Multiple Scattering and Radiation Damping. II*

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The analysis of multiple elastic scattering by a system of fixed scatterers in terms of two-body collision matrices is extended to large bodies. The dynamical theory of particle wave propagation in crystals is formulated and discussed from this viewpoint, as an extension of Ewald's theory to scatterers of finite size. A theory of wave propagation in a random medium, including disordered crystals, is obtained as the limiting case for a crystal with infinite unit cell. Contrary to the usual approach, this analysis considers a definite "frozen" configuration of the medium; statistical averaging over configurations may be used on the intensities obtained.

In crystals and dense liquids, radiation damping is compensated by the reaction of surrounding scatterers. Therefore, it is found convenient to restate the basic equations of multiple scattering in terms of a matrix β which is essentially the Green's function for the single scattering event, with "standing wave" rather than "outgoing wave" boundary conditions. This has the advantage of eliminating the spurious radiation damping terms at the outset; one obtains nonattenuated waves in nondissipative crystals or dense liquids by a simple expansion method.

Application is made to the following problems: the equivalent of the Lorentz-Lorenz formula for matter waves; index of refraction for an interaction of the form of a one-level Breit-Wigner formula; attenuation of long waves in a disordered crystal containing protons with random spins; double refraction effect in a liquid or crystal containing protons with partially oriented spin.

I. INTRODUCTION

IN Part I of this paper, the elastic scattering by a system of fixed centers of force was treated by replacing the interaction potential by a matrix α which describes rigorously the single scattering event. This reformulation has three advantages over the usual perturbation method: first, the iteration solution converges for short-range forces, where Born's approximation diverges; second, the first approximation to the present equations tends toward the correct solution for a system of widely separated scatterers, whereas the perturbation method gives only a first approximation; third, the results are applicable where the interaction potential is not known, but the single scattering amplitudes are.

The matrix α is essentially the Green's function of the single scattering problem, with the usual boundary condition describing outgoing (or retarded) waves. In this Part II, an alternative formulation is developed in which α is replaced by a matrix β which is essentially the Green's function for boundary conditions describing half-retarded, half-advanced (standing) waves. The new formulation which is rigorously equivalent to the previous one is advantageous for the following reasons: In crystals and in sufficiently dense liquids, the radiation damping effect on the single scatterer is compensated by the reaction of surrounding scatterers. Hence, it is logical that for the purpose of successive approximation, one should start with a description of the single scattering event from which radiation damping has been eliminated. In Sec. III, the theory of an infinite crystal is developed on this basis, and applied to the following problems: the equivalent of the Lorentz-Lorenz formula for scalar waves; the refractive index for long waves of a crystal consisting of absorbing scatterers; in particular, the case of a one-level Breit-Wigner dispersion formula for the single scatterer.

In Sec. IV, a dynamical theory of liquid scattering is formulated as a generalization of the crystal theory, starting from the remark that the most general configuration may be considered as the limit of a crystal with infinite unit cell. Boundary conditions for the semi-infinite medium are discussed, and an approximate solution is found.

Section V discusses the extension of the theory to the case where the scatterers are not simple centers of force, but many-body systems such as nuclei. In this case, the preceding results are approximately correct if the elastic submatrix of a more general matrix α or β is inserted. This approximation amounts to disregarding inelastically scattered particles. The properties of β for this general case are discussed.

II. INTRODUCTION OF HALF-ADVANCED, HALF-RETARDED SOLUTIONS

Let us examine more closely the convergence of the imaginary part of the expansion (I, 66).[1] The imaginary part of the forward scattered amplitude in the first approximation is $N\,\mathrm{Im}f(k_0)$, i.e., that corresponding to N non-interfering point scatterers. Since $\mathrm{Im}F(k_0)$ is proportional to the total scattering cross section, this approximation is good when the scatterers are located at random (total cross section $=N\times$ single cross section), but poor when interference is predominant. In the latter case, the first approximation is incorrect even in the event of vanishing single cross section, so that higher terms of the expansion in α must outweigh the first approximation. This situation is illustrated by the example of a spherical crystal discussed in (I). For a small phase δ_0,

$$\mathrm{Im}\alpha\approx\delta^2/2\pi^2k_0,\quad \alpha^2\approx\delta^2/(2\pi^2k_0)^2,$$

so that the first imaginary term, $iN\delta^2/2\pi^2k_0$ is compensated by the second term of the second approximation (I, 95), i.e., $-i\alpha^2 2\pi^2k_0N$; the first term of Eq. (I, 95) represents the correct first approximation to the

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[1] H. Ekstein, Phys. Rev. **83**, 721 (1951). Further referred to as I.

imaginary part of the forward scattering amplitude. In general, several terms of the expansion in terms of α will be needed to extricate the correct first approximation, because the diverse powers of δ are awkwardly distributed among the terms of the power series in α.

This awkwardness is due to the unusual circumstance that the parameter of expansion is complex, an undesirable feature for many purposes. For instance, the elegant method of calculating the nth order of the iteration from the previous orders[2] can be used only when the expansion parameter is real.

Physically, the complex nature of α is due to radiation damping, where the single scattering event is purely elastic. In a more extended body, most of the scatterers are symmetrically surrounded by other scatterers so that radiation damping is largely compensated by the reaction of the surrounding scatterers. This situation suggests that a more suitable approximation method could be obtained if the single scattering event were described by the sum of the retarded and advanced solutions of the Schrödinger equation, rather than by the retarded solution describing outgoing waves only.[3]

If the Green's function $\cos(k_0 r)/r$ is chosen for the unperturbed Schrödinger equation, the solution of the single scattering problem will appear as half-advanced, half-retarded spherical wave. The steps analogous to Eqs. (I, 24 to 29) lead to Eq. (I, 30) which defines the matrix β.

To relate the multiple scattering problem to β, we rewrite Eq. (I, 43) by using the definition (I, 7),

$$F_n(\mathbf{k})-P\int \frac{d\mathbf{k}'}{k'^2-k_0^2}\exp[-i\mathbf{r}_n\cdot(\mathbf{k}-\mathbf{k}')]u(\mathbf{k}-\mathbf{k}')F_n(\mathbf{k}')$$

$$=u(\mathbf{k}-\mathbf{k}_0)\exp[-i\mathbf{r}_n\cdot(\mathbf{k}-\mathbf{k}_0)]+\frac{i\pi k_0}{2}\int_S u(\mathbf{k}-\mathbf{k}')$$

$$\times\exp[-i\mathbf{r}_n\cdot(\mathbf{k}-\mathbf{k}')]F_n(\mathbf{k}')d\Omega$$

$$+\sum_{m\neq n}\int_h \frac{d\mathbf{k}'}{k'^2-k_0^2}\exp[-i\mathbf{r}_n\cdot(\mathbf{k}-\mathbf{k}')]$$

$$\times u(\mathbf{k}-\mathbf{k}')F_m(\mathbf{k}'). \quad (1)$$

If now Eq. (I, 30) is used for comparison, the procedure used in Eq. (I, 44 to 52) gives

$$F_n(\mathbf{k})=\beta(\mathbf{k},\mathbf{k}_0)\exp[-i\mathbf{r}_n\cdot(\mathbf{k}-\mathbf{k}_0)]$$

$$+\sum_{m\neq n}\int_h \frac{d\boldsymbol{\kappa}}{\kappa^2-k_0^2}\exp[-i\mathbf{r}_n(\mathbf{k}-\boldsymbol{\kappa})]\beta(\mathbf{k},\boldsymbol{\kappa})F_m(\boldsymbol{\kappa})$$

$$+\frac{i\pi k_0}{2}\int_S \exp[-i\mathbf{r}_n\cdot(\mathbf{k}-\boldsymbol{\kappa})]\beta(\mathbf{k},\boldsymbol{\kappa})F_n(\boldsymbol{\kappa})d\Omega, \quad (2)$$

so that now only the principal value of the nth term is missing from the sum over m, rather than the total hook integral as in Eq. (I, 52).

Equation (2) is merely a restatement of the original problem; the boundary conditions still describe outgoing waves, since $F(\mathbf{k})$ is connected with $\psi(\mathbf{r})$ by Eq. (I, 8). The identity of the two formulations can easily be seen for point scatterers. Equation (I, 52) can then be written, with the definition (I, 44),

$$G_n(\mathbf{k})=\alpha_0\exp(i\mathbf{r}_n\cdot\mathbf{k}_0)$$

$$+2\pi^2\alpha_0\sum_{m\neq n}\frac{\exp[ik_0|\mathbf{r}_n-\mathbf{r}_m|]}{|\mathbf{r}_n-\mathbf{r}_m|}G_m(\mathbf{k}). \quad (3)$$

Since, in this case $\alpha_0=f_0(k_0)$, $\beta_0=g_0(k_0)$, Eq. (I, 19) can be used:

$$G_n=\beta_0\Big[\exp(i\mathbf{r}_n\cdot\mathbf{k}_0)$$

$$+2\pi^2\sum_{m\neq n}\frac{\exp(ik_0|\mathbf{r}_n-\mathbf{r}_m|)}{|\mathbf{r}_n-\mathbf{r}_m|}G_m+2i\pi^2k_0G_n\Big], \quad (4)$$

and this is the form which Eq. (2) takes for this particular case.

