Conversion of an Amplified Dirac Equation to an Approximately Relativistic Form*

W. A. BARKER† AND Z. V. CHRAPLYVY
Physics Department, St. Louis University, St. Louis, Missouri
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A fermion is described by a one-body relativistic wave equation of the Dirac type in which scalar, vector, tensor, pseudovector, and pseudoscalar interactions are included phenomenologically. This equation is converted to an approximately relativistic form by means of the Foldy-Wouthuysen transformation. The resulting relativistic correction terms associated with the five types of interaction are tabulated. The tables may be used to convert any one-body Dirac equation to its corresponding approximately relativistic form. The fine structure of the hydrogenic atom including the Lamb-Retherford shift is discussed as an illustration.

I. INTRODUCTION

A FERMION of mass m interacting with an external field may be described by a four-component wave function ψ satisfying the equation

$$\{\gamma_\mu(p_\mu+g_{\rm I}\theta_\mu)-i(mc+g\theta) -(1/2!)g_{\rm II}\gamma_\mu\gamma_\nu\theta_{\mu\nu}-(1/3!)g_{\rm III}\gamma_\mu\gamma_\nu\gamma_\rho\theta_{\mu\nu\rho} -(i/4!)g^P\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma\theta_{\mu\nu\rho\sigma}\}\psi=0, \quad (1)$$

where $p_\mu=(-i\hbar\nabla,\ -i\hbar\partial/\partial x_4)$, $x_4=ict$, $\gamma_\mu=(-i\beta\boldsymbol{\alpha},\ \beta)$; β and $\boldsymbol{\alpha}$ are the familiar Dirac matrices; c, $\hbar$ are the velocity of light and Planck's constant divided by 2π. The usual summation convention applies in expressions involving repeated Greek indices with the restriction that no two of the subscripts are equal in the same term. The external field is characterized by a scalar θ, a vector θ_μ, an antisymmetric tensor $\theta_{\mu\nu}$, a pseudovector $\theta_{\mu\nu\rho}$, and a pseudoscalar $\theta_{\mu\nu\rho\sigma}=\theta^P$, which have been added to the free particle Dirac equation in such a way as to satisfy the invariance requirements of special relativity under a Lorentz transformation. The strength of the interaction is determined by the coupling constants g, $g_{\rm I}$, $g_{\rm II}$, $g_{\rm III}$, and g^P.

We do not identify the interaction terms with any particular type of external field nor the coupling constants with any specific coupling mechanism. For example, θ_μ may be identified with the electromagnetic potentials, the field strengths in scalar meson theory, or any other type of external field as long as it transforms like a four vector.

We propose to convert Eq. (1) into an approximately relativistic form, more amenable to physical interpretation and practical calculation, by a method recently developed by Foldy and Wouthuysen.[1] This technique has two advantages not shared by the method previously used to study this amplified Dirac equation[2]—the traditional method of eliminating the small components in terms of the large. First, the transformed Hamiltonian is Hermitian. Second, the transformed four-component wave function may be split into two sets of two-component wave functions which describe positive and negative energy states, respectively. The calculations are carried out to the second order in the expansion parameter, $(\hbar/mc)(\partial/\partial x_\mu)$.

By way of illustration, the results are applied to a study of the fine structure of the hydrogenic atom including the Lamb-Retherford shift.

II. REDUCTION OF THE AMPLIFIED DIRAC EQUATION TO AN APPROXIMATELY RELATIVISTIC FORM

In order to apply the Foldy-Wouthuysen method, it is necessary to write Eq. (1) in the form

$$H\psi=(\beta mc^2+\mathcal{O}+\mathcal{E})\psi=i\hbar\partial\psi/\partial t, \quad (2)$$

where the odd operators $\mathcal{O}$ and the even operators $\mathcal{E}$ are explicitly separated.[3] This may be readily done with the aid of the relations $\boldsymbol{\sigma}=(1/2i)(\boldsymbol{\alpha}\times\boldsymbol{\alpha})$, $\gamma_5=i\alpha_1\alpha_2\alpha_3$, and the following change of notation: $\theta_\mu\rightarrow(\boldsymbol{\theta},\theta_4)$, $\theta_{\mu\nu}\rightarrow(\mathbf{M},\mathbf{P})$, $\theta_{\mu\nu\rho}\rightarrow(\mathbf{S},S_4)$. The result is

