi/2 + \delta_{1\alpha}{}^2),$$
$$u_{3\alpha}{}^2 \sim a_{3\alpha}{}^2 \sin(y - 3\pi/2 + \delta_{3\alpha}{}^2).$$

The weighting influence of the $g_3(y)$ term (which is small throughout the region in which the nuclear potential is large) in the various integrands suggests that the "subdominant" amplitudes $a_{13}{}^2$, $a_{31}{}^2$ might best be kept small by placing $A_{13}{}^2 = A_{31}{}^2 = 0$, which amounts to choosing $\delta_{13}{}^2 = \delta_{31}{}^2 = \pi/2$. In Born approximation, for which the free particle trial functions $u_{11}{}^2 = a_{11}{}^2 g_1(y)$, $u_{13}{}^2 = u_{31}{}^2 \equiv 0$, $u_{33}{}^2 = a_{33}g_3(y)$ are used, Eqs. (A22) become

TABLE III. Triplet $p-p$ phase shifts and amplitudes at 350 Mev for singular tensor potential with short range square well cutoff (SRSW case).

	$\delta_{11}{}^2$	$a_{13}{}^2$	$a_{31}{}^2$	$\delta_{33}{}^2$	$\delta_{31}{}^2$	$\delta_{13}{}^2$
Repulsive exact	0.218	−0.101	−0.098	0.050	$\pi/2$	$\pi/2$
Attractive exact	0.018	0.084	0.085	−0.077	$\pi/2$	$\pi/2$
Repulsive Born	0.088	−0.099	−0.099	0.067	$\pi/2$	$\pi/2$
Attractive Born	−0.088	0.099	0.099	−0.067	$\pi/2$	$\pi/2$

$$^B\delta_{11}{}^2 = -\frac{2}{5}\lambda\int_0^\infty U(g_1)^2 dy,\ {}^Ba_{13}{}^2 = {}^Ba_{31}{}^2 = \frac{6}{5}\sqrt{6}\lambda\int_0^\infty Ug_1g_3 dy,$$
$$^B\delta_{33}{}^2 = -\frac{8}{5}\lambda\int_0^\infty U(g_3)^2 dy,\quad \delta_{13}{}^2 = \delta_{31}{}^2 = \pi/2. \tag{A23}$$

It is evident from the behavior of the functions $g_1(y)$ and $g_3(y)$ that, of the four quantities now describing the coupled state scattering ($\delta_{11}{}^2$, $a_{13}{}^2$, $a_{31}{}^2$, $\delta_{33}{}^2$), $\delta_{11}{}^2$ might be large but the other three are small. ($a_{11}{}^2$ and $a_{33}{}^2$ may be normalized to unity since only the ratios

$a_{13}{}^2/a_{33}{}^2$ and $a_{31}{}^2/a_{11}{}^2$ are relevant.)

In general, wherever a comparison of the exact solution with the Born approximation could be made, the latter was found to be quite accurate for the three small quantities ($a_{13}{}^2$, $a_{31}{}^2$, $\delta_{33}{}^2$), with only $\delta_{11}{}^2$ showing marked deviations. Table III gives the comparison at 350 Mev for the SRSW case.

To integrate Eqs. (A20) numerically, it is convenient to start with a power series solution near the origin (where the potential is constant). The roots of the indicial equations are

$$\begin{aligned} \alpha_1 &= 6, & \beta_1 &= 4,\\ \alpha_2 &= 2, & \beta_2 &= 4,\\ \alpha_3 &= -1, & \beta_3 &= 1,\\ \alpha_4 &= -1, & \beta_4 &= -3, \end{aligned}$$

where

$$u_i = \sum_{n=0}^\infty a_n{}^i y^{n+\alpha_i},$$
$$w_i = \sum_{n=0}^\infty b_n{}^i y_n{}^{+\beta_i}. \tag{A24}$$

The fundamental set of regular solutions can therefore be taken[12] to be

$$u_1 = \sum_0^\infty a_n y^{n+6},\quad u_2 = u_1 \log y + \sum_0^\infty C_n y^{n+2},$$
$$w_1 = \sum_0^\infty b_n y^{n+4},\quad w_2 = w_1 \log y + \sum_0^\infty d_n y^{n+4}. \tag{A25}$$

[12] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis* (The Macmillan Company, New York, 1946), Chap. X.

The recurrence relations are

$$a_n[(n+6)(n+5)-2]+K_1a_{n-2}+Kb_n=0,$$
$$b_n[(n+4)(n+3)-12]+K_2b_{n-2}+Ka_{n-4}=0,$$
$$a_{n-4}(2n+3)+C_n[(n+2)(n+1)-2]+K_1C_{n-2}+Kd_{n-4}=0,$$
$$b_n(2n+7)+d_n[(n+4)(n+3)-12]+K_2d_{n-2}+KC_n=0, \quad \text{(A26)}$$

where

$$K_0=\tfrac{2}{5}\lambda e^{-ay_0}/y_0^2;\quad K=\pm 3(6)^{\frac{1}{2}}K_0;\quad K_1=1\mp K_0;\quad K_2=1\mp 4K_0.$$

(Upper sign: "attractive," lower sign: "repulsive.")

The quantities a_0 and d_0 are undetermined; the former merely defines the normalization and the latter represents the arbitrary amount of solution $\alpha=1$ that may be mixed in solution $\alpha=2$.

In some cases the coupled equations were integrated on a differential analyzer; in others, a desk calculator was used. To check the phase shifts, the resulting radial functions were used as trial functions in the integral equations. For the uncoupled equations, a method recently described by Kynch[13] was used. Its advantage lies in the fact that the nuclear phase shift is integrated directly, whereas in an integration of the wave function most of the effort is "wasted" in obtaining the centrifugal phase shift. If the quantity $\tan^{-1}(-1^l\, S(y'))$ represents the phase shift which would obtain if the potential for $y\geq y'$ were placed equal to zero, then

$$(-1)^{l+1}\, dS_l/dy=V(g_l+Sg_{-l})^2 \quad \text{(A27)}$$

where $d^2u/dy^2-l(l+1)u/y^2+u=Vu$. $S(y)$ is either monotonically increasing or decreasing depending on whether the potential is repulsive or attractive. For a square well cutoff, the power series expansion for $S(y)$ (P-state) is given by

$$S(y)=S_0y^5+S_2y^7+S_4y^9+S_6y^{11}.$$

Let $V=\epsilon=$ constant.

$$S_0=\frac{\epsilon}{45},\quad S_4=\frac{\epsilon}{9}\left(S_0^2-\frac{4}{15}S_0-\tfrac{2}{3}S_2+\frac{1}{525}\right),$$
$$S_2=-\frac{2\epsilon}{21}\left(S_0+\frac{1}{30}\right),\quad S_6=\frac{\epsilon}{11}\left(S_0^2+2S_0S_2+\frac{4}{35}S_0-\frac{4}{15}S_2-\tfrac{2}{3}S_4-\frac{4}{42\,525}\right). \quad \text{(A28)}$$

Within a hard core $S(y)=-g_1(y)/g_{-1}(y)$ (P-state).

The author wishes to express his sincere thanks to Professor Robert Serber, Dr. R. S. Christian, Professor Owen Chamberlain, and Dr. Robert Jastrow for many stimulating discussions and helpful suggestions. The assistance of Mr. John Killeen and the operating crew of the differential analyzer was invaluable. The excellent computational work of Mr. Victor Brady was also much appreciated.

[13] G. J. Kynch, Proc. Phys. Soc. (London) 65, 2, 83 (1952).