The reformulation of the scattering problem given in Eq. (2) has the advantages of Born's approximation with respect to the relation between imaginary and real parts. For instance, the well-known theorem connecting the imaginary part of the forward scattering amplitude to the total cross section is satisfied also by first and second iteration approximations, i.e., the cross section to the first nonvanishing order can be calculated either by integrating the first approximation or by determining the imaginary part of the second approximation. To see this, we compare the cross section

$$\sigma=\int|\psi_{sc}|^2r^2d\Omega=(2\pi^2)^2\int|F(\mathbf{k})|^2d\Omega, \quad (5)$$

or, using the first approximation to Eq. (2),

$$\sigma^{(1)}=(2\pi^2)^2\int|\beta(\mathbf{k},\mathbf{k}_0)|^2|\sum\exp[-i\mathbf{r}_n\cdot(\mathbf{k}-\mathbf{k}_0)]|^2d\Omega, \quad (6)$$

with the imaginary part of the forward scattering amplitude,

$$\mathrm{Im}\psi_{sc}=2\pi^2\,\mathrm{Im}F(\mathbf{k}_0,\mathbf{k}_0). \quad (7)$$

If the second iteration of Eq. (2) is used,

$$F^{(2)}(\mathbf{k}_0)=\int_h\frac{d\boldsymbol{\kappa}}{\kappa^2-k_0^2}\beta(\mathbf{k}_0,\boldsymbol{\kappa})\sum\exp[-i\mathbf{r}_n\cdot(\mathbf{k}_0-\boldsymbol{\kappa})]$$

$$\times\beta(\boldsymbol{\kappa},\mathbf{k}_0)\textstyle\sum_n\exp[-i\mathbf{r}_n(\boldsymbol{\kappa}-\mathbf{k}_0)]$$

$$-PN\int\frac{d\boldsymbol{\kappa}}{K^2-k_0^2}\beta(\mathbf{k}_0,\boldsymbol{\kappa})\beta(\boldsymbol{\kappa},\mathbf{k}_0). \quad (8)$$

[2] Jost, Luttinger, and Slotnick, Phys. Rev. **80**, 189 (1950).

[3] That a many-body problem can be described by using half-retarded, half-advanced solutions for the single scatterer to obtain a retarded total effect, is shown by J. A. Wheeler and R. P. Feynman, Revs. Modern Phys. **17**, 157 (1945).

It is shown in Sec. V that the matrix β is Hermitean when the scattering is purely elastic. Hence,

$$\mathrm{Im}F^{(2)}(\mathbf{k}_0)=\frac{\pi k_0}{2}\int|\beta(\mathbf{k},\mathbf{k}_0)|^2 \times|\sum\exp[-i\mathbf{r}_n(\mathbf{k}-\mathbf{k}_0)]|^2 d\Omega, \quad (9)$$

and therefore

$$\sigma^{(1)}=(4\pi/k_0)\,\mathrm{Im}\psi_{sf}^{(2)}, \quad (10)$$

in agreement with the exact theorem, whereas the expansion in terms of α does not show this relation. We may conclude, in general, that for dense media the formulation of the scattering problem in terms of half-advanced, half-retarded single scattering solutions is more convenient for the purpose of approximations.

III. THE INFINITE CRYSTAL

The theory of wave propagation in an infinite periodic medium was developed by v. Laue[4] for x-rays and adapted to the case of electrons by Bethe and others.[5] For scalar particles, the solution has the form

$$\psi=\exp(i\mathbf{K}_0\cdot\mathbf{r})\sum c_n\exp(2\pi i\mathbf{A}_n\cdot\mathbf{r}), \quad (11)$$

where the $\mathbf{A}_n$ are the reciprocal lattice vectors. If the potential v is expanded into a Fourier series,

$$v=\sum V_n\exp(2\pi i\mathbf{A}_n\cdot\mathbf{r}), \quad (12)$$

the introduction of Eqs. (11) and (12) into the Schrödinger equation leads to

$$[(\mathbf{K}_0+2\pi\mathbf{A}_n)^2-\mathbf{k}_0^2]c_n+(2m/\hbar^2)\sum V_{n-m}c_m=0, \quad (13)$$

where $k_0^2=2mE/\hbar^2$. Laue's method for solving Eq. (13) is to disregard all but a small number of c_n's [for which $(\mathbf{K}_0+2\pi\mathbf{A}_n)^2\approx k_0^2$] and to solve the remaining equations. To determine the range of validity of this approximation, let us consider the case where only c_0 is retained in first approximation. We have

$$K_0^2-k_0^2+(2m/\hbar^2)V_0=0, \quad (14)$$

$$-c_n=\frac{2m}{\hbar^2}\frac{V_nc_0}{(\mathbf{K}_0+2\pi\mathbf{A}_n)^2-k_0^2}\quad(n\neq0). \quad (15)$$

Hence, in second approximation,

$$K_0^2-k_0^2+\frac{2m}{\hbar^2}\left(V_0-\frac{2m}{\hbar^2}\sum{}'\frac{|V_m|^2}{(\mathbf{K}_0+2\pi\mathbf{A}_m)^2-k_0^2}\right)=0. \quad (16)$$

If (14) is to be a good approximation, it is necessary that

$$\frac{2m}{\hbar^2}\sum{}'\frac{|V_m|^2}{(\mathbf{K}_0+2\pi\mathbf{A}_m)^2-k_0^2}\ll|V_0|. \quad (17)$$

Assuming that the geometrical orientation of $\mathbf{K}_0$ is such that the lower terms do not contribute much to the sum, the validity of Eq. (16) depends on the rapidity with which the Fourier coefficients V_m decrease, and on their numerical magnitude. For high values of m, the sum can be approximated by the integral

$$\int^\infty\frac{|V(k)|^2}{k^2}d\mathbf{k}.$$

If the potential v represents point-scatterers, the Fourier transform of the delta-function will become asymptotically constant and the sum diverges. More generally, for small scatterers, Laue's approximation method must fail. For the extreme case of point-scatterers, an alternative method is given by Ewald's method which has been applied to the case of scalar particles by Goldberger and Seitz.[6]

In addition to its mathematical convenience in these cases, Ewald's method has the more general advantage that it correlates the properties of the crystal with the scattering properties (polarizability) of the single ion, rather than with the potential or electronic density which is not directly measurable. However, the theory has been established only for point scatterers, so that a generalization of Ewald's method for finite scatterers seems desirable. An attempt to fill this gap has been made recently by Lax,[7] but it has not been quite successful. Following Schwinger and Lippmann, Lax describes the single scattering process by the equation

$$\psi_a=\phi_a+\frac{1}{E-H}T\phi_a,$$

where ψ_a is the total wave function, ϕ_a an eigenfunction of the unperturbed Hamiltonian H belonging to the energy E, and T is the matrix which is denoted by f in the present paper. Lax concludes that in a multiple scattering problem, the foregoing equation can be interpreted by

$$(\text{scattered wave})=(E-H)^{-1}T\ (\text{incident wave}).$$

However, since ϕ_a is an eigenfunction of H belonging to the energy E, this generalization implies that the field incident on one scatterer can be represented as a superposition of plane waves of definite energy E. In general, this assumption is unwarranted; for instance, the incident field

$$e^{i(r-r_0)k}/r-r_0$$

has all energies in its Fourier spectrum.

To represent the scattering under the influence of a general incident field, an operator more general than T is required, *viz.*, either α of which T is a submatrix or β, according to the boundary conditions chosen for the

[4] M. von Laue, *Roentgenstrahlinterferenzen* (Akademische Verlagsgesellschaft, Leipzig, 1941).
[5] A. Sommerfeld and H. Bethe, *Handbuch der Physik* (J. Springer, Berlin, 1933), **XXIV**, 2.
[6] M. L. Goldberger and F. Seitz, Phys. Rev. **71**, 294 (1947).
[7] M. Lax, Phys. Rev. **85**, 621 (1952).

single scattering event.[8] In the following, the generalization for finite scatterers will be formulated rigorously.

The usual theory which starts from single scattering amplitudes meets a certain difficulty in eliminating the effect of radiation damping for long waves. From elementary dispersion theory[9] one expects n^2-1 (n refractive index) to be proportional to the single scattering amplitude, which, because of radiation damping, is necessarily complex. It was pointed out already by Lorentz on intuitive grounds that in a nonabsorbing crystal the radiation damping of a single scatterer must be exactly compensated by the influence of the other scatterers, so that the absurdity of a single attenuated wave in a nonabsorbing crystal can be avoided. However, a rigorous proof of this compensation has been given only for the case of a simple lattice of electromagnetic point-dipoles by Ewald.[10] Any approximate treatment which starts from the retarded solution for a single scatterer, has difficulties to avoid the appearance of the spurious imaginary terms.[11]

It seems more natural, in view of the intuitive knowledge of the cancellation of radiation damping, to start from the half-retarded, half-advanced solutions in constructing the total field. In this manner, spurious imaginary parts are avoided at the outset, regardless of the degree of approximation.

