Multiple Scattering and the Many-Body Problem—Applications to Photomeson Production in Complex Nuclei*

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A study is made of the scattering of fast particles by atomic nuclei. A rigorous (formal) solution to the many-body Schrödinger equation is given which has the structure of a multiply scattered wave. The relation of this solution to the "impulse approximation" is discussed. A decomposition of the wave function into "coherent" and "incoherent" parts is effected. This makes it possible to derive the familiar "optical models" directly, as well as to find systematic corrections to these models. The theory is applied to a discussion of photomeson production in complex nuclei.

I. INTRODUCTION

THE existence of high energy accelerators has made it possible to study reactions in complex nuclei induced by bombarding particles whose energies are large compared to nuclear binding energies. For such processes it is expected that the binding energy of the nucleus plays only a secondary role and that the scattering of the incoming particle by the nucleus can be described in terms of its scattering by the constituent nucleons of the nucleus individually. That is, the dynamical treatment of the many-body problem is thereby reduced to that of the two-body problem.

A number of models have been used to treat these reactions. In their study of the scattering of high energy neutrons by nuclei, Fernbach, Serber, and Taylor[1] have considered the nucleus to be a continuous "optical medium" characterized by an index of refraction and an absorption coefficient. The phase change of the incident wave is calculated as if it had followed a well-defined geometrical path through the nucleus, the total wave being the algebraic sum of the wavelets resulting from all such paths.

A somewhat more detailed application of this model has been made by Byfield, Kessler, and Lederman[2] and by Steinberger[3] in the discussion of their own experiments of meson scattering by nuclei. This involved solving the Schrödinger equation which results from expressing the meson-nucleus interaction in terms of the "optical parameters" mentioned above.

On the other hand, an elegant reduction of the many-body interactions is afforded by the use of the "impulse approximation," which has been introduced by Chew[4] and discussed in detail by Chew and Wick[5] and Chew and Goldberger.[6] The treatment given by these authors did not appear to be in the most convenient form, however, for a systematic development of the multiple scattering aspects the problem—and thus for a derivation of the optical models mentioned above.

The purpose of the present paper is to attempt the description of the multiple scattering of a fast particle in a nucleus. A rigorous formal solution to the many-body Schrödinger equation will be obtained which has the structure to be expected of a multiply scattered wave. The relation of this solution to the impulse approximation will be discussed.

On introducing a further approximation, which assumes the number of scatterers to be large, it is possible to make a separation of "coherent" and "incoherent" effects in the above solution to the Schrödinger equation. This makes it simple to derive the "optical models" mentioned above and to discuss more generally some formal relations to be expected for the scattering cross sections. Since the optical models are obtained almost directly from a rigorous solution to the Schrödinger equation, systematic corrections to these models appear to be quite straightforward.

To make the discussion of the model more specific, we shall apply it to photomeson production in complex nuclei (which includes discussion of meson scattering in nuclei). This will illustrate the flexibility of the method, since we shall also include an interaction for meson absorption (as well as for scattering). The formal arguments are quite general, however, in spite of their rather specific application in the present paper.

In Sec. II we shall introduce phenomenological interactions (and evaluate some of their matrix elements) for the photoproduction and scattering of mesons by individual nucleons, as well as for absorption by a pair of nucleons. It will then be shown that the solution to the Schrödinger equation for photomeson production involves solving the Schrödinger equation for meson scattering by the nucleus.

There appears to be ample experimental evidence[7–9] that meson scattering[2,3,10] plays a significant role in

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[1] Fernbach, Serber, and Taylor, Phys. Rev. **75**, 1352 (1949). R. Serber [Phys. Rev. **72**, 1114 (1947)] has described the general features common to the various models employed to discuss high energy nuclear reactions.

[2] Byfield, Kessler, and Lederman, Phys. Rev. **86**, 17 (1952) (further references are given in this work).

[3] J. Steinberger (private communication).

[4] G. F. Chew, Phys. Rev. **80**, 196 (1950).

[5] G. F. Chew and G. C. Wick, Phys. Rev. **85**, 636 (1952).

[6] G. F. Chew and M. L. Goldberger, Phys. Rev. **87**, 778 (1952).

[7] McMillan, Peterson, and White, Science **110**, 579 (1949).

[8] R. F. Mozeley, Phys. Rev. **80**, 493 (1950).

[9] R. Littauer and D. Walker, Phys. Rev. **86**, 838 (1952).

[10] G. Bernardini and F. Levy, Phys. Rev. **84**, 610 (1951).

photomeson production. The observed variation of the cross section with atomic mass number A is $A^{\frac{2}{3}}$, which is proportional to the cross-sectional area of the nucleus and would seem to imply that only those mesons produced in the surface of the nucleus are able to get out. On the other hand, those mesons which are produced and re-absorbed in the nucleus will lead to high nuclear excitation. One would consequently expect to see an increase in the cross section for producing nuclear stars by γ-rays at about the threshold for meson production, which has also been observed.[11–13] (In a subsequent paper by Reff[14] a detailed comparison of the consequences of the present theory with the observed phenomena will be made.)

In Sec. III the scattering of a meson by the nucleus will be discussed. As a multiple scattering type of theory, we shall see that the model can be well described in terms of meson scattering at a point in the nucleus, followed by a scattering at a different point, followed by a scattering at a third point, etc. We shall neglect specifically "field effects" which correspond to re-creation of an absorbed meson, so once the meson is absorbed the multiple scattering process ends. We shall repeatedly encounter averages of certain operators with respect to the nuclear wave functions. These expectation values can be expressed in terms of nucleon density in the nucleus, nucleon momentum distributions, and correlation in nucleon positions within the nuclear structure.

The interpretation of such expectation values with respect to the ground state of the nucleus can be made on very reasonable physical grounds. The interpretation of expectation values with respect to excited nuclear states is more ambiguous in general. However, this will not cause difficulty for the following reason: We consider the meson to be sufficiently energetic that its velocity is of the order of c, the velocity of light. The velocity of the heavier nucleons will be appreciably less than this ($\simeq c/10$). We imagine the meson to be produced (or scattered) at a point. The meson leaves this point with a velocity of about c, while the "shock wave" carrying nuclear excitation will proceed from this same point at a much lower velocity. When the meson reaches the position at which it is to be scattered again, it is expected to have "outrun" the nuclear excitation and will find the structure of the nuclear medium in the vicinity of the point of subsequent scattering to be essentially the same as in the nuclear ground state. The expectation value of operators localized at this second point will thus be replaced by expectation values with respect to the ground state of the nucleus.

[11] R. D. Miller, Phys. Rev. **82**, 260 (1951).
[12] J. Keck, Phys. Rev. **85**, 410 (1952).
[13] S. Kikuchi, Phys. Rev. **86**, 41 (1952).
[14] I. Reff (to be published). See also, Phys. Rev. **87**, 207 (1952), for a preliminary account.

II. FORMULATION OF THE PROBLEM

A. Preliminary Definitions

We seek to describe the mesonic processes in complex nuclei in terms of elementary interactions between subunits of the many-body system. The first of these is the interaction of the electromagnetic field with individual nucleons to produce a meson. This we write as

$$H'=\sum_{l=1}^{A}\exp(-iq\cdot\mathbf{Z}_l)N_l, \tag{1}$$

where $\mathbf{q}$ is the momentum of the produced meson, $\mathbf{Z}_l$ is the coordinate of the lth nucleon, and N_l is a function of $\mathbf{q}$, spin and isotopic spin operators, and the photon energy and polarization. The sum is over the A nucleons in the nucleus. N_l has nonvanishing matrix elements for those charge states of the meson and nucleon which conserve charge.

As we are considering photomeson production from a nucleus, we shall need the Hamiltonian and eigenfunctions of the nuclear system. Let the Hamiltonian be H_N, so the eigenfunctions ψ_I satisfy

$$H_N\psi_I=\epsilon_I\psi_I \tag{2}$$

for a state of binding energy ϵ_I. The process of photomeson production leads in general from an initial nuclear state A to a final nuclear state F, possibly through intermediate nuclear states I.

The wave function for a meson in a plane wave state of momentum $\mathbf{q}$ is (we use as units $\hbar=c=1$)

$$\lambda_q=(2\pi)^{-\frac{3}{2}}\exp(i\mathbf{q}\cdot\mathbf{Z}). \tag{3}$$

The Schrödinger equation for this state is

$$h\lambda_q=\epsilon_q\lambda_q, \tag{4}$$

where h is the kinetic energy operator of the meson and $\epsilon_q\equiv q_0=(q^2+\mu^2)^{\frac{1}{2}}$ is the kinetic energy of the meson, whose rest mass is μ.

Expressing the energy operator of the electromagnetic field by H_γ, we define

$$H_0\equiv h+H_N+H_\gamma. \tag{5}$$

The eigenvalues of H_0 are written as E_I; and, in particular, the energy of the ground state of the nucleus plus the incident photon is called E_A.