$$H\psi=\{\beta mc^2+cg\beta\theta+icg_{\rm I}\theta_4+cg_{\rm II}\beta\boldsymbol{\sigma}\cdot\mathbf{M}+cg_{\rm III}\boldsymbol{\sigma}\cdot\mathbf{S} +c\boldsymbol{\alpha}\cdot(\mathbf{p}+g_{\rm I}\boldsymbol{\theta})-cg_{\rm II}\beta\boldsymbol{\alpha}\cdot\mathbf{P}-icg_{\rm III}\gamma_5S_4 +cg^P\beta\gamma_5\theta^P\}\psi=i\hbar\partial\psi/\partial t. \quad (3)$$

The terms involving the matrices I, β, $\boldsymbol{\sigma}$, and $\beta\boldsymbol{\sigma}$ are even, those involving $\boldsymbol{\alpha}$, $\beta\boldsymbol{\alpha}$, γ_5, and $\beta\gamma_5$ are odd. The transformed wave equation generated by the Hermitian operator $S=-(i/2mc^2)\beta\mathcal{O}$ is of the form[4]

$$H'\psi'=\left\{\beta mc^2+\mathcal{E}+\frac{\beta}{2mc^2}\mathcal{O}^2 -\frac{1}{8m^2c^4}\left[\mathcal{O},[\mathcal{O},\mathcal{E}]_-+i\hbar\frac{\partial\mathcal{O}}{\partial t}\right]_-+\cdots\right\}\psi' =i\hbar\partial\psi'/\partial t. \quad (4)$$

Carrying this prescription out explicitly for Eq. (3) requires the evaluation of a large number of commutators and anticommutators. The calculations are facilitated by the use of the following identities of which the first two are well known and the last two

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[1] L. L. Foldy and S. A. Wouthuysen, Phys. Rev. **78**, 29 (1950).
[2] G. Petiau, J. phys. et radium **10**, 264 (1949).
[3] An odd operator couples the upper and lower components of ψ; an even operator does not. See reference 1, p. 30, footnote 3.
[4] See reference 1, Eq. (32).

are easily verified.

$$[(\boldsymbol{\alpha}\cdot\mathbf{B}),\,(\boldsymbol{\alpha}\cdot\mathbf{C})]_{\mp}=\mathbf{B}\cdot\mathbf{C}+i\boldsymbol{\sigma}\cdot\mathbf{B}\times\mathbf{C}\mp\mathbf{C}\cdot\mathbf{B}\mp i\boldsymbol{\sigma}\cdot\mathbf{C}\times\mathbf{B},\quad(5)$$

$$(\boldsymbol{\sigma}\cdot\mathbf{B})(\boldsymbol{\sigma}\cdot\mathbf{C})=\mathbf{B}\cdot\mathbf{C}+i\boldsymbol{\sigma}\cdot\mathbf{B}\times\mathbf{C},\quad(6)$$

$$[(\boldsymbol{\alpha}\cdot\mathbf{B}),\,(\boldsymbol{\sigma}\cdot\mathbf{C})]_{\mp}=-\gamma_5\mathbf{B}\cdot\mathbf{C}+i\boldsymbol{\alpha}\cdot\mathbf{B}\times\mathbf{C}\pm\gamma_5\mathbf{C}\cdot\mathbf{B}\mp i\boldsymbol{\alpha}\cdot\mathbf{C}\times\mathbf{B},\quad(7)$$

$$[(\boldsymbol{\alpha}\cdot\mathbf{B}),\,(\gamma_5 C)]_{\mp}=\mp(\boldsymbol{\sigma}\cdot\mathbf{B})C\pm C(\boldsymbol{\sigma}\cdot\mathbf{B}).\quad(8)$$

It is assumed that $\mathbf{B}$, $\mathbf{C}$, and C commute with the Dirac matrices but not necessarily with one another.

The five types of interaction generate relativistic correction terms of zero, first, and second order in the expansion parameter. These we have placed into six classifications, of which three involve $\boldsymbol{\sigma}$ and three do not.