We first generalize Eq. (2) for the case where there are N different scattering centers described by matrices β_m. In an obvious way, one finds

$$F_n(\mathbf{k})=\beta_n(\mathbf{k},\mathbf{k}_0)\exp[-i\mathbf{r}_n\cdot(\mathbf{k}-\mathbf{k}_0)]$$
$$+\sum_{m\neq n}\int_h\frac{d\boldsymbol{\kappa}}{\kappa^2-k_0^2}\exp[-i\mathbf{r}_n\cdot(\mathbf{k}-\boldsymbol{\kappa})]\beta_n(\mathbf{k},\boldsymbol{\kappa})F_m(\boldsymbol{\kappa})$$
$$+\frac{i\pi k_0}{2}\int_S\beta_n(\mathbf{k},\boldsymbol{\kappa})\exp[-i\mathbf{r}_n\cdot(\mathbf{k}-\boldsymbol{\kappa})]F_n(\boldsymbol{\kappa})d\Omega. \quad (18)$$

We consider an infinite crystal with translation vectors $\mathbf{a}_n$ and scatterers of type j located with their centers at $\mathbf{a}_n+\mathbf{r}_j$, where $\mathbf{r}_j$ is located in the first unit cell. We label the scatterers by the numbers n, j, specifying the unit cell and type. Further, we omit now the first term in Eq. (18) which describes the incident wave, since we are now interested in self-sustained waves in the infinite crystal. Equation (18) becomes

$$F_{nj}=\sum_{ml}{}'\int_h\frac{d\boldsymbol{\kappa}}{\kappa^2-k_0^2}\exp[-i(\mathbf{a}_n+\mathbf{r}_l)\cdot(\mathbf{k}-\boldsymbol{\kappa})]$$
$$\times\beta_j(\mathbf{k},\boldsymbol{\kappa})F_{ml}(\boldsymbol{\kappa})+\frac{i\pi k_0}{2}\int_S\beta_j(\mathbf{k},\boldsymbol{\kappa})$$
$$\times\exp[-i(\mathbf{r}_n+\mathbf{a}_j)\cdot(\mathbf{k}-\boldsymbol{\kappa})]F_{nj}(\boldsymbol{\kappa})d\Omega, \quad (19)$$

where the prime sign means now omission of the term $m=n$, $l=j$. This infinite set of integral equations can be reduced to a finite set by assuming the solutions to have the form of lattice waves, i.e.,

$$F_{nj}=G_j\exp[-i(\mathbf{a}_n+\mathbf{r}_j)\cdot(\mathbf{k}+\mathbf{K}_0)], \quad (20)$$

where the propagation vector $\mathbf{K}_0$ is to be determined. By inserting Eq. (20) into Eq. (19) and making the transformation $\mathbf{a}_n\rightarrow\mathbf{a}_{n+1}$, one can verify that Eq. (20) is a solution provided that it satisfies the equation

$$G_j(\mathbf{k})=\sum_{ml}{}'\exp[i\mathbf{K}_0\cdot(\mathbf{a}_m+\mathbf{r}_l-\mathbf{r}_j)]\int_h\frac{d\boldsymbol{\kappa}}{\kappa^2-k_0^2}$$
$$\times\exp[i\boldsymbol{\kappa}\cdot(\mathbf{r}_j-\mathbf{a}_m-\mathbf{r}_l)]\beta_j(\mathbf{k},\boldsymbol{\kappa})G_l(\boldsymbol{\kappa})$$
$$+\frac{i\pi k_0}{2}\int_S\beta_j(\mathbf{k},\boldsymbol{\kappa})G_j(\boldsymbol{\kappa})d\Omega, \quad (21)$$

where the prime means omission of $m=0$, $l=j$. To carry out the summation we use the well-known formula,

$$\sum_n\exp(i\mathbf{a}_n\cdot\mathbf{k})=[(2\pi)^3/\tau]\sum_h\delta(\mathbf{k}-2\pi\mathbf{A}_h), \quad (22)$$

where $\mathbf{A}_h$ are the reciprocal lattice vectors and τ is the volume of the unit cell. Hence,

$$G_j(\mathbf{k})=\frac{(2\pi)^3}{\tau}\sum_{hl}\frac{\exp[-i2\pi\mathbf{A}_h\cdot(\mathbf{r}_l-\mathbf{r}_j)]\times\beta_j(\mathbf{k},\mathbf{K}_0+2\pi\mathbf{A}_h)G_l(\mathbf{K}_0+2\pi\mathbf{A}_h)}{(\mathbf{K}_0+2\pi\mathbf{A}_h)^2-k_0^2}$$
$$-P\int\frac{d\boldsymbol{\kappa}\beta_j(\mathbf{k},\boldsymbol{\kappa})}{\kappa^2-k_0^2}G_j(\boldsymbol{\kappa})d\boldsymbol{\kappa}. \quad (23)$$

It is clear that $\mathbf{K}_0$ must be such that $(\mathbf{K}_0+2\pi\mathbf{A}_h)^2$ is different from k_0^2, for every h. This remark permits a simplification of Eq. (21). If the two contributions to the hook-integral are separated, the surface integrals will be

$$\frac{i\pi k_0}{2}\sum_{\text{all }m,l}\exp[i\mathbf{K}_0(\mathbf{a}_m+\mathbf{r}_l-\mathbf{r}_j)]$$
$$\times\int_S\exp[i\boldsymbol{\kappa}\cdot(\mathbf{r}_j-\mathbf{a}_m-\mathbf{r}_l)]\beta_j(\mathbf{k},\boldsymbol{\kappa})G_l(\boldsymbol{\kappa})d\Omega,$$

and if Eq. (22) is used, all terms must vanish, since the requirement $\kappa^2=k_0^2$ implicit in the symbol S conflicts with $\boldsymbol{\kappa}=\mathbf{K}_0+2\pi\mathbf{A}_h$. Therefore, the surface integrals do

[8] For point scatterers, the distinction between T and α becomes irrelevant. Hence, Lax's calculations are approximations valid within the same range as Ewald's, i.e., for scatterers very small as compared to the interatomic distance and to the wavelength, or, more generally, whenever $\alpha\approx f$.

[9] E. Fermi, *Nuclear Physics* (University of Chicago Press, Chicago, 1950).

[10] P. P. Ewald, Ann. Inst. Henri Poincaré **VIII**, 79 (1932).

[11] An ingenious attempt to avoid the paradox in the general case has been made by M. Lax, Phys. Rev. **85**, 621 (1952).

not contribute to Eq. (21), and it can be written

$$G_j(\mathbf{k})=\sum_{m,l}{}' \exp[i\mathbf{K}_0\cdot(\mathbf{a}_m+\mathbf{r}_l-\mathbf{r}_j)]P\int \frac{d\boldsymbol{\kappa}}{\kappa^2-k_0^2} \times\exp[i\boldsymbol{\kappa}\cdot(\mathbf{r}_j-\mathbf{a}_m-\mathbf{r}_l)]\beta_j(\mathbf{k},\boldsymbol{\kappa})G_l(\boldsymbol{\kappa}). \quad (24)$$

To illustrate the meaning of this equation, consider the case of point scatterers in a Bravais lattice:

$$G=\frac{\tan\delta_0}{k_0}\sum{}'\frac{\cos|a_n|k_0}{|a_n|}\exp[i\mathbf{a}_n\cdot\mathbf{K}_0]G, \quad (25)$$

which is obtained from Eq. (24) by letting all β_l's except $\beta_1=\beta$ vanish, and letting β and hence G tend toward constants. The value of β is then determined by Eq. (I, 22). This is to be compared with

$$G=\frac{\exp(2i\delta_0)-1}{2ik_0}\sum{}'\frac{\exp(i|\mathbf{a}_n|k_0)}{|\mathbf{a}_n|}\exp(i\mathbf{a}_n\cdot\mathbf{K}_0)G, \quad (26)$$

which is obtained from Eq. (3). The two equations, although mathematically identical, differ in that all radiation damping effects are eliminated from Eq. (25).