Now, if the meson were to leave the nucleus without interaction, the Hamiltonian would be

$$H=H_0+H',$$

and the cross section for photomeson production would be

$$\sigma=(2\pi)^4S\langle H'\delta(E_A-H_0)H'\rangle. \tag{6}$$

By the symbol $\langle\cdots\rangle$ we mean the average with respect to the ground state of the nucleus. The symbol S designates an average with respect to initial polarization states. If we neglect correlations between nucleon positions and the energy of excitation of the nucleus, use of Eq. (1) leads to

$$\sigma=(2\pi)^4 S\sum_l \langle N_l^+\delta(\epsilon_q-h)N_l\rangle$$
$$=A\sigma_f, \qquad (7)$$

where

$$\sigma_f=A^{-1}[Z(\sigma_p^{+}+\sigma_p^{0})+(A-Z)(\sigma_n^{-}+\sigma_n^{0})].$$

σ_p^+ is the cross section for producing a π^+ meson from a free proton, σ_p^0 that for a π^0 meson, etc. Z is the number of protons in the nucleus, so σ_f is the effective "average cross section" per nucleon for producing a meson. For brevity of notation in Eq. (7), we have summed over individual cross sections.

Equation (7) is based on the assumption that the produced meson does not interact with the nucleus. Finding the corrections to Eq. (7) resulting from such interactions will be our primary concern. It is known that the meson can be scattered as well as absorbed in nuclear matter. The scattering will be assumed to arise from an interaction of the meson with one nucleon at a time (i.e., we neglect many-body interactions). The absorption (to conserve energy and momentum) must, however, be a many-body interaction. We assume for reasons discussed previously[15] that it involves an interaction of the meson with a pair of nucleons. Although these interactions could be formally derived from a field theoretic approach, it seems preferable to introduce them in a phenomenological manner, as was the electromagnetic interaction of Eq. (1).

The scattering interaction of the meson with the αth nucleon we designate by

$$V_\alpha=(\mathbf{Z}'-\mathbf{Z}_\alpha|V_\alpha|\mathbf{Z}-\mathbf{Z}_\alpha), \qquad (8)$$

which is assumed to be diagonal in the nucleon coordinate $\mathbf{Z}_\alpha$ because the mass of the nucleon is considerably larger than is that of a meson. $\mathbf{Z}$ and $\mathbf{Z}'$ represent the space coordinate of the meson. From the algebraic analysis of Chew and Goldberger[6] we obtain the transition operator for the scattering arising from V as

$$t_\alpha^0=V_\alpha+V_\alpha(\epsilon_b+i\eta-h-k_\alpha-V_\alpha)^{-1}V_\alpha. \qquad (9)$$

Here ϵ_b is the energy of the meson and nucleon and k_α is the kinetic energy operator of the αth nucleon. The use of "η," as a positive parameter which goes to zero after the integrations are done, follows the convention of Lippmann and Schwinger.[16] Again, because of the large mass of the nucleon, we shall assume that in a coordinate representation t_α^0 takes the form[17]

$$t_\alpha^0=(\mathbf{Z}'-\mathbf{Z}_\alpha|t_\alpha^0|\mathbf{Z}-\mathbf{Z}_\alpha). \qquad (9)$$

Transforming to a meson momentum representation, we have

$$(2\pi)^{-3}\int \exp(-i\mathbf{q}'\cdot\mathbf{Z}')t_\alpha \exp(i\mathbf{q}\cdot\mathbf{Z})d^3Zd^3Z'$$
$$\equiv\exp[-i(\mathbf{q}'-\mathbf{q})\cdot\mathbf{Z}_\alpha](q'|t_\alpha^0|q), \qquad (10)$$

where $(q'|t_\alpha^0|q)$ is independent of $\mathbf{Z}_\alpha$.

The scattering interaction for a nucleus containing A nucleons is

$$V=\sum_{\alpha=1}^{A} V_\alpha. \qquad (11)$$

It is understood that the isotopic spin dependence of V_α and t_α^0 corresponds to the observed scattering phenomena for the various charge states of the meson and nucleon.

The interaction for absorption has been discussed previously[15,18] and is obtainable from that for producing mesons in nucleon-nucleon collisions. We assume that the absorbing pair of nucleons has a relative coordinate $\mathbf{r}$ and a center-of-mass coordinate $\mathbf{x}$. Then the absorption operator R_μ, corresponding to absorption by the μth pair of nucleons is approximately[19]

$$(\mathbf{r}',\mathbf{x}'|R_\mu|\mathbf{Z},\mathbf{r},\mathbf{x})=R_\mu^0(\mathbf{r}',\mathbf{Z}-\mathbf{x})\delta(\mathbf{x}-\mathbf{x}')\delta(\mathbf{r}). \qquad (12)$$

The $\delta(\mathbf{r})$ approximates a short-range interaction and has been discussed in detail in references 15 and 18. R_μ also contains a corresponding term to produce mesons. When the meson is absorbed in the nucleus by two nucleons, they are expected to recoil with an energy of about 140 Mev. We suppose that these two recoil nucleons have a relative momentum $\mathbf{p}$ and a total momentum $\mathbf{G}$ and that their wave function is a plane wave. Transforming R_μ^0, we have

$$(2\pi)^{-3}\int \exp(-i\mathbf{p}\cdot\mathbf{r}')R_\mu^0(\mathbf{r}',\mathbf{Z}-\mathbf{x}) \exp(i\mathbf{q}\cdot\mathbf{Z})d^3r'd^3Z$$
$$\equiv\exp(i\mathbf{q}\cdot\mathbf{x})R_\mu^0(\mathbf{p},\mathbf{q}). \qquad (13)$$

The absorption operator for the nucleus is the sum of R_μ over all nucleon pairs:

$$R=\sum_\mu R_\mu. \qquad (14)$$

To avoid a discussion of field emission and absorption effects, we shall include R in the Hamiltonian of the system, but shall treat it as a small perturbation so that it will indeed be a transition operator. One-half the transition rate for absorption in the nucleus is then (neglecting correlations between pairs)

$$\Delta_0\equiv\pi\sum_\mu\langle R_\mu\delta(E_A-H_0)R_\mu\rangle, \qquad (15)$$

where it is understood that the first R_μ absorbs and the second R_μ reproduces the meson. As before[15] we approximate E_A-H_0 by ϵ_q-p^2/M, where M is the nu-

[15] Brueckner, Serber, and Watson, Phys. Rev. **84**, 258 1(951).
[16] B. Lippmann and J. Schwinger, Phys. Rev. **79**, 469 (1950).
[17] A discussion of this point has been given by Fernbach, Green, and Watson, Phys. Rev. **84**, 1084 (1951).
[18] K. Watson and K. Brueckner, Phys. Rev. **83**, 1 (1951).
[19] This follows from the form given in reference 15 when $\mu/2M$ is neglected compared to unity (W is the nucleon mass). A specific assumption about the form of t and R is not, of course, necessary for the general theory of the next section.

cleonic mass. To calculate the matrix element $(q'|\Delta_0|q)$, we first consider the absorption of the meson by nucleons "1" and "2," so [see Eq. (12)]

$$\mathbf{r}=\mathbf{Z}_1-\mathbf{Z}_2, \quad \mathbf{x}=\tfrac{1}{2}(\mathbf{Z}_1+\mathbf{Z}_2). \tag{16}$$

Then

$$\begin{aligned}\langle R_{(12)}\delta(E_A-H_0)R_{(12)}\rangle &= \int\psi_A(\mathbf{r}', \mathbf{x}, \mathbf{Z}_3\cdots\mathbf{Z}_A)\delta(\mathbf{r}')R^{0+}{}_{(12)}(\mathbf{p}, \mathbf{q}')\\ &\quad\times d^3p\,\delta(\epsilon_q-p^2/M)R^0{}_{(12)}(\mathbf{p}, \mathbf{q})\delta(\mathbf{r})\exp[-i(\mathbf{q}'-\mathbf{q})\cdot\mathbf{x}]\\ &\quad\times\psi_A(\mathbf{r}, \mathbf{x}, \mathbf{Z}_3\cdots\mathbf{Z}_A)d^3r'd^3rd^3xd^3Z_3\cdots d^3Z_A\\ &=\int\langle R^{0+}{}_{(12)}(\mathbf{p}, \mathbf{q}')d^3p\,\delta(\epsilon_q-p^2/M)R^0{}_{(12)}(\mathbf{p}, \mathbf{q})\rangle\\ &\quad\times P(O, x)\exp[-i(\mathbf{q}'-\mathbf{q})\cdot\mathbf{x}]d^3x,\end{aligned} \tag{17}$$

where $P(O, x)$ is the joint probability of finding the nucleons "1" and "2" at the same point and of finding their center-of-mass at the point $\mathbf{x}$. It is convenient to write

$$P(O, x)=P_0v(x)/V_A. \tag{18}$$

P_0 is a factor describing correlation in nuclear structure and was discussed in reference (15), $v(x)$ is the probability density of the center-of-mass of the nucleons "1" and "2" when they are at the same point. It is normalized according to

$$\int v(x)d^3x=V_A, \tag{19}$$

the nuclear volume. Equations (18) and (19) are in agreement with the unit normalization of the nuclear wave functions.