A "Schrödinger term" is one which has the appearance of one or more of the terms in the ordinary Schrödinger equation, $(p^2/2m+V)\psi=E\psi$.

A "divergence term" is one which is similar to the term used by Foldy and Wouthuysen[1] to account for the "Darwin correction," to the S levels in a hydrogen atom, $-(e\hbar^2/8m^2c^4)\nabla\cdot\mathbf{E}$.

A "Darwin term" is one which is analogous to the term originally used by Darwin[5] to account for the "Darwin correction," $-(e\hbar^2/4m^2c^4)\mathbf{E}\cdot\nabla$.

A "spin-field term" is one which resembles the well-known term involving the normal magnetic moment of the electron, $(\beta e\hbar/2mc)\boldsymbol{\sigma}\cdot\mathbf{H}$.

A "spin-orbit term" is one which is similar to the well-known Thomas term,[6] $-(e\hbar/4m^2c^4)\boldsymbol{\sigma}\cdot(\mathbf{E}\times\mathbf{p})$.

Finally, a "spin-momentum term" is one which involves the dot product of $\boldsymbol{\sigma}$ and the linear momentum $\mathbf{p}$.

The results are presented in tabular form. In Table I are entered all those correction terms which do not involve mixed products between different types of interaction. In Table II are entered correction terms which do involve mixed products. It must always be considered along with Table I when a study is made of the correction terms arising from two or more types of interaction present in the original relativistic equation, say the vector and the tensor. The coupling constants have been suppressed in the tables, since how they should be reinserted is quite evident when a given term is to be used. Of course, the order of a term will be changed if the coupling constant includes any power of the Compton wavelength of the particle.

III. THE HYDROGENIC ATOM

In this section, we illustrate the use of the tables in demonstrating how one may obtain expressions for the major contributions to the Lamb-Retherford shift by applying the Foldy-Wouthuysen transformation to a relativistic one-body wave equation in which appropriate quantum-electrodynamic effects are included phenomenologically.

$$H\psi=\{\beta mc^2+e\phi+c\boldsymbol{\alpha}\cdot\mathbf{p}+(\mu_e-1)(e\hbar/2mc)i\beta\boldsymbol{\alpha}\cdot\mathbf{E}-e_1(\hbar/mc)^2\nabla\cdot\mathbf{E}\}\psi=i\hbar\partial\psi/\partial t.\quad(9)$$

The first three terms are the usual ones which occur in the wave equation for a Dirac particle of charge e coupled to the potential ϕ of an external electrostatic field. In addition, the anomalous magnetic moment of the electron (μ_e-1) Bohr magnetons is coupled to the external electric field $\mathbf{E}$, and e_1, a state-dependent quantity arising from a quantum-electrodynamic correction to the rest energy, is coupled to the d'Alembertian of the external field, which in the electrostatic case is simply $\nabla\cdot\mathbf{E}$. Two terms analogous to these have been used by Foldy[7] in a relativistic wave equation for the neutron to account for the magnetic and electrostatic contributions to the electron-neutron interaction.

Equation (9) is, of course, a specialization of Eq. (3) under the substitutions, $\theta_4=i\phi$, $g_{\mathrm{I}}=-e/c$; $\mathbf{P}=i\mathbf{E}$, $g_{\mathrm{II}}=-(\mu_e-1)(e\hbar/2mc^2)$; $\theta_4'=i\nabla\cdot\mathbf{E}$, $g_{\mathrm{I}}'=e_1\hbar^2/m^2c^3$; we assume all the remaining coupling constants to be zero. The corresponding approximately relativistic wave equation may be written by using the vector and tensor columns in Table I. Mixed product terms in Table II are neglected as they are of third or higher order. The result is

$$\begin{aligned}H'\chi'=\Bigg\{&mc^2+\frac{p^2}{2m}+e\phi-\frac{e\hbar^2}{8m^2c^2}\nabla\cdot\mathbf{E}\\&-\frac{e\hbar}{4m^2c^2}\boldsymbol{\sigma}\cdot\mathbf{E}\times\mathbf{p}-\frac{p^4}{8m^3c^2}-e_1\left(\frac{\hbar}{mc}\right)^2\nabla\cdot\mathbf{E}\\&-\frac{(\mu_e-1)}{4m^2c^2}e\hbar^2\nabla\cdot\mathbf{E}-\frac{(\mu_e-1)}{2m^2c^2}e\hbar\boldsymbol{\sigma}\cdot\mathbf{E}\times\mathbf{p}\Bigg\}\chi'\\=&\,i\hbar\partial\chi'/\partial t,\end{aligned}\quad(10)$$

where H' is the transformed Hamiltonian and χ' is a two-component wave function corresponding to positive energies. (The p^4 term in Eq. (10) does not come from the tables. It is obtained from $-\beta\mathcal{O}^4/8m^3c^6$ which is the next even term in the expansion (4).)