Rigorous solutions can be obtained only for point-scatterers, by evaluating the right-hand side of Eq. (23) through a limiting process which lets the β and G tend toward constants. Following Ewald,[10] we may choose

$$G_j=\lim_{E\to\infty}\left\{\frac{(2\pi)^3}{\tau}\sum_{hl}\frac{\exp[-2\pi i\mathbf{A}_h\cdot(\mathbf{r}_l-\mathbf{r}_j)-(1/4E^2) \times\{(\mathbf{K}_0+2\pi\mathbf{A}_h)^2-k_0^2\}]G_l\beta_j}{(\mathbf{K}_0+2\pi\mathbf{A}_h)^2-k_0^2} -P\int\frac{d\boldsymbol{\kappa}\exp[-(1/4E^2)(\kappa^2-k_0^2)]G_j\beta_j}{\kappa^2-k_0^2}\right\}. \quad (27)$$

We consider the limiting case of large wavelengths, i.e., k_0, $K_0\to 0$ in a Bravais lattice. Then

$$1=\frac{(2\pi)^3\beta}{\tau(K_0^2-k_0^2)}+\beta\lim_{E\to\infty}\left\{\frac{(2\pi)^3}{\tau}\sum{}'\frac{\exp[-\pi^2A_h^2/E^2]}{4\pi^2A_h^2} -P\int\frac{d\boldsymbol{\kappa}\exp(-\kappa^2/4E^2)}{\kappa^2}\right\}. \quad (28)$$

For evaluation of the sum, it is desirable to have E as small as possible. To obtain the error due to using a finite value of E, we consider the curly bracket as a function of $1/E$ and write

$$f\left(\frac{1}{E}\right)=f(0)+\int_0^{1/E}\frac{df}{d\eta}d\eta, \quad (28)$$

$$\frac{df}{d\eta}=-\frac{2\pi}{\tau}2\pi^2\eta\sum{}'\exp(-\pi^2\eta^2A_h^2)+\frac{4\pi^3}{\eta^2}, \quad (29)$$

or by theta-transformation of the sum

$$\frac{df}{d\eta}=-\frac{4\pi^3}{\eta^2}\sum_n\exp(-a_n^2/\eta^2)+\frac{4\pi^3}{\tau}\eta+\frac{4\pi^3}{\eta^2}. \quad (30)$$

If η is large as compared to the lattice distance, only the term $n=0$ contributes to this sum, and we have

$$f\left(\frac{1}{E}\right)\approx f(0)+\int_0^{1/E}\frac{4\pi^3}{\tau}\eta d\eta=f(0)+\frac{2\pi^3}{\tau}\frac{1}{E^2}. \quad (31)$$

In the simplest case, i.e., a cubic Bravais lattice, one finds that $aE=10$ is sufficient to make the corrective term one percent. One finds

$$\{\ \}=-(2\pi/a)9.4. \quad (32)$$

With $n^2=K_0^2/k_0^2$, Eq. (28) gives

$$n^2-1=\frac{4\pi\cdot 2\pi^2\beta/a(k_0a)^2}{1+(2\pi^2\beta/a)\cdot 3}, \quad (33)$$

the equivalent of the Lorentz-Lorenz formula for matter waves. Since, for small cross section, $2\pi^2\beta$ is the scattering length, it is clear that the corrective term in the denominator is noticeable only when the scattering length is of the order of the distance between scatterers. This does not happen for any slow-neutron scattering process, so that the Lorentz correction is unimportant for this case.

The experience obtained with the electromagnetic Lorentz correction leads to the belief that Eq. (33) remains substantially correct for a more general dense assembly of scatterers, even when they are not rigorously point scatterers. Hence, it is plausible that in a phenomenological theory of meson scattering by heavy nuclei, if the nucleus is considered as a refracting sphere, Eq. (33) should be used in estimating the refracting index, using for β the observed scattering amplitude of the single nucleon. The main effect of the correction will be to smooth out the maxima of β in the heavy nucleus as compared to the single nucleon.

Returning to the general case, we seek to solve Eq. (23) by approximation. In general, a few terms of the sum will have small denominators, *viz.*, those for which the Laue-Bragg condition,

$$(\mathbf{K}_0+2\pi\mathbf{A}_n)^2\approx k_0^2,$$

is approximately realized. If N large terms are retained, the remaining equation can be written

$$[k^2-k_0^2]c_j(\mathbf{k})=\frac{(2\pi)^3}{\tau}\sum_{h,l}^{h=N}\exp[i(\mathbf{K}_h-\mathbf{k})\cdot\mathbf{r}_j] \times\beta_j(\mathbf{k},\mathbf{K}_h)G_l(\mathbf{K}_h), \quad (34)$$

where

$$\mathbf{K}_h=\mathbf{K}_0+2\pi\mathbf{A}_h, \quad (35)$$

and

$$G_j(\mathbf{k})=c_j(\mathbf{k})[k^2-k_0^2]\exp[i(\mathbf{k}-K_0)\cdot\mathbf{R}_j] \quad (36)$$

In this approximation, a further simplification is possible, by summing over the index j:

$$[k^2-k_0^2]C(\mathbf{k})=\frac{(2\pi)^3}{\tau}\sum_h^N\beta(\mathbf{k},\mathbf{K}_h)C(\mathbf{K}_h), \quad (37)$$

where

$$C(\mathbf{k})=\sum_j c_j(\mathbf{k}) \tag{38}$$

and

$$\beta(\mathbf{k},\boldsymbol{\kappa})=\sum_j \exp[i(\boldsymbol{\kappa}-\mathbf{k})\cdot\mathbf{r}_j]\beta_j(\mathbf{k},\boldsymbol{\kappa}). \tag{39}$$

To solve Eq. (34), only the values at $\mathbf{k}=\mathbf{K}_h$ are required. Hence, with the notation

$$C(\mathbf{K}_h)=C_h, \tag{40}$$

we have

$$[K_h^2-k_0^2]C_h=\frac{(2\pi)^3}{\tau}\sum_m^N \beta(\mathbf{K}_h,\mathbf{K}_m)C_m. \tag{41}$$

Equation (41) is similar to Laue's equations, if all but N terms are omitted in the sum. β takes the place of the structure factor V_{n-h} and β_j the place of the atomic form factor. However, Eq. (41) is more general in two respects: (1) the value of β depends on the unknown K_0, (2) β is not merely a function of the difference of its two arguments: this would be true only if β was identified with the two-body interaction potential, which is an inadequate approximation for nuclear interactions.[12]

Comparison with Eq. (13) suggests the following simple interpretation: in Laue's equations, substitute $\beta(\mathbf{K}_h,\mathbf{K}_m)$ as "pseudopotential," and proceed by Laue's approximation method. This rule has a particularly simple form when the scatterers are very small: β becomes a constant and can be considered as having been derived from an effective potential $\delta(\mathbf{r}-\mathbf{a}_n)$ in coordinate representation with an appropriate constant coefficient.

However, this simple rule does not hold for higher approximations. In particular, when β and, hence, G are constants, the second approximation would diverge if one used only the reinterpreted Laue equation. However, Eq. (23) shows that the contributions for $|K|\gg k_0$ are small: indeed, when the denominator has become a smooth function, sum and integral are equal.

Thus, we can formulate a heuristic rule similar to that found in (I), but not deducible from it: for small scatterers, substitute β as pseudopotential in Laue's equation, and proceed by Laue's approximation method, cutting off all integrals and sums at some point $|\mathbf{A}_m|\gg k_0$.

This situation provides an explanation for a curious coincidence found by Goldberger and Seitz.[6] Besides a correct calculation of neutron refraction and diffraction, based on Ewald's method, they present another derivation which uses delta-functions as effective potentials to describe the nuclei. They point out the inadequacy of this method, but find that it gives the same result as the correct method.

This coincidence is now explained by the mutual cancellation of two omissions: they first disregard, in effect, the integral in Eq. (23), calculate the first approximation by Eq. (41) and omit the second approximation which, with their formalism, would have been infinite. It can be seen now that the integral in Eq. (23) serves precisely to make this second approximation finite, and in their case, very small.

Returning to the general case of finite scatterers, we consider the simplest case, where only the contribution of the first term is large:[13] we have as a first approximation

$$K_0^2-k_0^2-[(2\pi)^3/\tau]\beta(\mathbf{K}_0,\mathbf{K}_0)=0. \tag{42}$$

It is shown in Sec. V that β is Hermitean where no inelastic scattering is present: hence, n^2 is real. If, as it happens often, the third term is small, we can solve this transcendental equation by successive approximations:

$$K_0^2-k_0^2=[(2\pi)^3/\tau]\beta(\mathbf{k}_0,\mathbf{k}_0), \tag{43}$$

where

$$\mathbf{k}_0=(\mathbf{K}_0/|\mathbf{K}_0|)k_0, \tag{44}$$

and so forth. Equation (43) is a sufficient approximation for all known cases of slow neutron scattering.

For point scatterers

$$n^2-1=(4\pi/k_0^3\tau)\tan\delta_0. \tag{45}$$

If we had used the retarded solution α instead of β we would have obtained

$$n^2-1=[(2\pi)^3/\tau k_0^2]\alpha(\mathbf{k}_0,\mathbf{k}_0), \tag{46}$$

and for point scatterers

$$n^2-1=\frac{4\pi}{k_0^3\tau}\frac{\exp(2i\delta_0)-1}{2i}. \tag{47}$$

Equation (47) is a slight generalization of the elementary theory[9] which predicts wave attenuation in a case where it obviously cannot exist. The use of the half-retarded, half-advanced solution has avoided this difficulty.