To obtain $(q'|\Delta_0|q)$ we multiply Eq. (17) by the number of absorbing nucleon pairs, N_p. We now evaluate Δ_0 by three methods, the first two being approximations to Eq. (17). It is apparent that (except for the smallest nuclei) Δ_0 will be nearly diagonal in q. Consequently, for the two approximations we replace $R^{0+}(\mathbf{p}, \mathbf{q}')$ by $R^{0+}(\mathbf{p}, \mathbf{q})$ in Eq. (17). Then following the arguments of reference 15, Δ_0 is related to the mean free path for absorption, λ_a, in the nucleus by

$$(q'|\Delta_0|q)=\frac{v_\pi}{2\lambda_a}(2\pi)^{-3}\int v(x)\exp[-i(\mathbf{q}'-\mathbf{q})\cdot\mathbf{x}]d^3x, \tag{20}$$

where v_π is the velocity of the pion before absorption. If constant nuclear density were assumed, we would have

$$\begin{aligned}v(x)&=1 \quad \text{inside nucleus,}\\ v(x)&=0 \quad \text{outside nucleus.}\end{aligned} \tag{21}$$

This assumption is not necessary, but is the one most commonly chosen. For specific calculations in this paper we shall assume Eq. (21).

Then, if the nucleus is very large, the integral in Eq. (20) becomes a δ-function:

(Approximation I)

$$(q'|\Delta_0|q)=(v_\pi/2\lambda_a)\delta(\mathbf{q}'-\mathbf{q}). \tag{22}$$

As will be seen later, this approximation leads to the model of Fernbach, Serber, and Taylor[1] and to that of reference 15.

The second approximation involves using Eq. (20) as it stands. This equation can be simplified in a coordinate representation, if we consider $v_\pi/2\lambda_a$ to be evaluated at the energy of the meson in the nuclear medium and remove it from under the integral below:

(Approximation II)

$$\begin{aligned}(Z'|\Delta_0|Z)&=(2\pi)^{-3}\int\exp(i\mathbf{q}'\cdot\mathbf{Z}')(q'|\Delta_0|q)\\ &\qquad\times\exp(-i\mathbf{q}\cdot\mathbf{Z})d^3q'd^3q\\ &\simeq(v_\pi/2\lambda_a)v(Z)\delta(\mathbf{Z}'-\mathbf{Z}).\end{aligned} \tag{23}$$

We shall see in Sec. III that this leads to the model used by Lederman[2] and by Steinberger.[3] Evaluation of Eq. (17) keeping an arbitrary q' in $R^{0+}(p, q')$ leads to an expression which can be formally written as

(Approximation III)

$$(Z'|\Delta_0|Z)=(v_\pi/2\lambda_a)v(Z', Z). \tag{24}$$

(We have assumed that the nucleus is infinitely heavy and that its center is at the origin of the coordinate system.)

The diagonal element of $t_\alpha{}^0$ [Eq. (9)] with respect to the nuclear coordinates can be found by the same method. Writing $t_C\equiv\sum_\alpha\langle t_\alpha{}^0\rangle$ we obtain

$$(q'|t_C|q)=\sum_\alpha\int\langle(q'|t_\alpha{}^0|q)\rangle\times\exp[-i(\mathbf{q}'-\mathbf{q})\cdot\mathbf{Z}_\alpha]P(Z_\alpha)d^3Z_\alpha. \tag{25}$$

$P(Z_\alpha)$ is the probability per unit volume of finding nucleon "α" at the point Z_α. We express $P(Z_\alpha)$ as

$$P(Z_\alpha)=v_0(Z_\alpha)/V_A\simeq v(Z_\alpha)/V_A. \tag{26}$$

On summing over α, we obtain a factor of A. Approximations I and II are made as for Δ_0. Using the general theorem[16] relating the imaginary part of the scattering amplitude in the forward direction to the total scattering cross section, we obtain (since the total spin of the nucleus is much less than $A/2$)

$$(A/V_A)(2\pi)^3\langle(q|t_\alpha{}^0|q)\rangle=V_0-iv_\pi/2\lambda_S\equiv B_S, \tag{27}$$

where λ_S is the mean free path for a scattering in the nucleus and V_0 is

$$V_0=(2\pi/q_0)\ \mathrm{Re}(a_S)A/V_A.$$

$\mathrm{Re}(a_S)$ is the real part of the meson scattering amplitude in the forward direction (actually the average for the neutrons and protons in the nucleus). Then by our three approximations for Δ_0, we have [Eq. (27)]

$$\begin{aligned} \text{I:}\quad & (q'|t_C|q)=B_S\delta(\mathbf{q}'-\mathbf{q});\\ \text{II:}\quad & (Z'|t_C|Z)=B_Sv_0(Z)\delta(\mathbf{Z}'-\mathbf{Z}); \qquad (28)\\ \text{III:}\quad & (Z'|t_C|Z)=B_Sv_0(Z',Z). \end{aligned}$$

To anticipate the conclusions of Sec. III, we note that V_0 is the "effective well depth" of the nucleus as seen by the meson.[20] It should be remarked that V_0, λ_S, and λ_a are "operators" in the charge coordinates of the meson, their eigenvalues being the appropriate numerical values of these parameters for the three charge states of the meson.

B. Solution of the Schrödinger Equation

The Hamiltonian for the system is

$$H=H_0+R+V+H', \qquad (29)$$

where the various terms are defined by Eqs. (1), (5), (11), and (14). The wave function of the system corresponding to an initial state, ϕ_A, containing a photon and the initial nucleus is

$$\chi_A=\Phi\phi_A. \qquad (30)$$

Φ is the wave matrix introduced by Møller.[21] In the Lippmann-Schwinger[16] formulation, the Schrödinger equation for Φ can be written as (we omit the symbol "+" on Φ)

$$\Phi=1+a^{-1}(H'+R+V)\Phi, \qquad (31)$$

it being understood that this equation operates on ϕ_A. The quantity a is defined as

$$a\equiv E_A+i\eta-H_0. \qquad (32)$$

Spontaneous creation of virtual mesons by R is expected to be small, so we assume that R vanishes when operating on the state ϕ_A. (V obviously does so.) We also treat H' as a small perturbation, since it involves an electromagnetic interaction. Then

$$\Phi=1+\Omega a^{-1}H', \qquad (33)$$

where

$$\Omega=1+a^{-1}(R+V)\Omega. \qquad (34)$$

Ω is seen to describe the scattering of a meson in the nucleus, so our next problem is to study the solution to Eq. (34).

III. THE SCATTERING OF A MESON BY THE NUCLEUS

A. Basic Multiple Scattering Equations

We turn our attention to the solution of Eq. (34). The quantity R is to be considered as a small perturbation.[22] Spontaneous creation of mesons by R is to be neglected. Thus the first occurrence of R in Ω (reading from right to left) will correspond to absorption of a meson. The next occurrence of R must then correspond to creation of a meson. Neglect of "spontaneous creation" implies that the re-creation must occur immediately from the same pair of nucleons. This means that an expression such as

$$\Delta\equiv Ra^{-1}R \qquad (35)$$

can be written as

$$\textstyle\sum_\mu R_\mu a^{-1}R_\mu$$

[Eq. (14)]. Also, following a single R interaction, there is no meson so V vanishes. Now, Δ is the lowest order transition operator for meson scattering arising from R. This is a many-body effect and is expected to be negligible compared to the scattering arising from V. (Experimental studies of meson scattering in hydrogen and deuterium[23] support this hypothesis that the interaction is primarily between the meson and one nucleon at a time.) However, we cannot quite neglect terms like Δ, because this quantity contains the "shadow" cast by true absorption. This arises from the imaginary part of Δ that is diagonal in the nuclear coordinates. We shall then assume that ["Im(···)" means "imaginary part of (···)"]

$$\begin{aligned}\Delta\simeq i\,\mathrm{Im}(\langle\Delta\rangle)&\simeq -i\pi\textstyle\sum_\mu\langle R_\mu\delta(E_A-H_0)R_\mu\rangle\\ &=-i\Delta_0. \qquad (36)\end{aligned}$$

Δ_0 is defined by Eq. (15). We shall henceforth use Δ and $-i\Delta_0$ interchangeably in our equations.

A quantity b is defined as

$$b=a-\Delta. \qquad (37)$$

We note that, following an R which absorbs a meson,

$$b=a, \qquad (38)$$

since Δ vanishes when operating on a state that does not contain a meson.

As a first step in the solution of Eq. (34), we solve the integral equation

$$\Omega_S=1+b^{-1}V\Omega_S. \qquad (39)$$

In the special case that $R=0$, Eqs. (34) and (39) are identical. That is, Ω_S evaluated for $\Delta=0$ describes the scattering of the meson in the absence of absorption.

To solve for Ω_S we introduce two subsidiary functions:

$$t_\alpha'=V_\alpha+V_\alpha(a-V_\alpha)^{-1}V_\alpha, \qquad (40)$$

$$t_\alpha=V_\alpha+V_\alpha(b-V_\alpha)^{-1}V_\alpha. \qquad (41)$$

We shall wish to identify t_α' and t_α with t_α^0 [Eq. (9)]—the scattering from a free nucleon. The identification of t_α' with t_α^0 is the impulse approximation. We shall not discuss this in detail since it would be largely a

[20] From their experiments Byfield *et al.* (reference 2) and Steinberger (reference 3) find that $V_0\simeq 20$ Mev and $(1/\lambda_S+1/\lambda_a)\simeq 10^{+12}$ cm^{-1}. These values are dependent upon the "optical model" which was used—i.e., approximation II in our notation.

[21] C. Møller, Kgl. Danske. Videnskab. Selskab, Mat.-fys. Medd **23**, No. 1 (1945).

[22] This restriction is not necessary [K. Brueckner and K. Watson, to be published].