The last six terms in Eq. (10) may be considered as perturbations on the Balmer energy levels. Their individual contributions to the energy are given in Table III.

The kinetic energy, Darwin, and Thomas corrections are well known[8] and account for all the $Rch\alpha^2$ terms in the expansion of the Sommerfeld fine structure formula. R is the Rydberg, and α is the fine structure constant.

[5] C. G. Darwin, Proc. Roy. Soc. (London) **118**, 654 (1928).

[6] L. H. Thomas, Nature **117**, 514 (1926).

[7] L. L. Foldy, Phys. Rev. **83**, 688 (1951); Phys. Rev. **86**, 646 (1952).

[8] See, for example, E. U. Condon and G. Shortley, *The Theory of Atomic Spectra* (Cambridge University Press, New York, 1951), Chap. 5.

TABLE I. Relativistic correction terms associated with single-type interactions.

	Scalar	Vector	Tensor	Pseudovector	Pseudoscalar
Zero-Order Terms					
"Schrödinger" and "Spin-Field"	$\beta c\theta$	$ic\theta_4$	$\beta c\boldsymbol{\sigma}\cdot\mathbf{M}$	$c\boldsymbol{\sigma}\cdot\mathbf{S}$	
1st-Order Terms					
"Schrödinger"		$(\beta/2m)(\boldsymbol{\theta}\cdot\boldsymbol{\theta})$	$-(\beta/2m)(P^2)$	$-(\beta/2m)(S_4^2)$	$-(\beta/2m)(\theta^P)^2$
"Divergence"		$-(\beta i\hbar/2m)(\nabla\cdot\boldsymbol{\theta})$	$-(i\hbar/2m)(\nabla\cdot\mathbf{P})$		
"Darwin"		$(\beta/m)(\boldsymbol{\theta}\cdot\mathbf{p})$			
"Spin-Field"		$(\beta\hbar/2m)(\boldsymbol{\sigma}\cdot\nabla\times\boldsymbol{\theta})$	$(\hbar/2m)(\boldsymbol{\sigma}\cdot\nabla\times\mathbf{P})$	$(\beta\hbar/2m)(\boldsymbol{\sigma}\cdot\nabla S_4)$	$-(i\hbar/2m)(\boldsymbol{\sigma}\cdot\nabla\theta^P)$
"Spin-Orbit"			$-(i/m)(\boldsymbol{\sigma}\cdot\mathbf{P}\times\mathbf{p})$		
"Spin-Momentum"				$(i\beta/m)(S_4\boldsymbol{\sigma}\cdot\mathbf{p})$	
2nd-Order Terms					
"Schrödinger"	$-\dfrac{\beta}{2m^2c}(\theta p^2)$				
"Divergence"	$\dfrac{\beta\hbar^2}{8m^2c}(\nabla^2\theta)$	$-\dfrac{\hbar^2}{8m^2c}\left\{\nabla\cdot\left(-i\nabla\theta_4+\dfrac{1}{c}\dfrac{\partial\boldsymbol{\theta}}{\partial t}\right)\right\}$	$-\dfrac{\beta\hbar^2}{8m^2c^2}\left(\nabla\cdot\dfrac{\partial\mathbf{P}}{\partial t}\right)$	$\dfrac{\hbar}{4m^2c}(\nabla\cdot\mathbf{S}\times\mathbf{p})$	
"Darwin"	$\dfrac{\beta i\hbar}{2m^2c}(\nabla\theta\cdot\mathbf{p})$		$-\dfrac{\beta\hbar}{4m^2c}\left(\nabla\times\mathbf{M}+\dfrac{i}{c}\dfrac{\partial\mathbf{P}}{\partial t}\right)\cdot\mathbf{p}$		