As an illustration of the effect of radiation damping, we consider again the case of small scatterers, *viz.*, nuclei whose scattering is described by a one-level Breit-Wigner formula. In this case[14] with the usual notation

$$e^{2i\delta_0}-1=4\pi^2ik_0f=e^{2i\eta}-1-\frac{i\Gamma_1e^{2i\eta}}{E-E_c+\frac{1}{2}i\Gamma}. \tag{48}$$

From Eq. (I, 19) we have[15]

$$n^2-1=\frac{(2\pi)^3}{\tau k_0^2}\frac{1}{2\pi^2ik_0}\frac{e^{i\eta}[E-E_c+\frac{1}{2}i(\Gamma_2-\Gamma_1)]-e^{-i\eta}[E-E_c+\frac{1}{2}i(\Gamma_2+\Gamma_1)]}{e^{i\eta}[E-E_c+\frac{1}{2}i(\Gamma_2-\Gamma_1)]+e^{-i\eta}[E-E_c+\frac{1}{2}i(\Gamma_2+\Gamma_1)]}. \tag{49}$$

[12] The question as to whether the generalization of Ewald's method to finite scatterers leads precisely to the form of Laue's equations has been discussed, but not entirely clarified by Ewald (Z. Krist. **97**, 1 (1937)).

[13] This approximation does not mean that the wave function is essentially a plane wave, as in Eq. (14). On the contrary, for point scatterers the wave function has singularities of the type $1/r$. The higher terms $G(\mathbf{K}_h)$ are omitted not because they are small, but because they are compensated by the integral on the right hand side of Eq. (23).

[14] N. E. Mott and H. S. W. Massey, *The Theory of Atomic Collisions* (The Clarendon Press, Oxford, 1949).

[15] The use of Eq. (I, 19), which has been derived for pure potential scattering only, will be justified in Sec. V.

In particular, if Γ_2 is very small, we can neglect η near resonance,

$$n^2-1=\frac{-2\pi}{\tau k_0^3}\frac{\Gamma_1}{E-E_c+\frac{1}{2}i\Gamma_2}, \tag{50}$$

whereas, by using the retarded solution, we would have obtained

$$n^2-1=-\frac{2\pi}{k_0^3\tau}\frac{\Gamma_1}{E-E_c+\frac{1}{2}i(\Gamma_1+\Gamma_2)}. \tag{51}$$

The correction for radiation damping has the effect of removing the natural line width from the terms composing the total line width. If no competing processes are present, i.e., $\Gamma_2=0$, the correct formula (50) shows that the index of refraction is either real or imaginary whereas the uncorrected Eq. (51) predicts attenuated waves. Hence, a good mirror for slow neutrons should have nuclei with large scattering, but small absorption cross section.

It is shown in Sec. V that for finite scatterers, and including absorption

$$\mathrm{Im}\beta(\mathbf{k}_0,\mathbf{k}_0)=[k_0/(2\pi)^3]\sigma_{\mathrm{in}}, \tag{52}$$

where σ_{in} is the cross section for inelastic and absorption processes of the single, fixed scatterer. Hence, we have, for long waves or, more generally, far from Laue-Bragg reflections

$$\mathrm{Im}(n^2-1)=\sigma_{\mathrm{in}}/k_0\tau. \tag{53}$$

Equation (53) is just what one would expect intuitively; since in the present approximation all inelastically scattered particles are considered lost, the attenuation is clearly proportional to the inelastic cross-section under conditions where no attenuation by radiation damping occurs.

Equation (53) is important in interpreting slow neutron transmission measurements. The attenuation coefficient varies from σ_{in} (long waves) to σ_{total} (e.g., in case of exact Bragg reflection from a crystal).

We discuss the second approximation to Eq. (42). We have, from Eq. (34), as a first approximation

$$(k^2-k_0^2)c_j(\mathbf{k})=[(2\pi)^3/\tau]\beta_j(\mathbf{k},\mathbf{K}_0)\times\exp[i\mathbf{r}_j\cdot(\mathbf{K}_0-\mathbf{k})]C_0, \tag{54}$$

and, by introducing this into the small terms of Eq. (23),

$$K_0^2-k_0^2=\frac{(2\pi)^3}{\tau}\beta(\mathbf{k}_0,\mathbf{k}_0)+\frac{(2\pi)^6}{\tau^2}\sum_{h\neq 0}\frac{\beta(\mathbf{k}_0,\mathbf{K}_h)\beta(\mathbf{K}_h,\mathbf{k}_0)}{K_h^2-k_0^2}-\frac{(2\pi)^3}{\tau}P\int d\boldsymbol{\kappa}\sum_j\frac{\beta_j(\mathbf{k}_0,\boldsymbol{\kappa})\beta_j(\boldsymbol{\kappa},\mathbf{k}_0)}{\kappa^2-k_0^2}, \tag{55}$$

It is apparent that the second approximation does not diverge, even for point scatterers. Just as in the case of finite scattering systems, the integral compensates the nonoscillatory terms of the sum, because the self-action of each scatterer has been eliminated at the outset, whereas in Laue's approximation method the self-action causes divergence.

IV. THE RANDOM MEDIUM

In this section, the random medium will be studied from the viewpoint of a dynamical theory, i.e., we seek the properties of solutions in the infinite medium rather than considering the scatterer as a cause of perturbation of an incident beam.

There are, at present, two approaches to wave propagation in random media: (1) the perturbation (or geometric) theory which calculates the scattered wave for a definite configuration, and averages intensities by weighting all possible configurations with their probability. (2) a dynamical theory originated by Foldy[16] and extended by Lax[7,11] which determines the properties of a mean wave obtained by statistical averaging of some scatterer positions. Now, the outstanding fact about a random medium such as glass is that measurable properties do not vary from sample to sample: one can say that the sample is so large that its various parts constitute already a statistical ensemble. Hence, it seems that the purpose of a deductive theory is to show that observable properties don't depend on the precise configuration, but only on certain autocorrelation functions, in other words, to show that scattering is not "structure sensitive."

The present analysis considers a definite "frozen" configuration of the random medium, and does not make use of any statistical averaging. It is found that the only assumption necessary is the mathematical existence of a correlation function. To avoid a misunderstanding, it may be remembered that the autocorrelation function has by itself nothing to do with statistics; it is a definite functional of a given function.

We consider first a definite time-independent potential $v(\mathbf{r})$. The "random medium" is defined by the existence of the correlation function

$$\lim_{\tau\to\infty}\frac{1}{\tau}\int v(\mathbf{r}'+\mathbf{r})v(\mathbf{r})d\mathbf{r}=w(\mathbf{r}), \tag{56}$$

where τ is the volume of integration. To define the problem completely, we must select a class of solutions by some means. We do not wish to this by the introduction of boundary conditions at infinity, because it is clear from analogy with the crystal that the solutions needed to satisfy boundary conditions in a problem involving a finite scatterer are of very different types, increasing exponentially in various space directions.[17] In the case of the crystal, the selection of solutions is made by Floquet's theorem, which is expressed by Eq. (11). What we need for the random medium is evidently an analog of Floquet's theorem.

[16] L. L. Foldy, Phys. Rev. **67**, 107 (1945).

[17] In the simplest case of a homogeneous medium, the solutions needed to solve the problem of total reflection are exponentially increasing normal to the boundary.

Our choice is based on the simple remark that the crystal can be considered as the most general medium provided that the unit cell is extended to infinity. Hence, we shall follow the formalism of Sec. III, going to the limit $\tau\to\infty$ at the appropriate point. This selection is not equivalent to the periodic boundary conditions frequently used in physics, mainly because complex values of the wave numbers must be admitted

We consider now the second-order solution to the crystal problem. By inversion of Eq. (12) we have

$$V_n=\frac{1}{\tau}\int_\tau \exp(-2\pi i A_n\cdot\mathbf{r})v(\mathbf{r})d\mathbf{r}. \tag{57}$$

In the limit the reciprocal lattice vectors will form a dense set $\mathbf{R}$ and V_n will tend toward $V(\mathbf{R})$. But

$$\lim_{\tau\to\infty}\frac{1}{\tau}\left|\int \exp(-2\pi i\mathbf{R}\cdot\mathbf{r})v(\mathbf{r})d\mathbf{r}\right|^2=\lim\tau|V(\mathbf{R})|^2 \tag{58}$$

will exist in the limit, if the correlation function $w(\mathbf{r})$ exists, and will be given by the Wiener-Khinchin formula

$$\lim\tau|V(\mathbf{R})|^2=\int d\mathbf{r}\exp(-2\pi i\mathbf{r}\cdot\mathbf{R})w(\mathbf{r})=G(\mathbf{R}). \tag{59}$$

The Fourier coefficients V_n tend to zero as $1/\sqrt{\tau}$, except V_0, the mean value, which remains finite. Therefore, V_0 and the coefficient a_0 must be singled out. Instead of Eqs. (14) and (15), we have

$$[K_0^2+(2m/\hbar^2)V_0-k_0^2]c_0+\sum{}' v_m c_m=0, \tag{60}$$

$$(n\neq 0)[(\mathbf{K}_0+2\pi\mathbf{A}_n)^2+(2m/\hbar^2)V_0-k_0^2]c_n +V_n c_0+\sum{}' V_{n-m}c_m=0. \tag{61}$$

Assuming again c_0 to be large, we obtain

$$c_n=-\frac{2m}{\hbar^2}\frac{V_n c_0}{(\mathbf{K}_0+2\pi\mathbf{A}_n)^2+(2m/\hbar^2)V_0-k_0^2}, \tag{62}$$

and

$$K_0^2-k_0^2+\frac{2m}{\hbar^2}V_0-\left(\frac{2m}{\hbar^2}\right)^2\sum{}' \times\frac{|V_m|^2}{(\mathbf{K}_0+2\pi\mathbf{A}_n)^2+(2m/\hbar^2)V_0-k_0^2}=0. \tag{63}$$

For finite cell dimensions, $\mathbf{K}_0$ is obviously such that none of the denominators in Eqs. (62, 63) vanishes. As the vectors $\mathbf{A}_n$ become a dense set, it is no longer possible to avoid the vanishing of the denominators, and a suitable limiting process must be defined. By analogy with usual scattering theory, we define the limiting process so that k_0^2 has a small positive imaginary part, which is made to vanish after the sum has been converted into an integral.