[23] Anderson, Fermi, Nagle, and Yodh, Phys. Rev. **86**, 413 (1952).

repetition of the analysis of Chew and Goldberger[6] (although our statement of the problem differs somewhat from that of these authors). The essential point is the observation that a [Eq. (32)] and $\epsilon_b+i\eta-h-k_\alpha$ [Eq. (9)] differ only by the excitation imparted to the remainder of the nucleus. This is expected to be a small correction for high energy scatterings, such as we are considering. That $t_\alpha \simeq t_\alpha'$ is demonstrated later in this section.

The solution to Eq. (39) is

$$\Omega_S=1+\sum_{(\alpha)}\frac{1}{b}\left\{t_{\alpha1}+t_{\alpha1}\frac{1}{b}t_{\alpha2}+\cdots+t_{\alpha1}\frac{1}{b}t_{\alpha2}\frac{1}{b}\cdots\frac{1}{b}t_{\alpha n}+\cdots\right\}. \quad (42)$$

Here the index "α_i" on $t_{\alpha i}$ ($\alpha_i=1, 2, \cdots A$) refers to the scattering of the meson by the α_ith nucleon. The summation over the α's (designated by $\sum_{(\alpha)}$) is a summation over all indices "α_i" independently (from 1 to A), except that *no two adjacent* indices can have the same value. Thus, if there were only the one nucleon "α," Ω_S would reduce to

$$\Omega_S=1+(1/b)t_\alpha.$$

Once we identify t_α with $t_\alpha{}^0$, we see that Ω_S represents true multiple scattering,[24] since the meson is scattered first by one nucleon then by another, etc.—and this is summed over all possible ways that such can occur. To see that Eq. (42) is a rigorous solution to Eq. (39), we substitute the former into the right hand side of the latter. We obtain terms like

$$\frac{1}{b}V\sum_{(\alpha)}\frac{1}{b}t_{\alpha1}\frac{1}{b}t_{\alpha2}\cdots\frac{1}{b}t_{\alpha n}=\sum_{(\alpha)}\frac{1}{b}V_{\alpha0}\frac{1}{b}t_{\alpha1}\frac{1}{b}t_{\alpha2}\cdots\frac{1}{b}t_{\alpha n}+\sum_{(\alpha)}\frac{1}{b}V_{\alpha1}\frac{1}{b}t_{\alpha1}\frac{1}{b}t_{\alpha2}\cdots\frac{1}{b}t_{\alpha n}, \quad (43)$$

where the index α_0 is never equal to α_1 in the summation. Using the identity

$$\frac{1}{b-V_{\alpha1}}-\frac{1}{b}=\frac{1}{b}V_{\alpha1}\frac{1}{b-V_{\alpha1}}, \quad (44)$$

we find that (from Eq. (41))

$$V_{\alpha1}\frac{1}{b}t_{\alpha1}=t_{\alpha1}-V_{\alpha1}$$

holds as an identity. Thus the second term on the right-hand side of Eq. (43) is

$$\sum_{(\alpha)}\frac{1}{b}t_{\alpha1}\frac{1}{b}t_{\alpha2}\cdots\frac{1}{b}t_{\alpha n}-\sum_{(\alpha)}\frac{1}{b}V_{\alpha1}\frac{1}{b}t_{\alpha2}\cdots\frac{1}{b}t_{\alpha n}. \quad (46)$$

[24] We might call t_α the "effective scattering amplitude" from a bound nucleon.

Relabeling the summation indices of the first term of Eq. (43) as $\alpha_0\to\alpha_1$, $\alpha_1\to\alpha_2\cdots\alpha_n\to\alpha_{n+1}$ and summing over n, we find that the second terms of Eq. (46) and the first terms of Eq. (43) cancel in pairs. The first terms of Eq. (46) are just those terms occurring in Eq. (42), so Eq. (39) is indeed satisfied.

Let us return now to the basic scattering equation (34). The solution is

$$\Omega=(1+a^{-1}R)\Omega_S(1+b^{-1}\Delta). \quad (47)$$

To see that this satisfies Eq. (34), we substitute into the right-hand side of (34) to obtain

$$\Omega=1+(1/a)\{V\Omega_S+R\Omega_S+R(1/a)R\Omega_S\}\{1+(1/b)\Delta\}. \quad (48)$$

[$V(1/a)R=0$ since R absorbs the meson.] Now

$$1/a=(1/b)-(1/a)\Delta(1/b) \quad (49)$$

and $(1/b)V\Omega_S=\Omega_S-1$ [Eq. (39)]. We also write the term $R(1/a)R$ in Eq. (48) as

$$R(1/a)R=\Delta \quad (50)$$

by Eq. (35). Equation (48) then reduces to

$$\Omega=(1+a^{-1}R)\Omega_S(1+b^{-1}\Delta)-(1/b)\Delta+(1/a)\Delta+(1/a)\Delta(1/b)\Delta=\Omega, \quad (51)$$

by Eq. (49). Thus Eq. (47) is the required solution to Eq. (34).

To show that t_α' and t_α are nearly equal, we observe that

$$t_\alpha-t_\alpha'=t_\alpha'(1/a)\Delta(1/b)t_\alpha. \quad (52)$$

By Eq. (36) we replace Δ by $-i\Delta_0$. In b we use Eq. (22) for Δ_0. In the numerator, we use Eq. (20). Referring to Eq. (10) and to the definitions of Δ_0, we see that (if we neglect the energy of excitation of the nucleus)

$$\int \exp(i\mathbf{q}_1\cdot\mathbf{Z}_\alpha)\left(q_1\left|\frac{1}{a}\Delta\frac{1}{b}\right|q_2\right)\exp(-i\mathbf{q}_2\cdot\mathbf{Z}_\alpha)d^3q_1d^3q_2$$
$$=(2\pi)^{-3}\left(\frac{-iv_\pi}{2\lambda_a}\right)\int_{V_A}d^3xd^3q_1d^3q_2\frac{\exp[i\mathbf{q}_1\cdot(\mathbf{Z}_\alpha-\mathbf{x})]}{\epsilon_q+i\eta-q_{01}}\times\frac{\exp[-i\mathbf{q}_2\cdot(\mathbf{Z}_\alpha-\mathbf{x})]}{\epsilon_q+i\eta-q_{02}+i\dfrac{v_\pi}{2\lambda_a}}\simeq-(2\pi)^2\frac{\epsilon_q}{2\lambda_a}, \quad (53)$$

on the assumption that the nucleus is large. In terms of the scattering amplitude a_S,

$$t_\alpha'=a_S/[(2\pi)^2\epsilon_q]. \quad (54)$$

Therefore

$$t_\alpha-t_\alpha'=t_\alpha'(1/a)\Delta(1/b)t_\alpha\simeq(a_S/2\lambda_a)t_\alpha. \quad (55)$$

This is a correction to t_α of about 2 to 3 percent at the meson energies for which a_S and λ_a have been measured.

B. Decomposition of the Scattering into Coherent and Incoherent Parts

Referring to Eq. (42) we see that Ω_S can be written as

$$\Omega_S = 1 + \frac{1}{b}\sum_{\alpha_1} t_{\alpha 1}\Omega_S(\alpha_1),$$

$$\Omega_S(\alpha_1) = 1 + \frac{1}{b}\sum_{\alpha_2 \neq \alpha_1} t_{\alpha 2}\Omega_S(\alpha_2). \quad (56)$$

These equations have the formal structure of the multiple scattering equations introduced by Foldy[25] and generalized by Lax.[26] These authors introduced the equations on grounds of physical plausibility and considered the coordinates of the scatterers to be adiabatic parameters. On the other hand, Eqs. (56) represent rigorous (formal) solutions to the many-body problem. The adiabatic approximation is obtained from Eqs. (56) by (1) assuming the impulse approximation and identifying t_α with t_α^0; (2) neglecting nuclear excitation in the energy denominators. The use of Eqs. (56) to improve the second approximation leads to corrections which can be expressed in terms of momentum distributions of the nucleons in the nucleus.

The coherent scattering arises from that part of Ω_S which is diagonal in the nuclear coordinates, or

$$\Omega_{SC} \equiv \langle \Omega_S \rangle. \quad (57)$$

Using the first of Eqs. (56) to find Ω_{SC}, we encounter

$$\langle t_{\alpha 1}\Omega_S(\alpha_1)\rangle.$$

This represents scattering by nucleon α_1 of the incident wave plus the scattered waves from all the other nucleons. We expect that these previous scatterings will also have been coherent to a good approximation, for otherwise the scattering at particle α_1 would have to react in such a way as to return to their place in the ground state of the nucleus the other nucleons which had been raised to excited levels by previous inelastic scatterings. This possibility would require very strong correlations in nuclear structure and will be neglected by us. Thus we can write

$$\langle t_{\alpha 1}\Omega_S(\alpha_1)\rangle = \langle t_{\alpha 1}\rangle\langle \Omega_S(\alpha_1)\rangle. \quad (58)$$

Now $\langle \Omega_S(\alpha_1)\rangle$ differs from $\langle \Omega_S\rangle$ by the removal of one nucleon for the last scattering. If the number of nucleons is large, we can replace $\langle \Omega_S(\alpha_1)\rangle$ by $\langle \Omega_S\rangle$. Then Ω_{SC} satisfies the equation [t_C is defined in connection with Eq. (25)]

$$\Omega_{SC} = 1 + (1/b)t_C\Omega_{SC}, \quad (59)$$

since a and b are "coherent quantities." In this equation (when operating on the ground state of the nucleus) b has the value

$$\epsilon_q + i\eta - h + i\Delta_0.$$

[25] L. Foldy, Phys. Rev. **67**, 107 (1945).
[26] M. Lax, Revs. Modern Phys. **23**, 287 (1951).