"Spin-Field"		$-\dfrac{i\hbar^2}{8m^2c^2}\left(\boldsymbol{\sigma}\cdot\nabla\times\dfrac{\partial\boldsymbol{\theta}}{\partial t}\right)$ $-\dfrac{\hbar}{4m^2c}\left\{\boldsymbol{\sigma}\cdot\left(-i\nabla\theta_4+\dfrac{1}{c}\dfrac{\partial\boldsymbol{\theta}}{\partial t}\right)\times\boldsymbol{\theta}\right\}$	$\dfrac{\beta}{8m^2c}\Big\{-\hbar^2\boldsymbol{\sigma}\cdot\nabla\times\nabla\times\mathbf{M}+2i\hbar(\boldsymbol{\sigma}\cdot\nabla)(\mathbf{M}\cdot\mathbf{p})$ $+\hbar^2(\boldsymbol{\sigma}\cdot\nabla)(\nabla\cdot\mathbf{M})-i\hbar^2\boldsymbol{\sigma}\cdot\nabla\times\dfrac{\partial\mathbf{P}}{\partial t}-\dfrac{2\beta\hbar}{c}$ $\times\boldsymbol{\sigma}\cdot\mathbf{P}\times\dfrac{\partial\mathbf{P}}{\partial t}-2\beta i\hbar(\boldsymbol{\sigma}\cdot\nabla)(\mathbf{M}\cdot\mathbf{P})+2\beta i\hbar$ $\times(\mathbf{M}\cdot\nabla)(\boldsymbol{\sigma}\cdot\mathbf{P})+2\beta i\hbar\boldsymbol{\sigma}\cdot\mathbf{P}\times(\nabla\times\mathbf{M})$ $+4(\boldsymbol{\sigma}\cdot\mathbf{P})(\mathbf{M}\cdot\mathbf{p})\Big\}$	$(\hbar^2/8m^2c)\{(\boldsymbol{\sigma}\cdot\nabla)(\nabla\cdot\mathbf{S})-\boldsymbol{\sigma}\cdot\nabla\times(\nabla\times\mathbf{S})$ $-(i/c)(\boldsymbol{\sigma}\cdot\nabla\partial S_4/\partial t)\}-(i\hbar/4m^2c)(\boldsymbol{\sigma}\cdot\mathbf{S}\times\nabla S_4)$	$-\dfrac{\beta\hbar^2}{8m^2c^2}\left(\boldsymbol{\sigma}\cdot\nabla\dfrac{\partial\theta^P}{\partial t}\right)$
"Spin-Orbit"	$-\dfrac{\beta\hbar}{4m^2c}(\boldsymbol{\sigma}\cdot\nabla\theta\times\mathbf{p})$	$-\dfrac{\hbar}{4m^2c}\left\{\boldsymbol{\sigma}\cdot\left(-i\nabla\theta_4+\dfrac{1}{c}\dfrac{\partial\boldsymbol{\theta}}{\partial t}\right)\times\mathbf{p}\right\}$		$(1/2m^2c^2)\{\boldsymbol{\sigma}\cdot(\mathbf{S}\times\mathbf{p})\times\mathbf{p}\}$ $+(i\hbar/4m^2c)\{\boldsymbol{\sigma}\cdot(\nabla\times\mathbf{S})\times\mathbf{p}+\boldsymbol{\sigma}\cdot\nabla\times(\mathbf{S}\times\mathbf{p})\}$	
"Spin-Momentum"			$\dfrac{\beta}{4m^2c}(i\hbar\nabla\cdot\mathbf{M}-2\mathbf{M}\cdot\mathbf{p})(\boldsymbol{\sigma}\cdot\mathbf{p})$		$-\dfrac{\beta i\hbar}{4m^2c^2}\left(\dfrac{\partial\theta^P}{\partial t}\boldsymbol{\sigma}\cdot\mathbf{p}\right)$

TABLE II. Relativistic correction terms associated with mixed-type interactions.