In converting the sum in Eq. (63) into an integral, we note that the number of reciprocal lattice points in the unit volume of Fourier space is τ. Hence, we obtain

$$K_0^2-k_0^2+\frac{2m}{\hbar^2}V_0-\left(\frac{2m}{\hbar^2}\right)^2\int_h \times\frac{G(\mathbf{R})d\mathbf{R}}{(\mathbf{K}_0+2\pi\mathbf{R})^2-k_0^2+(2m/\hbar^2)V_0}=0, \tag{64}$$

to the second order.

We now turn to the boundary value problem of a semi-infinite medium, with an incident plane wave. Rigorously, the solution of the boundary value problem would require an infinite number of solutions for the infinite medium, and not only the one approximated by Eqs. (62, 64), but we follow the usual procedure[18] in omitting all but the strongest. The tangential components of $\mathbf{K}_0$ are then determined by the requirements of continuity: they must be equal to those of the incident wave, and the only unknown is the component of $\mathbf{K}_0$ normal to the boundary which is determined by Eq. (64). The reflected and transmitted strong waves are determined as in homogeneous refraction, with a refractive index

$$n^2-1=-\frac{2m}{k_0^2\hbar^2}V_0+\frac{(2m)^2}{k_0^2\hbar^4}\int_h \times\frac{G(\mathbf{R})d\mathbf{R}}{(\mathbf{k}_0+2\pi\mathbf{R})^2-k_0^2+(2m/\hbar^2)V_0}, \tag{65}$$

where we have substituted $\mathbf{k}_0$, the wave vector of the incident wave, for $\mathbf{K}_0$ in the denominator of the integral. By the definition of the integral, the imaginary part of n^2 is given by

$$\mathrm{Im}n^2=\frac{m^2}{(2\pi)^2k_0\hbar^4}\int_S G\left(\frac{\mathbf{R}-\mathbf{k}_0}{2\pi}\right)d\Omega. \tag{66}$$

which is the equivalent of the formulas for turbidity in the optics of liquids.[19]

In contrast to the crystal case, the weak reflected or scattered vacuum waves cannot be disregarded, because the resonance denominator vanishes for some of the internal weak waves.

The calculation of the weak reflected waves is somewhat facilitated by remarking that those internal waves for which the tangential component of $\mathbf{K}_h$ is larger than k_0, can be only continued into the vacuum by waves decaying exponentially from the surface as in total reflection of homogeneous bodies. Hence, most of the weak waves will give rise only to surface waves which are hardly observable. Furthermore, it is clear that waves with propagation vectors greatly different

[18] M. von Laue, *Materiewellen und ihre Interferenzen* (Akademische Verlagsgesellschaft, Leipzig, 1948).

[19] S. Bhagavantam, *Scattering of Light and the Raman Effect* (Chemical Publishing Company, Brooklyn, 1942).

from k_0 suffer strong, if not total, reflection in penetrating through the boundary. It is, hence, plausible to assume that the only important contribution to the weak external field is given by internal waves with propagation vectors $|\mathbf{K}_h| \approx |\mathbf{K}_0| \approx k_0$. But these waves suffer little refraction, so that we may simply continue them into the vacuum without change.

By Eqs. (11) and (62), the wave function in the vacuum is

$$\psi \approx -\exp(i\mathbf{K}_0\cdot\mathbf{r})\sum_h \frac{2m}{\hbar^2}\frac{V_n c_0}{K_h^2-k_0^2}\exp(2\pi i\mathbf{A}_h\cdot\mathbf{r}), \quad (67)$$

the summation being extended over a small range $|\mathbf{K}_h| \approx k_0$. Passing to the continuum, we have for $c_0=1$

$$\psi = -\frac{2m}{\hbar^2}\tau\int_h \frac{\exp[i(\mathbf{K}_0+2\pi\mathbf{R})\cdot\mathbf{r}]V(\mathbf{R})}{(\mathbf{K}_0+2\pi\mathbf{R})^2-k_0^2}d\mathbf{R}. \quad (68)$$

For large values of r, one obtains the asymptotic value

$$\psi \sim -\frac{2m}{\hbar^2}\frac{\tau}{4\pi}V\left(\frac{\mathbf{k}-\mathbf{K}_0}{2\pi}\right)\frac{\exp(ik_0 r)}{r},$$

where $\mathbf{k}$ is a vector of length k_0 and parallel to $\mathbf{r}$. Hence, by Eq. (59)

$$|\psi|^2 \sim \left(\frac{2\pi m}{\hbar^2}\right)^2\frac{\tau}{r^2}G\left(\frac{\mathbf{k}-\mathbf{K}_0}{2\pi}\right). \quad (69)$$

Equation (69) is almost identical with Born's approximation for the scattered waves, but it contains instead of the propagation vector of the incident wave $\mathbf{k}_0$, the vector $\mathbf{K}_0$ which has a complex normal component, so that only a surface layer of the medium contributes to scattering.

This approach to the dynamical theory of the random medium may become inadequate either because the potential has strong singularities (point scatterers) so that the second term in Eq. (64) becomes very large even diverges; or because the interaction cannot be described by a potential. In either case, we can use the alternative formulation in terms of two-body matrices β.

Conversion of the sums into integrals, in the manner previously described, gives by Eq. (55)

$$K_0^2-k_0^2=\frac{(2\pi)^3}{\tau}\beta(\mathbf{K}_0,\mathbf{K}_0)$$
$$+\frac{(2\pi)^3}{\tau}P\int\frac{\beta(\mathbf{K}_0,\boldsymbol{\kappa})\beta(\boldsymbol{\kappa},\mathbf{K}_0)-\sum\beta_j(\mathbf{K}_0,\boldsymbol{\kappa})\beta_j(\boldsymbol{\kappa},\mathbf{K}_0)}{\kappa^2-k_0^2}d\boldsymbol{\kappa}$$
$$+\frac{(2\pi)^3}{\tau}\frac{i\pi k_0}{2}\int\beta(\mathbf{K}_0,\boldsymbol{\kappa})\beta(\boldsymbol{\kappa},\mathbf{K}_0)d\Omega. \quad (70)$$

This expression is meaningful if the limits

$$\lim_{\tau\to\infty}\frac{\beta(\mathbf{K}_0,\mathbf{K}_0)}{\tau} \quad \text{and} \quad \lim_{\tau\to\infty}\frac{\beta(\mathbf{K}_0,\boldsymbol{\kappa})\beta(\boldsymbol{\kappa},\mathbf{K}_0)}{\tau}(\boldsymbol{\kappa}\neq\mathbf{K}_0) \quad (71)$$

exist. The first limit is clearly the mean value of $\beta_i(\mathbf{K}_0,\mathbf{K}_0)$ and exists by assumption. To investigate the second limit, we may disregard the matrices β_i and consider

$$\lim_{\tau}\frac{1}{\tau}|\sum\exp(i\mathbf{k}\cdot\mathbf{r}_i)|^2.$$

The existence of this limit is implied in the existence of $G(\mathbf{R})$ as defined in Eq. (59), for, if the potential is due to individual scatterers, with potentials $v_i(\mathbf{r})$, V has the form

$$V=(1/\tau)\textstyle\sum_i\exp(i\mathbf{R}\cdot\mathbf{r}_i)V_i(\mathbf{R}), \quad (72)$$

with

$$V_i=\int_\infty v_i(\mathbf{r})\exp(i\mathbf{R}\cdot\mathbf{r})d\mathbf{r}. \quad (73)$$

Hence, the refractive index is

$$n^2-1=(2\pi)^3\left[\frac{\langle\beta_i(\mathbf{K}_0,\mathbf{K}_0)\rangle}{k_0^2}+P\frac{1}{k_0^2}\int\frac{G''(\mathbf{K}_0,\boldsymbol{\kappa})}{\kappa^2-k_0^2}d\boldsymbol{\kappa}\right.$$
$$\left.+\frac{i\pi}{2k_0}\int_S G'(\mathbf{K}_0,\boldsymbol{\kappa})d\Omega\right], \quad (74)$$

where

$$G'(\mathbf{K}_0,\boldsymbol{\kappa})=\lim(1/\tau)\beta(\mathbf{K}_0,\boldsymbol{\kappa})\beta(\boldsymbol{\kappa},\mathbf{K}_0), \quad (75)$$

$$G''(\mathbf{K}_0,\boldsymbol{\kappa})=\lim(1/\tau)[\beta(\mathbf{K}_0,\boldsymbol{\kappa})\beta(\boldsymbol{\kappa},\mathbf{K}_0)$$
$$-\textstyle\sum\beta_j(\mathbf{K}_0,\boldsymbol{\kappa})\beta_j(\boldsymbol{\kappa},\mathbf{K}_0)], \quad (76)$$

and

$$\langle\beta_i(\mathbf{K}_0,\mathbf{K}_0)\rangle=\lim(1/\tau)\textstyle\sum\beta_i(\mathbf{K}_0,\mathbf{K}_0). \quad (77)$$

If one or several scatterers are point scatterers (β_i constant), the principal value still exists in this formulation because the constant part of the first term of G'' is canceled by the second term.