Thus [using the techniques of Chew and Goldberger[6] to solve Eq. (59)]

$$\Omega_{SC} = 1 + \frac{1}{b - t_C}t_C$$
$$= 1 + (1/e)t_C, \quad (60)$$

where we have introduced the new quantity

$$e \equiv b - t_C. \quad (61)$$

To find the coherent part of Ω [Eq. (47)] we observe that the R term is incoherent and recall that only the coherent part of Δ (i.e., $-i\Delta_0$) is kept in any case. So

$$\Omega_C \equiv \langle \Omega \rangle = (1 + (1/e)t_C)(1 + (1/b)\Delta)$$
$$= 1 + (1/e)(t_C + \Delta). \quad (62)$$

The wave function of the scattered meson is

$$\phi_q(Z) = \Omega_C \lambda_q(Z)$$

[see Eq. (3)]. From Eq. (62) we see that ϕ_q satisfies the Schrödinger equation

$$[h + t_C + \Delta]\phi_q = \epsilon_q \phi_q, \quad (63)$$

where [Eqs. (23), (29), and (36)]

$$t_C + \Delta = \left[V_0 - \frac{iv_\pi}{2}(1/\lambda_S + 1/\lambda_a)\right]v(Z)$$
$$\equiv Bv(Z). \quad (64)$$

Equation (63) has been studied by Lederman *et al.*[2] and by Steinberger[3] in connection with their own experiments.

The expression[27]

$$C \equiv \langle t_\alpha \rangle \quad (65)$$

($t_C = AC$) is independent of α.

$$I_\alpha \equiv t_\alpha - C \quad (65')$$

represents purely inelastic scattering. We replace t_α by $I_\alpha + C$ in Eqs. (56). Quantities such as $C\Omega_S(\alpha_1)$ occur, which we approximate by

$$C\Omega_S(\alpha_1) \simeq C\Omega_S. \quad (66)$$

This assumes that the number of scatterers is large (as can be seen from the arguments of Appendix A). Thus Eqs. (56) become

$$\Omega_S = 1 + (1/b)t_C\Omega_S + (1/b)\sum_{\alpha_1} I_{\alpha 1}\Omega_S(\alpha_1);$$

$$\Omega_S(\alpha_1) = 1 + (1/b)t_C\Omega_S + (1/b)\sum_{\alpha_2 \neq \alpha_1} I_{\alpha 2}\Omega_S(\alpha_2). \quad (67)$$

[27] Actually, C should be defined as $C = \Sigma_I \langle I | t_\alpha | I \rangle \Gamma_I$, where Γ_I is the projection operator on the nuclear state I. This does not modify the formal arguments which follow. The approximation of Eq. (65) follows from the arguments of the Introduction.

To solve these equations, we introduce the functions

$$F=1+(1/e)\sum_{\alpha_1} I_{\alpha 1}F_{\alpha 1},$$

$$F_{\alpha 1}=1+(1/e)\sum_{\alpha_2\neq\alpha_1} I_{\alpha 2}F_{\alpha 2}, \tag{68}$$

where e is defined by Eq. (61). The desired solutions to Eq. (67) are

$$\Omega_S=F(1+(1/e)t_C),$$
$$\Omega_S(\alpha_1)=F_{\alpha 1}(1+(1/e)t_C), \tag{69}$$

as may be seen from substituting these expressions into the right-hand side of Eqs. (67). [To satisfy the second of these equations we must approximate $t_C F_\alpha$, by $t_C F$, essentially the same approximation as that made in Eq. (66).]

Finally,

$$\Omega=(1+(1/a)R)\Omega_S(1+(1/b)\Delta)$$
$$=(1+(1/a)R)F\Omega_C, \tag{70}$$

where Ω_C is given by Eq. (62).

We note that the operator F is

$$F=1+(1/e)\sum_{(\alpha)}\{I_{\alpha 1}+I_{\alpha 1}(1/e)I_{\alpha 2}+\cdots\}, \tag{71}$$

an expression much like that of Eq. (42), except that all coherent effects are in the propagation functions, $1/e$. The physical interpretation of Eq. (71) is that the wave is propagated in a refracting medium between inelastic scatterings. That R is only on the left in Eq. (70) reflects the fact that once the meson is absorbed there will of course be no more meson scattering. The appearance of Ω_C on the right in Eq. (70) shows that the "effective incident wave" is the coherent wave rather than the actual wave of incoming mesons.

It is instructive to substitute Eq. (70) into the right-hand side of Eq. (34) to verify that we have indeed found a solution to the original equation (to within the approximation that the number of scatterers is large). This is done in Appendix A. In Appendix B it is shown that Eq. (69) can be very simply obtained when the scattering can be treated in Born approximation.

Before returning to the photomeson phenomena, we shall briefly discuss the scattering according to approximation I [Eqs. (22) and (29)] for $t_C+\Delta$. We consider first the coherent scattering:

$$\Omega_C=1+(1/e)(t_C+\Delta). \tag{62'}$$

In e we keep only the diagonal part of $t_C+\Delta$:

$$(q'|t_C+\Delta|q)=B\delta(\mathbf{q}'-\mathbf{q}). \tag{72}$$

[B is defined in Eq. (64).] In the numerator we can use approximation II:

$$(q'|t_C+\Delta|q)=\frac{B}{(2\pi)^3}\int_{V_A}\exp[-i(\mathbf{q}'-\mathbf{q})\cdot\mathbf{x}]d^3x. \tag{73}$$

[See, for instance, Eq. (20)]. Then

$$\int(q'|(1/e)(t_C+\Delta)|q)\exp(i\mathbf{q}'\cdot\mathbf{Z})d^3q'$$
$$=\frac{B}{(2\pi)^3}\int d^3q'\int_{V_A}d^3x\frac{\exp[i\mathbf{q}'\cdot(\mathbf{Z}-\mathbf{x})]}{q_0+i\eta-q_0'-B}\exp(i\mathbf{q}\cdot\mathbf{x}). \tag{74}$$

Now,

$$\int d^3q'\frac{\exp[i\mathbf{q}'\cdot(\mathbf{Z}-\mathbf{x})]}{q_0+i\eta-q_0'-B}=-(2\pi)^2(q_0/\Lambda)$$
$$\times\exp(iq\Lambda)\exp\left(-i\frac{q_0}{q}B\Lambda\right), \tag{75}$$

where $\mathbf{\Lambda}=\mathbf{x}-\mathbf{Z}$, and it is assumed that the nucleus is large enough that Λ is much greater than the wavelength of the meson. By this same assumption, we find

$$\int\left(q'\left|\frac{1}{e}(t_C+\Delta)\right|q\right)\exp(i\mathbf{q}'\cdot\mathbf{Z})d^3q'$$
$$=-\exp(i\mathbf{q}\cdot\mathbf{Z})\left[1-\exp\left(-i\frac{q_0}{q}BD\right)\right], \tag{76}$$

where use is made of the fact that B has a negative imaginary part. D is the distance from the point $\mathbf{Z}$ to the boundary of the nucleus[28] along the direction of the vector—$\mathbf{q}$. The coherent scattering is described by the transition operator:

$$T_C=(t_C+\Delta)\Omega_C. \tag{77}$$

Using Eqs. (73) and (76),

$$(q'|T_C|q)=\frac{B}{(2\pi)^3}\int_{V_A}\exp[-i(\mathbf{q}'-\mathbf{q})\cdot\mathbf{Z}]$$
$$\times\exp(-iq_0/qBD)d^3Z$$
$$=(-i/(2\pi)^2q_0)f(\theta), \tag{78}$$

where θ is the angle between $\mathbf{q}$ and $\mathbf{q}'$ and $f(\theta)$ is given by Eq. (7) of the paper by Fernbach, Serber, and Taylor.[1] The model of these authors thus follows from the use of approximation I in the propagation function $1/e$.