	Scalar-Vector	Scalar-Tensor	Tensor-Pseudovector	Tensor-Pseudoscalar	Pseudovector-Pseudoscalar
1st-Order Terms "Spin-Field"				$-(\beta/m)(\theta^P\boldsymbol{\sigma}\cdot\mathbf{P})$	
2nd-Order Terms "Schrödinger"	$(\beta i\hbar/2m^2c)(\nabla\theta\cdot\boldsymbol{\theta})-(\beta/2m^2c)(\theta\boldsymbol{\theta}\cdot\boldsymbol{\theta})$	$(i\hbar/4m^2c)(\nabla\theta\cdot\mathbf{P})+(\beta/2m^2c)(\theta P^2)$	$-(\beta\hbar/4m^2c)(2\mathbf{M}\cdot\nabla S_4-\mathbf{P}\cdot\nabla\times\mathbf{S}+\mathbf{S}\cdot\nabla\times\mathbf{P})$	$(i\hbar/2m^2c)(\mathbf{M}\cdot\nabla\theta^P)+(\beta/m^2c)(\theta^P\mathbf{M}\cdot\mathbf{P})$	$\dfrac{\beta\hbar}{4m^2c^2}\left(S_4\dfrac{\partial\theta^P}{\partial t}-\theta^P\dfrac{\partial S_4}{\partial t}\right)$
"Divergence"	$(\beta i\hbar/2m^2c)(\theta\nabla\cdot\boldsymbol{\theta})$	$(i\hbar/2m^2c)(\theta\nabla\cdot\mathbf{P})$	$-(\beta\hbar/4m^2c)\{\nabla\cdot(\mathbf{P}\times\mathbf{S})+2S_4\nabla\cdot\mathbf{M}\}$	$(i\hbar/4m^2c)(\nabla\cdot\mathbf{M})\theta^P$	$-(\beta i\hbar/4m^2c)(\theta^P\nabla\cdot\mathbf{S})$
"Darwin"	$-(\beta/m^2c)(\theta\boldsymbol{\theta}\cdot\mathbf{p})$		$-(i\beta/m^2c)\{(\mathbf{P}\times\mathbf{S})+S_4\mathbf{M}\}\cdot\mathbf{p}$		
"Spin-Field"	$-(\beta\hbar/4m^2c)\times(\boldsymbol{\sigma}\cdot\nabla\theta\times\boldsymbol{\theta}+2\theta\boldsymbol{\sigma}\cdot\nabla\times\boldsymbol{\theta})$	$-(\hbar/2m^2c)\times(\boldsymbol{\sigma}\cdot\nabla\theta\times\mathbf{P}+\theta\boldsymbol{\sigma}\cdot\nabla\times\mathbf{P})$	$-\dfrac{\beta i\hbar}{4m^2c}\Big\{\boldsymbol{\sigma}\cdot\nabla S_4\times\mathbf{M}+2S_4\boldsymbol{\sigma}\cdot\nabla\times\mathbf{M}+(\boldsymbol{\sigma}\cdot\mathbf{P})\times(\nabla\cdot\mathbf{S})+\boldsymbol{\sigma}\cdot\nabla\times(\mathbf{P}\times\mathbf{S})+\boldsymbol{\sigma}\cdot(\mathbf{S}\times\nabla)\times\mathbf{P}-\dfrac{i}{c}\times\dfrac{\partial}{\partial t}\boldsymbol{\sigma}\cdot(S_4\mathbf{P})\Big\}+\dfrac{\beta}{2m^2c}\{S_4\boldsymbol{\sigma}\cdot\mathbf{P}\times\mathbf{S}+S_4^2\boldsymbol{\sigma}\cdot\mathbf{M}-\beta\boldsymbol{\sigma}\cdot\mathbf{P}\times(\mathbf{P}\times\mathbf{S})\}$	$-(\hbar/4m^2c)(\theta^P\boldsymbol{\sigma}\cdot\nabla\times\mathbf{M}+\boldsymbol{\sigma}\cdot\nabla\theta^P\times\mathbf{M})+(\beta/2m^2c)(\theta^{P2}\boldsymbol{\sigma}\cdot\mathbf{M})$	$-(\beta\hbar/4m^2c)\times\{\boldsymbol{\sigma}\cdot(S\times\nabla\theta^P-\theta^P\nabla\times\mathbf{S})\}$