The first term in Eq. (74) has the same form as in the crystal case, and needs no further discussion. The third term has a simple meaning only when the single scattering event is purely elastic. In this case, the integral is essentially the cross section, in first approximation, of a unit volume of liquid, by comparison with Eqs. (9) and (10). This is in agreement with well-known results on turbidity.[19] The second term is purely real for nonabsorbing liquids, and is evidently the equivalent of the Lorentz correction. From the comparison with the Lorentz correction for crystals we may infer that its order of magnitude, relative to the first term is β/a, where a is a mean distance between scatterers. This term constitutes the main contribution to the first nonvanishing correction for the refractive index of a liquid as used in the experiments of Burgy, Ringo, and Hughes.[20]

Since the third term vanishes when the density of scatterers is so large that destructive interference prevails, we find that the refractive index is real for very dense, nonabsorbing liquids. This result is in agreement

[20] Burgy, Ringo and Hughes, Phys. Rev. 84, 1160 (1951).

with theoretical results usually reported.[19] These, however, are obtained by arbitrarily disregarding radiation damping contrary to the present derivation.

As an example, we consider the imaginary part of n^2 for the case of a crystal with protons having random spins. The disordered crystal is a special case of a random medium, and can be treated dynamically by the present method. However, the particular approximation method used to derive Eq. (70) assumes that no Bragg reflection has importance, so that the following result is only valid for a wavelength large enough to avoid Bragg reflection in a polycrystal, or for a nonreflecting position of a single crystal.

The elements of the matrix β for the n-p scattering are derived in Sec. V.

The contribution of the first term in the bracket [Eq. (74)] is

$$(a_1-a_3)^2\nu_p/16\pi^2k_0,$$

where ν_p is the number of protons per unit volume. This contribution is in agreement with Eq. (53), since the inelastic (usually called incoherent) cross section of the unpolarized proton is $(4\pi/8)(a_1-a_3)^2$. The second term in the bracket is real to the second order in the scattering length, and does not contribute. The integrand of the third term can be easily evaluated on the assumption of random spins.[21] It is

$$\nu_p(\langle\beta^2\rangle-\langle\beta\rangle^2)=\frac{\nu_p}{16\cdot4\pi^4}(a_1-a_3)^2.$$

Hence,

$$\mathrm{Im}n^2=\frac{4\pi\nu_0}{k_0}\frac{3}{2}\frac{1}{8}(a_1-a_3)^2. \tag{77}$$

In other words, the apparent cross section of the protons, as measured by a transmission experiment, is $\frac{3}{2}$ times the "incoherent" n-p cross section. This result could also have been obtained from the elementary theory, i.e., by determining in first approximation, the number of incident neutrons scattered by a thin sheet.

Specifically dynamical phenomena, not described by a kinematic theory, are to be expected when the scatterers have their spins partially aligned by an external field. In this case, the spinor character of the neutrons cannot be disregarded, and the coefficients C_n in Eq. (41) must be considered as spinors. For a liquid, all coefficients but C_0 may be disregarded in first approximation, so that two refractive indices must be considered, for neutron spin parallel and antiparallel to the field direction, as in the theory of neutron propagation in a ferromagnet.[22] We have

$$n^2-1=-\frac{4\pi}{k_0^2}\left[\frac{\nu_0}{4}(3a_3+a_1)\pm\frac{\epsilon}{2}\nu_p(a_3-a_1)+\nu_Ma_M\right], \tag{78}$$

where ν_p and ν_M are the numbers per unit volume, of protons and other spinless nuclei, respectively, a_M the scattering length of these, and ϵ the relative excess of protons aligned:

$$\epsilon=(\nu_+-\nu_-)/\nu-\tfrac{1}{2}. \tag{79}$$

A measurement of the angle of total reflection provides a means for determining the difference of triplet and singlet amplitude.

V. PROPERTIES OF THE MATRICES α AND β

The basic assumption of this paper is that equations derived for the simplest case of potential scattering remain approximately valid if the matrices α and β describing the elastic scattering of a more general single scatterer are substituted. We shall now express these quantities in terms of the general theory of scattering and derive some of their properties.

It would be desirable to relate the multiple scattering to a general time-independent theory of single scattering. Neither of the three presentations known to us[23–25] is quite sufficient to describe absorption, creation and rearrangement processes which occur in the nuclear phenomena under study. We have to make the further assumption that the general properties of the scattering matrices are preserved in a more general theory, and we shall use Pauli's formalism.

The scattering of an incident plane wave is described by the matrix

$$(k|f|0)=U(k|\phi|0), \tag{80}$$

where U is the interaction Hamiltonian, and $(k|\phi|0)$ the wave function in momentum representation. If U is a scalar potential in coordinate space,

$$(\mathbf{k}|f|0)=\frac{1}{(2\pi)^3}\int u(\mathbf{r})\psi(\mathbf{r})\exp(-i\mathbf{k}\cdot\mathbf{r})d\mathbf{r}, \tag{81}$$

where ψ is that wave function which has the asymptotic form $\exp(i\mathbf{k}_0\cdot\mathbf{r})$. Hence, it is seen by comparison with Eq. (I, 4) that

$$f(\mathbf{k})=-(2m/\hbar^2)(\mathbf{k}|f|0) \tag{82}$$

in the simplest case.[26] The generalization consists in substituting for f that submatrix which describes elastic scattering, i.e.,

$$f(\mathbf{k})=-(2m/\hbar^2)(\mathbf{k}|f|0)\delta_{nn_0}, \tag{83}$$

where n and n_0 characterize two unperturbed states of the scatterer. In the following, it will be useful to

[21] Reference 4, p. 177.
[22] H. Ekstein, Phys. Rev. **76**, 1328 (1949).
[23] C. Møller, Kgl. Danske Videnskab. Selskab Mat.-fys. Medd. **23**, No. 1 (1945); **22**, No. 19 (1946).
[24] W. Pauli, *Meson Theory of Nuclear Forces* (Interscience Publishers, Inc., New York, 1948).
[25] B. Lippmann and J. Schwinger, Phys. Rev. **79**, 469 (1950).
[26] Pauli's equation giving the connection between the asymptotic value of the wave function in coordinate space and the matrix $(k|f|0)$ differs from our Eq. (I, 8) by a factor $(-(2\pi)^{-\frac{1}{2}})$. The minus sign is presumably omitted by mistake; the constant factor is due to our use of unnormalized plane waves $\exp(i\mathbf{k}\cdot\mathbf{r})$ as usual in the theory of scalar scattering, instead of normalized plane waves $(2\pi)^{-\frac{1}{2}}\exp(i\mathbf{k}\cdot\mathbf{r})$.

distinguish between variables characterizing the scatterer (n) and the incident particle (k). Thus, we can write

$$f(\mathbf{k}) = -(2m/\hbar^2)(k, n_0|f|k_0, n_0). \quad (84)$$

Generalizing Eq. (I, 28) we can define a matrix $(k|\alpha|\kappa)$ for the general scattering problem by the integral equation

$$(k|\alpha|\kappa) - \sum_l \frac{(k|U|l)}{E_0 - E_l}(l|\alpha|\kappa) + i\pi \int_A (k|U|A)(A|\alpha|\kappa) = (k|U|\kappa), \quad (85)$$

which reduces to Pauli's equation (5) for $(k|f|0)$ in the special case where $E(\kappa) = E_0$. In Pauli's notation, the symbol A stands for all variables except the energy, and the integral is extended over the "energy shell" $E = E_0$. The matrix $\alpha(\mathbf{k}, \boldsymbol{\kappa})$ used in the theory of multiple scattering is proportional to that submatrix of $(k|\alpha|\kappa)$, which describes elastic processes only:

$$\alpha(\mathbf{k}, \boldsymbol{\kappa}) = -(2m/\hbar^2)(nk|\alpha|n_0\kappa)\delta_{nn_0}. \quad (86)$$

The matrix $(k|\alpha|\kappa)$ has no direct physical meaning, but it can be connected with the scattering of a beam of particles issued from a point source at a finite distance, in the manner described by Eqs. (I, 23, 29).