We can also, in the same approximation, obtain the integral equation describing the diffusion of the meson density in the nuclear medium. To simplify matters, we assume isotropic scattering and neglect the difference in the scattering by neutrons and protons, as well as charge exchange scattering (we intend merely to illustrate the application of our equations). Referring

[28] Eq. (76) has a direct physical interpretation. The first term is canceled by the "unity" term in Eq. (62′), showing that the incident wave is "extinguished" within the nucleus. The remaining term shows a net phase shift, from which the index of refraction is seen to be $n=1-(q_0/q^2)B$.

to Eqs. (68) and (69) for Ω_S, we have (for a large nucleus)

$$\psi(Z)\equiv\int(q'|\Omega_S|q)\exp(i\mathbf{q}'\cdot\mathbf{Z})d^3q'$$
$$=\int\exp(i\mathbf{q}'\cdot\mathbf{Z})(q'|\Omega_C|q)d^3q'$$
$$+\sum_{\alpha_1}\int\frac{\exp[i\mathbf{q}'\cdot(\mathbf{Z}-\mathbf{Z}_{\alpha1})]}{\epsilon_q+i\eta-q_0'-B}\bar{I}_{\alpha1}\psi_{\alpha1}(Z_{\alpha1}), \quad (79)$$

where $\bar{I}_{\alpha1}$ is the matrix of $I_{\alpha1}$ on the energy shell and $\psi_{\alpha1}(Z_{\alpha1})$ represents the scattered wave from the nucleons other than that indicated by α_1. The meson density is given by

$$\rho(Z)=\langle\psi^*\psi\rangle. \quad (80)$$

Substituting Eq. (79) into Eq. (80), the cross terms approximately vanish (as they are essentially incoherent). Using Eq. (75), we obtain

$$\rho(Z)=\rho_C(Z)+\frac{\sigma_{in}}{4\pi}\frac{A}{V_A}\times\int_{V_A}\frac{\exp(-|\mathbf{Z}-\mathbf{Z}_{\alpha1}|1/\lambda)}{|\mathbf{Z}-\mathbf{Z}_{\alpha1}|^2}\rho(Z_{\alpha1})d^3Z_{\alpha1}, \quad (81)$$

where ρ_C is the "coherent density," $1/\lambda\equiv1/\lambda_S+1/\lambda_a$, and $\sigma_{in}=2(2\pi)^5q_0^2\langle\bar{I}_{\alpha1}{}^\dagger\bar{I}_{\alpha1}\rangle$ is the scattering cross section from a single nucleon (minus the coherent scattering). This is Foldy's integral equation.[29] We have approximated $\psi_{\alpha1}(Z_{\alpha1})$ by $\psi(Z_{\alpha1})$ in Eq. (81) for reasons similar to those leading to Eq. (58).

IV. PHOTOMESON PHENOMENA

A. Evaluation of the Cross Sections

The wave matrix for the photomeson problem is given by Eq. (33):

$$\begin{aligned}\Phi&=1+\Omega(1/a)H'\\&=1+[1+(1/a)R]F\Omega_C(1/a)H'\\&=1+[1+(1/a)R]F(1/e)H',\end{aligned} \quad (33')$$

since $\Omega_C(1/a)=(1/e)$ is an algebraic identity.

The transition operator is

$$\begin{aligned}T&=(H'+R+V)\Phi\\&=H'+(V+\Delta)F\Omega_C(1/a)H'+RF(1/e)H',\end{aligned} \quad (82)$$

to first order in H', remembering that $(R+V)$ vanishes when operating on the initial state (which does not contain a meson). To simplify the second term, we note that $(V+\Delta)F\Omega_C$ is just the transition operator for meson scattering in the nucleus (less the absorbing part). So

$$(V+\Delta)F(1/e)H'=\{(t_C+\Delta)+[1+(t_C+\Delta)(1/e)]\sum_{\alpha1}I_{\alpha1}F_{\alpha1}\}(1/e)H'. \quad (83)$$

Equation (82) is consequently split into three parts:

$$\begin{aligned}T&=T_\pi+T_S+T_a,\\T_\pi&=[1+(t_C+\Delta)(1/e)]H',\\T_S&=[1+(t_C+\Delta)(1/e)]\sum_{\alpha1}I_{\alpha1}F_{\alpha1}(1/e)H',\\T_a&=RF(1/e)H'.\end{aligned} \quad (84)$$

To find the cross section for producing a meson which is neither absorbed nor scattered inelastically, we need just T_π. For a transition to a state of nuclear excitation I, this is

$$\langle I|T_\pi|A\rangle=\langle I|1+(t_C+\Delta)(1/e)|I\rangle\langle I|H'|A\rangle. \quad (85)$$

According to the arguments advanced in the Introduction (that the meson "outruns" nuclear excitation), we can replace $\langle I|\cdots|I\rangle$ in Eq. (85) by $\langle|\cdots|\rangle$, or the average with respect to the ground state of the nucleus. The transition operator for "elastic" photomeson production is now

$$(Iq|T_\pi|A\gamma)=(\lambda_q,\ \Omega_C{}^{(-)\dagger}\langle I|H'|A\rangle), \quad (86)$$

where

$$\Omega_C{}^{(-)\dagger}=1+(t_C+\Delta)(1/e), \quad (87)$$

and λ_q is a plane wave. Recalling the wave function ϕ_q given by Eq. (63), we define

$$\phi_q{}^{(-)}=\Omega_C{}^{(-)}\lambda_q, \quad (88)$$

as another "coherent scattering" wave function. By approximation II, $\phi_q{}^{(-)}$ is related to ϕ_q by

$$\phi_q{}^{(-)}=\phi_{-q}{}^* \quad (89)$$

("$-q$" means $-\mathbf{q}$). Equation (86) thus can be expressed as

$$(Iq|T_\pi|A\gamma)=(\phi_q{}^{(-)},\ \langle I|H'|A\rangle). \quad (90)$$

For the sake of illustration, let us assume that [Eq. (1)] the photomesons are produced into S-states (not unreasonable for charged mesons) and that

$$H'\simeq\sum_l\delta(\mathbf{Z}-\mathbf{Z}_l)N_l \quad (91)$$

in a coordinate representation. Then

$$\begin{aligned}\sigma_\pi&=4\pi(2\pi)^4qq_0S\sum_I|(Iq|T_\pi|A\gamma)|^2\\&=A\sigma_f\frac{1}{V_A}\int_{V_A}d^3x|\phi_q{}^{(-)}(x)|^2,\end{aligned} \quad (92)$$

if we keep only the diagonal terms in the sum over l [Eq. (91)] in Eq. (92). Uniform nuclear density is

[29] See, for instance, Eq. (6.37) of Lax's paper (reference 24).

assumed here. Equation (92) is more general than is apparent from the simplifying assumptions made [such as Eq. (91)]. If there were neither scattering nor absorption, $|\phi_q^{(-)}|^2=1$, so Eq. (92) would agree with Eq. (7).

(It might be emphasized that we need not of course have summed our cross sections over meson charge states, but could have written separate cross sections for the three meson charge states. The equations were written as given in order to save enumeration of the individual cross sections.)

One-half the probability that the meson is absorbed is obtained from

$$P_a\equiv\pi\langle T_a{}^\dagger\delta(E_A-H_0)T_a\rangle$$
$$=\pi\langle H'(1/e^\dagger)F^\dagger R\delta(E_A-H_0)RF(1/e)H'\rangle. \quad (93)$$

Writing $F=1+(1/e)\sum_{\alpha_1}I_{\alpha_1}F_{\alpha_1}\equiv 1+(F-1)$, and neglecting the small contribution from the cross terms (which are essentially "incoherent"), P_a becomes

$$P_a=\langle H'(1/e^\dagger)\Delta_0(1/e)H'\rangle$$
$$+\langle H'(1/e^\dagger)(F^\dagger-1)\Delta_0(F-1)(1/e)H'\rangle. \quad (94)$$

We have replaced $\pi R\delta(E_A-H_0)R$ by Δ_0 in Eq. (94).

Similarly, one-half the probability that the meson is scattered inelastically before getting outside the nucleus is

$$P_S\equiv\pi\langle T_S{}^\dagger\delta(E_A-H_0)T_S\rangle. \quad (95)$$

T_S is given by Eq. (84). An evaluation of P_a or P_S would involve a discussion of the multiple scattering in some detail. This might be approximated, for instance, by the methods leading to Eq. (81). On the other hand, the sum of P_a and P_S is relatively simple. $2(P_a+P_S)$ is the probability that the meson is either scattered inelastically or absorbed. For want of a better name, we shall call $2(P_S+P_a)$ the probability for "star production," since it is expected to correspond to considerable nuclear excitation. To evaluate P_S+P_a, we first simplify Eq. (95). Now,

$$\pi\delta(E_A-H_0)=\tfrac{1}{2}i(1/a-1/a^\dagger). \quad (96)$$

Using this relation, we obtain after some algebra [see Eq. (87)]:

$$\pi\Omega_C{}^{(-)}\delta(E_A-H_0)\Omega_C{}^{(-)\dagger}=\tfrac{1}{2}i\{(1/e-1/e^\dagger)$$
$$+(1/e^\dagger)[t_C{}^\dagger+\Delta^\dagger-(t_C+\Delta)](1/e)\}, \quad (97)$$

which occurs in Eq. (95). We assume that approximation I or II is valid, so [see Eq. (64)]

$$\tfrac{1}{2}i[t_C{}^\dagger+\Delta^\dagger-(t_C+\Delta)]=\mathrm{Im}(t_C)-\Delta_0. \quad (98)$$

Then

$$P_S=\sum_{\alpha_1,\alpha_2}\tfrac{1}{2}i\langle H'(1/e^\dagger)F_{\alpha_1}{}^\dagger I_{\alpha_1}{}^\dagger(1/e-1/e^\dagger)I_{\alpha_2}F_{\alpha_2}(1/e)H'\rangle$$
$$+\langle H'(1/e^\dagger)(F^\dagger-1)[\mathrm{Im}(t_C)-\Delta_0](F-1)(1/e)H'\rangle. \quad (99)$$

The major contribution to the first term above comes from the terms for which $\alpha_1=\alpha_2$ (in the double summation) and will be the coherent (i.e., diagonal in the nuclear coordinates) part of

$$\tfrac{1}{2}iI_{\alpha_1}{}^\dagger(1/e-1/e^\dagger)I_{\alpha_1}\simeq\pi I_{\alpha_1}{}^\dagger\delta(E_A-H_0)I_{\alpha_1}, \quad (100)$$

since $1/e$ can be replaced by $1/a$ for essentially the same reason that $t_\alpha\simeq t_\alpha'$ [Eq. (55)]. F_{α_1} operating to the left on a coherent quantity can be replaced by F, for reasons discussed in connection with Eq. (66). The coherent part of Eq. (100) when summed over α_1 becomes $(-1)\mathrm{Im}(t_C)$. Thus, the first term of Eq. (99) is

$$-\langle H'(1/e^\dagger)F^\dagger[\mathrm{Im}(t_C)]F(1/e)H'\rangle.$$

Replacing F by $1+(F-1)$ and neglecting the "incoherent" cross terms, we see that the part bilinear in $(F-1)$ is canceled by the corresponding term involving "$\mathrm{Im}t_C$" in the second half of Eq. (99). On adding P_S and P_a, the Δ_0 term in Eq. (99) cancels the corresponding term in Eq. (94) to give

$$P_S+P_a=\langle H'(1/e^\dagger)[\Delta_0-\mathrm{Im}(t_C)](1/e)H'\rangle, \quad (101)$$

in which the complex functions F do not appear.

The cross section for "star formation" is

$$\sigma_{\mathrm{star}}=2(2\pi)^3(P_S+P_a). \quad (102)$$

A straightforward evaluation along the lines of the analysis leading to Eq. (78) (using approximation I for the propagation functions, $1/e$) gives

$$\sigma_{\mathrm{star}}=A\sigma_f(1-f_a), \quad (103)$$

where σ_f is defined in connection with Eq. (7). The function f_a was given by Eq. (6) of reference 15 and goes to zero as $A^{-\frac{1}{3}}$ for large nuclei. A similar analysis, using approximation I, of the "elastic" photoproduction leads to

$$\sigma_\pi=A\sigma_f f_a \quad (104)$$

in agreement with Eq. (5) of reference 15. Comparison with Eq. (92) shows that in approximation I, f_a and the average of $|\phi_q^{(-)}(x)|^2$ over the nucleus, are identical. We notice that the sum of σ_π and σ_{star} is equal to Eq. (7). This is proved more generally below.

B. General Relationships between the Cross Sections

We shall continue to assume that approximation I or II is valid. Then one-half the probability for "elastic" meson production is

$$P_\pi\equiv\pi\langle T_\pi{}^\dagger\delta(E_A-H_0)T_\pi\rangle$$
$$=\tfrac{1}{2}i\langle H'(1/e-1/e^\dagger)H'\rangle$$
$$+\langle H'(1/e^\dagger)[\mathrm{Im}(t_C)-\Delta_0](1/e)H'\rangle, \quad (105)$$

using Eq. (84) for T_π and Eqs. (97) and (98). Then, using Eq. (101),

$$P_\pi+P_S+P_a=\tfrac{1}{2}i\langle H'(1/e-1/e^\dagger)H'\rangle. \quad (106)$$

If only the diagonal terms in the sum over nucleons are kept in Eq. (106), $1/e$ can be replaced by $1/a$, as in

Eq. (100). By Eq. (96), we then obtain

$$P_\pi+P_S+P_a=\pi\langle H'\delta(E_A-H_0)H'\rangle, \qquad (107)$$

or, in the notation of Eqs. (103) and (104),

$$\sigma_{\text{total}}=A\sigma_f. \qquad (108)$$

This result depends upon the use of Eq. (98) which holds only in approximations I and II.

Corresponding to a true absorption process, there must be elastic scattering. In the present case this is the elastic scattering of photons by the entire nucleus. From Eqs. (33′) and (82), it is seen that the transition operator for this is

$$T_\gamma=H'F(1/e)H'. \qquad (109)$$

The coherent (and diagonal) part of T_γ is

$$\begin{aligned}\langle T_\gamma\rangle&=\langle H'F(1/e)H'\rangle\\&\simeq\langle H'(1/e)H'\rangle\\&=\tfrac{1}{2}\langle H'(1/e+1/e^\dagger)H'\rangle+\tfrac{1}{2}\langle H'(1/e-1/e^\dagger)H'\rangle, \qquad (110)\end{aligned}$$

[neglecting $(F-1)$, as before]. Comparing with Eq. (106), we have

$$\text{Im}\langle T_\gamma\rangle=-[P_\pi+P_S+P_a] \qquad (111)$$

in agreement with the corresponding general theorem[16] relating the scattering amplitude in the forward direction to the total scattering.

It should be observed that the functions F which describe the real complexity of multiple scattering phenomena have not appeared in the final results of the problems considered in this section. This is because we have asked only the simplest questions of the theory—i.e., we have studied just total cross sections and coherent phenomena. Had we studied the multiple scattering in detail (for instance, had we sought the angular and energy distribution of the emitted mesons), then the greater complexity of the F's would have been called for.

V. FINAL COMMENTS

We have confined applications of the theory largely to a derivation of the conventional "optical models." Since the discussion has been rather formal, it appears worthwhile to review the approximations involved and to consider the possibility of improving these.

Phenomenological interactions, V and R, were introduced to describe the meson scattering and absorption. We can suppose these to have been derived from a field theory by solving the one meson—one nucleon problem to find V. R can be imagined to be the leading term left over which is not diagonal in the meson occupation numbers.

Equations (56) are rigorous (but formal) solutions to Eq. (39). The solution for Ω given by Eq. (47) is also a rigorous solution to the Schrödinger equation.

The second of Eqs. (56) represents a set of A coupled integral equations. When the number of scatterers A is small it would appear feasible to attack these directly. On the other hand, when A is large the form (69) for Ω_S seems to be more useful. As is shown in Appendix A the error involved in this expression for Ω_S is of the order of A^{-1} times Ω_S. In Appendix B it is shown that Eq. (69) is valid for all values of A if the individual scatterings can be treated in the Born approximation. The fact that either of these two independent conditions is sufficient to insure the validity of Eq. (69) suggests that the error in this equation may be less than we have estimated it to be.

For a detailed study of the inelastic scattering the diffusion equation (81) provides probably the simplest approach, but is of limited validity in that it involves approximation I. A straightforward approach using Eqs. (69) and (70) is feasible if the multiplicity of the scatterings is small, and if one can either neglect the excitation energy of the nucleus or assume that it is that of a plane wave state of the struck nucleon.[30]

The optical model [Eq. (63)] is obtainable directly from Eqs. (69) and (70) if the effects of correlation between nucleon positions are neglected. Lax[31] has shown that a correction for these effects can be obtained by modifying $(t_C+\Delta)$ by a numerical factor. Using Eq. (69), such correlations arise from the coherent contribution from $(F-1)$.[32] The correlations may be taken into account exactly in the optical model, as will be shown in a subsequent publication in which a quantitative study of this model will be made.

It would thus seem that the optical model has considerable validity for the analysis of meson phenomena in complex nuclei. In this connection the term "optical model" is somewhat ambiguous. This ambiguity arises in connection with our approximations I, II, and III [Eq. (28)]. Approximation III takes into account the variation of the matrix elements of $(t_C+\Delta)$ for scatterings off the energy shell. The mean free path seems to be sufficiently long, however, that this is probably not a very important correction. On the other hand, Lederman *et al.*[2] and Steinberger[3] have found considerable differences between approximations I and II. In the analysis of their data they found that approximation II implies a mean free path for interaction which is about twice as large as that obtained from approximation I. We thus are led to expect that approximation II is fairly reliable, but that approximation I is of limited applicability.

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[30] E. Henley, Phys. Rev. **85**, 204 (1952).

[31] M. Lax, Phys. Rev. **85**, 621 (1952).

[32] For instance,

$$\left(q''\left|\left\langle I_{\alpha_1}\frac{1}{e}I_{\alpha_2}\right\rangle\right|q\right)\simeq\delta(\mathbf{q}''-\mathbf{q})(2\pi)^3\int d^3q'\frac{t_{qq'}t_{q'q}}{\epsilon_q+i\eta-q_0'}\times\int d^3Z_{\alpha_1}d^3Z_{\alpha_2}[P(Z_{\alpha_1},Z_{\alpha_2})-P(Z_{\alpha_1})P(Z_{\alpha_2})]\times\exp[-i(q'-q)\cdot(\mathbf{Z}_{\alpha_1}-\mathbf{Z}_{\alpha_2})]$$

Here the P's are nucleon probability densities. In the absence of correlation this expression vanishes, since then $P(Z_{\alpha_1}, Z_{\alpha_2})=P(Z_{\alpha_1})P(Z_{\alpha_2})$.

APPENDIX A

Remarks Concerning the Validity of Equation (69)

We wish to substitute Eq. (69) for Ω_S into Eq. (39) to see how well the latter equation is satisfied. However, a better solution is obtained if we modify the definition of t_α. Instead of using the definition given by Eq. (41), we shall in what follows assume that

$$t_\alpha = V_\alpha + V_\alpha \frac{1}{e - V_\alpha} V_\alpha. \quad \text{(A–1)}$$

The relation between the present t_α and the previous t_α is similar to that between the definitions given in Eqs. (40) and (41) [see also Eq. (55)].

The expression (69) for Ω_S is modified in that we suppose t_α now to be defined by Eq. (A–1). The physical interpretation of this is that the scattering amplitude is evaluated for the energy of the particle in the scattering medium.