For the theory of multiple scattering in extended bodies, we need the generalization of the matrix $\beta(\mathbf{k}, \boldsymbol{\kappa})$ introduced in Eq. (I, 30).

This matrix differs from α by the elimination of radiation damping in elastic, but only in elastic scattering. Hence, the appropriate generalization is given by the equation

$$(n, k|\beta|n_0, \kappa) - \sum_{l,m} \frac{(n, k|U|m, l)(m, l|\beta|n_0, \kappa)}{E_0 - E_{ml}} + i\pi \sum_{m \neq n_0} \int_A (n, k|U|m, A)(m, A|\beta|n_0, \kappa) = (n, k|U|n_0, \kappa), \quad (87)$$

in which the term describing radiation damping by loss of the elastically scattered particles has been omitted. The matrix $\beta(\mathbf{k}, \boldsymbol{\kappa})$ is again proportional to the elastic submatrix of $(n, k|\beta|n_0, \kappa)$:

$$\beta(\mathbf{k}, \boldsymbol{\kappa}) = -(2m/\hbar^2)(n, k|\beta|n_0, \kappa)\delta_{nn_0}. \quad (88)$$

If the Hamiltonian of the single scattering process is not known, the matrix α can, at least in principle, be measured by a set of double-scattering experiments, using Eq. (I, 52), but β is not directly related to an experiment using a small number of scatterers. Therefore, it is useful to show how β can be calculated if α is known.

We write the equation defining α (Eq. (85)) under the form

$$(n, k|\alpha|n_0, \kappa) - \sum_{l,m} \frac{(n, k|U|m, l)(m, l|\alpha|n_0, \kappa)}{E_0 - E_m} + i\pi \sum_{m \neq n_0} (n, k|U|m, A)(m, A|\alpha|n_0, \kappa) = (n, k|U|n_0, \kappa) - i\pi \int_A (n, k|U|n_0, A) \times (n_0, A|\alpha|n_0, \kappa). \quad (89)$$

Comparison with Eq. (87) shows that

$$(n, k|\alpha|n_0, \kappa) = (n, k|\beta|n_0, \kappa) - i\pi \int_A (n, k|\beta|n_0, A)(n_0, A|\alpha|n_0, \kappa). \quad (90)$$

This equation allows us to calculate β if α is known. It is similar to Heitler's integral equation, except for the circumstance that in general $E_0 \neq E_{n_0\kappa}$. The solution can be obtained in two steps: since only values of β on the energy shell occur under the integral, we can write, for the particular set of values $\kappa = k_0$ such that $E_0 = E_{n_0\kappa}$:

$$(n, k|f|n_0, k_0) = (n, k|g|n_0k_0) - i\pi \int_A (n, k|g|n_0, A)(n_0, A|f|n_0, k_0). \quad (91)$$

If g is determined by solving this equation, β is given by

$$(n, k|\beta|n_0, \kappa) = (n, k|\alpha|n_0, \kappa) + i\pi \int_A (n, k|g|n_0, A)(n_0, A|\alpha|n_0, \kappa). \quad (92)$$

In first approximation, when β differs only slightly from α, we have

$$(n, k|\beta|n_0, \kappa) = (n, k|\alpha|n_0, \kappa) + i\pi \int_A (n, k|\alpha|n_0, A)(n_0, A|\alpha|n_0, \kappa). \quad (93)$$

This equation can be used to estimate the importance of the correction for radiation damping. For point scatterers, Eq. (90) becomes

$$(n|\alpha|n_0) = (n|\beta|n_0) - i(4\pi^2k_0m/\hbar^2)(n|\beta|n_0)(n_0|\alpha|n_0), \quad (94)$$

or, if we write, in accordance with Eqs. (86, 88),

$$\alpha_n = -(2m/\hbar^2)(n|\alpha|n), \quad \beta_n = -(2m/\hbar^2)(n|\beta|n), \quad (95)$$

$$\beta_n = \alpha_n/(1 + 2i\pi^2k_0\alpha_n). \quad (96)$$

Hence, if α_n is written under the form $1/4\pi^2ik_0$

$\times[\exp(2i\delta_n)-1]$ where δ is now complex, we have

$$\beta_n=(1/2\pi^2k_0)\tan\delta_n, \tag{97}$$

which justifies the generalized use of Eq. (I, 22) made in Sec. III.

As an example, we calculate β for n-p scattering. The only nonvanishing matrix elements of α are, with obvious notation:[27]

$$\downarrow\downarrow \rightarrow \downarrow\downarrow \text{ or } \uparrow\uparrow \rightarrow \uparrow\uparrow : {}^3\alpha,$$
$$\downarrow\uparrow \rightarrow \downarrow\uparrow \text{ or } \uparrow\downarrow \rightarrow \uparrow\downarrow : \tfrac{1}{2}({}^3\alpha+{}^1\alpha),$$
$$\downarrow\uparrow \rightarrow \uparrow\downarrow \text{ or } \uparrow\downarrow \rightarrow \downarrow\uparrow : \tfrac{1}{2}({}^3\alpha-{}^1\alpha),$$

where ${}^3\alpha$ and ${}^1\alpha$ are the corresponding quantities for triplet and singlet scattering. The diagonal elements of β are, by Eq. (97)

$${}^3\alpha/(1+2i\pi^2k_0\,{}^3\alpha) \quad \text{(parallel spins)},$$
$$\tfrac{1}{2}({}^3\alpha+{}^1\alpha)/[1+i\pi^2k_0({}^3\alpha+{}^1\alpha)] \quad \text{(antiparallel spins)}.$$

By Eq. (I, 21) we have, for each of the elements ${}^3\alpha$ and ${}^1\alpha$,

$$2\pi^2\alpha=\delta_0/k_0+i\delta_0^2/k_0 \tag{98}$$

to the second order of the small phase angle. In terms of the scattering length,

$$a=-\delta_0/k_0, \tag{99}$$

we have then

$$2\pi^2\beta=-a_3 \quad \text{(parallel spins)}, \tag{100}$$

and

$$2\pi^2\beta=-\tfrac{1}{2}(a_3+a_1)+\tfrac{1}{4}ik_0(a_3-a_1)^2 \quad \text{(antiparallel spins)}. \tag{101}$$

The scattering lengths a_1 and a_3 are here those of the bound proton, i.e., twice the scattering length for the free collision.

Equation (87) can be written

$$\beta=U(1-T), \tag{102}$$

where

$$(nk|T|n'k')=\frac{(nk|\beta|n'k')}{E(nk)-E_0}$$
$$+i\pi(nk|\beta|n'k')\delta\{E(nk)-E_0\}(1-\delta_{nn'}). \tag{103}$$

[27] L. Rosenfeld, *Nuclear Forces* (North-Holland Publishing Company, New York, 1948).

Because of the Hermitean character of U, we have also

$$\beta^*=(1-T^*)U. \tag{104}$$

Multiplication of Eq. (102) by $(1-T^*)$ from the left and of Eq. (104) by $(1-T)$ from the right, gives

$$(1-T^*)\beta-\beta^*(1-T)=0. \tag{105}$$

In particular, when only elastic scattering exists, i.e., all matrix elements of U except $(n_0k|U|n_0k')$ vanish,

$$\beta-\beta^*=0, \tag{106}$$

which proves the statement made in Sec. II. In general we obtain from Eq. (105) for the matrix element n, $k=n'$, $k'=n_0$, k_0:

$$2\,\mathrm{Im}(n_0,k_0|\beta|n_0,k_0)$$
$$=-2\pi\sum_{m\neq n_0}\int_A|(m,A|\beta|n_0,k_0)|^2. \tag{107}$$

By Eq. (93) β differs from α only by second-order terms. Hence, we can substitute α for β in the right-hand side, with an error of third order in β. We can use this approximation consistently to evaluate $\beta(\mathbf{k}_0,\mathbf{k}_0)$ in Eq. (43). The elements of α appearing under the integral belong to the submatrix f of α, since $E_0=E(n_0,k_0)$. The relation between the asymptotic form of the scattered wave and f is[24]

$$\psi_m(\mathbf{r})\sim-(2\pi)^{\frac{1}{2}}\frac{\exp(ik_m)}{r}k_m\left(\frac{dk}{dE}\right)_{k=k_m}(m,A|f|n_0k_0). \tag{108}$$

Hence, the partial differential cross section $d\sigma_m$ for the process $n_0\rightarrow m$ is[28]

$$d\sigma_m=(2\pi)^4\frac{k_m}{k_0}\frac{m^2}{\hbar^4}|(m,A|f|n_0,k_0)|^2d\Omega. \tag{109}$$

Therefore,

$$\mathrm{Im}\beta(\mathbf{k}_0,\mathbf{k}_0)=[k_0/(2\pi)^3]\sigma_{\mathrm{in}}, \tag{110}$$

where

$$\sigma_{in}=\sum_{m\neq n_0}\int d\sigma_m. \tag{111}$$

This result was anticipated in Sec. III.

The author is indebted to Miss Irene Corvin for numerical calculations leading to Eq. (33).

[28] The factor $(2\pi)^3$ is due to different normalization, as pointed out in footnote 26.