If then the result of substituting Eq. (69) into the right-hand side of Eq. (30) is designated by $(\Omega_S)_1$, we have

$$(\Omega_S)_1 \equiv 1 + (1/b) V \Omega_S$$
$$= 1 + \frac{1}{b} V \left[1 + \frac{1}{e} \sum_{\alpha_1} I_{\alpha_1} F_{\alpha_1} \right] \left(1 + \frac{1}{e} t_C \right). \quad \text{(A–2)}$$

Now,

$$V \frac{1}{e} I_{\alpha_1} = \sum_{\alpha_0 \neq \alpha_1} V_{\alpha_0} \frac{1}{e} I_{\alpha_1} + V_{\alpha_1} \frac{1}{e} I_{\alpha_1}. \quad \text{(A–3)}$$

Expressing I_{α_1}, in terms of t_{α_1}, and C by Eq. (65′), we obtain

$$V_{\alpha_1} \frac{1}{e} I_{\alpha_1} = V_{\alpha_1} \frac{1}{e} t_{\alpha_1} - V_{\alpha_1} \frac{1}{e} C$$
$$= t_{\alpha_1} - V_{\alpha_1} - V_{\alpha_1} \frac{1}{e} C, \quad \text{(A–4)}$$

using Eq. (45) with b replaced by e. (This is the reason for the definition (A–1) of t_α.) We note that

$$V + \sum_{\alpha_0, \alpha_1 (\alpha_0 \neq \alpha_1)} V_{\alpha_0} (1/e) I_{\alpha_1} F_{\alpha_1} - \sum_{\alpha_1} V_{\alpha_1} F_{\alpha_1} = 0 \quad \text{(A–5)}$$

These arise from the first part of the second term of Eq. (A–2), the first term of Eq. (A–3), and the second of Eq. (A–4).

Inserting the t_{α_1} term of Eq. (A–4) back into Eq. (A–2) and writing $t_{\alpha_1} = I_{\alpha_1} + C$, we obtain

$$\sum_{\alpha_1} (1/b) t_{\alpha_1} F_{\alpha_1} = (1/b) t_C + (1/e) \sum_{\alpha_1} I_{\alpha_1} F_{\alpha_1}$$
$$+ (1/A)(1/b - 1/e) \sum_{\alpha_1} I_{\alpha_1} F_{\alpha_1}. \quad \text{(A–6)}$$

The last term of Eq. (A–4) becomes

$$\sum_{\alpha_1} V_{\alpha_1} (1/e) C F_{\alpha_1} = (1/A) \sum_{\alpha_1} V_{\alpha_1} (1/e) t_C F_{\alpha_1}. \quad \text{(A–7)}$$

By Eq. (76), $(1/e)t_C$ is expected to be of order unity. The quantities V_{α_1} and t_{α_1} can be assumed to be of the same order of magnitude, so Eq. (A–7) is of order $A^{-1}(\Omega_S)$. The same is true of the last term of Eq. (A–6). Finally,

$$1 + \frac{1}{b} \left(1 + \frac{1}{e} t_C \right) = 1 + \frac{1}{e} t_C. \quad \text{(A–8)}$$

Combining our results, we see that

$$(\Omega_S)_1 = \Omega_S + (1/A) \text{ times the order of } (\Omega_S). \quad \text{(A–9)}$$

Our modified form of Eq. (69) is thus seen to be a solution to the Schrödinger equation if A is large. In Appendix B it is shown that this solution (with a minor modification) holds independently of A if the Born approximation is valid—i.e., if $t_\alpha \simeq V_\alpha$.

APPENDIX B

The Born Approximation

The multiple scattering analysis is considerably simplified when the individual scattering processes can be treated in the Born approximation. For simplicity we assume only scattering without absorption, so the Hamiltonian is

$$H = H_0 + V. \quad \text{(B–1)}$$

As before,

$$V = \sum_\alpha V_\alpha. \quad \text{(B–2)}$$

Let the coherent part of V, that is $\langle V \rangle$, be called V_d. Then

$$V = \sum_\alpha v_\alpha + V_d, \quad \text{(B–3)}$$

where v_α is the "incoherent part" of V_α (i.e., is not diagonal with respect to the nuclear states). Then

$$\Omega = 1 + (1/a) V \Omega \quad \text{(B–4)}$$

is the integral equation for the scattering, where a is the same as previously.

Now define

$$f_\alpha = v_\alpha (1/d_\alpha) v_\alpha,$$
$$F_\alpha = \sum_{\beta \neq \alpha} f_\beta,$$
$$F = F_\alpha + f_\alpha,$$

and

$$d_\alpha = a - F_\alpha - V_d. \quad \text{(B–5)}$$
$$d = a - F - V_d.$$

Then

$$\Omega = 1 + (1/d) V + \{ \sum_{(\alpha)} (1/d_{\alpha_1}) [v_{\alpha_1} + v_{\alpha_1} (1/d_{\alpha_2}) v_{\alpha_2} + \cdots$$
$$+ v_{\alpha_2} (1/d_{\alpha_1}) v_{\alpha_2} \cdots (1/d_{\alpha_n}) v_{\alpha_n} + \cdots] \} (1/d) V. \quad \text{(B–6)}$$

Here we use the same summation convention as was used for Eq. (42).

This is a rigorous formal solution to Eq. (B–4), as can be seen from substitution into the right-hand side of that equation. When we can apply the Born approximation to treat V_α as the inelastic scattering amplitude (i.e., set $v_\alpha = I_\alpha$) from the αth nucleon, we can also replace f_α in the energy denominators by

$$f_\alpha \simeq -i\pi\langle v_\alpha \delta(E_A - H_0) v_\alpha\rangle. \qquad \text{(B–7)}$$

Then $V_d + F_\alpha$ coincides with the t_C used in the text [except that $\mathrm{Im}(t_C)$ is corrected by a factor of $(1-A^{-1})$], and the equivalence of Eq. (B–6) to Eq. (69) readily follows.

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A Quasi-Relativistic Theory of Gravitation

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Rays of gravitons, as well as rays of photons in a gravitational field, have transverse vibrations, and a light ray has its velocity decreased by the presence of a gravitational ray. The decrease is twice as much when they are traveling in the same direction as when they are perpendicular (and, therefore, share one dimension of vibration rather than two). Introduction of a harmonic function for the square of the velocity of light leads to Schwarzschild's equation for, first, a light ray, and then, by way of its electromagnetic field, for a particle of matter. If the sun is moving through the ether, the action of the field is relativistic, except that the angle between the light ray and the gravitational ray is measured with respect to the ether. The acceptance of Miller's ether drift data would lead, then, to perturbations of the planets, including the major part of that observed in the nodes of Venus.

1. INTRODUCTION

WE introduce here a theory of the interaction of gravitons and photons which leads to results very similar to those of the general theory of relativity. If we take the gravitational field of the sun to be stationary, the departure from Einstein's equations is especially small, and, in particular, his three well-known results must follow. Our theory leads naturally to the view, however, that motion of the field through space has significance for gravitational phenomena and gives new results dependent on the direction and magnitude of such motion.

Since ether drift is to play an important role in our theory, we consider past experiments which have given indications of the magnitude and direction of such a drift.

It is not generally realized that the fringe shifts of the Michelson-Morley experiment were not believed by all experimenters to be negligibly small, and that Morley and Miller, and then Miller alone, continued the work carefully and repeatedly over a period of 30 years. The results were still small, but the variation in the direction of the effect as the earth moved in its orbit led Miller finally to reach the definite conclusion that the solar system is moving through the ether with a velocity of 208 km/sec toward a point in the southern sky with right ascension 4 hr, 54 min and declination $-70°\ 33'$.[1]

It is possible that Miller's results may prove to be spurious, especially in view of their apparent contradiction of those of other experimenters, particularly Joos and Kennedy. It is also possible, however, that the differences in design (and perhaps in method) made Miller's interferometers better able to detect his effect. Miller himself has suggested that others might have obtained positive results if they had sought to analyze their small fringe-shifts in terms of direction in the detailed manner in which he did his own. The question of the admissibility of Miller's evidence is considered by informed opinion to be still open, and, in view of the unsettled situation, our use of it in the present paper must be considered to be highly speculative.

It may be well to recall here that the special theory of relativity has discarded the ether, or privileged frame of reference, not because of any lack of compatibility, but only because such an ether is without value to the special theory.[2]

2. THE STATIONARY GRAVITATIONAL FIELD

For simplicity, we consider the sun to be concentrated at a point which is stationary in the ether. We further suppose that the gravitons of its field, moving in rays directed out from and in toward the sun, decrease the velocity of photons through an interaction of the transverse vibrations which we find it convenient to associate with the two types of elementary particles. We suppose further that the velocity of a plane-polar-

[1] D. C. Miller, Revs. Modern Phys. **5**, 203 (1933). Miller includes accounts of several similar experiments by others, including Joos and Kennedy. See also N. Rosen, Phys. Rev. **57**, 154 (1940), where the idea is presented that motion with respect to the stars may be responsible for the Miller effect.

[2] See R. B. Lindsay, Sci. Monthly **67**, 50 (1948). Also R. B. Lindsay and H. Margenau, *Foundations of Physics* (John Wiley and Sons, Inc., New York, 1936), p. 354, where reference is made to work by Page and Sparrow connected with the Miller effect.