"Spin-Orbit"		$(i/m^2c)(\theta\boldsymbol{\sigma}\cdot\mathbf{P}\times\mathbf{p})$	$(\beta/2m^2c)(S_4\boldsymbol{\sigma}\cdot\mathbf{M}\times\mathbf{p})$		$-(i\beta/2m^2c)(\theta^P\boldsymbol{\sigma}\cdot\mathbf{S}\times\mathbf{p})$
	Scalar-Pseudovector	Scalar-Pseudoscalar	Vector-Tensor	Vector-Pseudovector	Vector-Tensor-Pseudovector
"Schrödinger"	$(\beta/2m^2c)(\theta S_4^2)$	$(\beta/2m^2c)(\theta^P)^2\theta$	$\dfrac{\beta i\hbar}{4m^2c}\left\{\mathbf{P}\cdot\left(-i\nabla\theta_4+\dfrac{1}{c}\dfrac{\partial\boldsymbol{\theta}}{\partial t}\right)\right\}-\dfrac{\beta\hbar}{4m^2c}\left\{\nabla\times\mathbf{M}+\dfrac{i}{c}\dfrac{\partial\mathbf{P}}{\partial t}\right\}\cdot\boldsymbol{\theta}$		$-(i\beta/m^2c)\times\{(\mathbf{P}\times\mathbf{S})+S_4\mathbf{M}\}\cdot\boldsymbol{\theta}$
"Divergence"				$(\hbar/4m^2c)(\nabla\cdot\mathbf{S}\times\boldsymbol{\theta})$	
"Spin-Field"	$-(\beta\hbar/2m^2c)\times(\theta\boldsymbol{\sigma}\cdot\nabla S_4+S_4\boldsymbol{\sigma}\cdot\nabla\theta)$	$(i\hbar/4m^2c)\times(\theta^P\boldsymbol{\sigma}\cdot\nabla\theta+2\theta\boldsymbol{\sigma}\cdot\nabla\theta^P)$	$-(i/m)(\boldsymbol{\sigma}\cdot\mathbf{P}\times\boldsymbol{\theta})$[1st order] $\dfrac{\beta i\hbar}{4m^2c}\{(\nabla\cdot\mathbf{M})(\boldsymbol{\sigma}\cdot\boldsymbol{\theta})+(\boldsymbol{\sigma}\cdot\nabla)(\mathbf{M}\cdot\boldsymbol{\theta})+(\mathbf{M}\cdot\nabla)\times(\boldsymbol{\sigma}\cdot\boldsymbol{\theta})\}-\dfrac{\beta}{2m^2c}(\mathbf{M}\cdot\boldsymbol{\theta})(\boldsymbol{\sigma}\cdot\boldsymbol{\theta})$	$(i\beta/m)(S_4\boldsymbol{\sigma}\cdot\boldsymbol{\theta})$[1st order] $\dfrac{1}{2m^2c}\{\boldsymbol{\sigma}\cdot(\mathbf{S}\times\boldsymbol{\theta})\times\boldsymbol{\theta}\}+(i\hbar/4m^2c)\times\{\boldsymbol{\sigma}\cdot(\nabla\times\mathbf{S})\times\boldsymbol{\theta}+\boldsymbol{\sigma}\cdot\nabla\times(\mathbf{S}\times\boldsymbol{\theta})-2\boldsymbol{\sigma}\cdot(\mathbf{S}\times\nabla)\times\boldsymbol{\theta}\}$	$(\beta/2m^2c)(S_4\boldsymbol{\sigma}\cdot\mathbf{M}\times\boldsymbol{\theta})$
"Spin-Orbit"				$\dfrac{1}{2m^2c}\{\boldsymbol{\sigma}\cdot(\mathbf{S}\times\boldsymbol{\theta})\times\mathbf{p}-\boldsymbol{\sigma}\cdot\boldsymbol{\theta}\times(\mathbf{S}\times\mathbf{p})\}$	
"Spin-Momentum"	$-(i\beta/m^2c)(\theta S_4\boldsymbol{\sigma}\cdot\mathbf{p})$		$-(\beta/2m^2c)(\mathbf{M}\cdot\boldsymbol{\theta})(\boldsymbol{\sigma}\cdot\mathbf{p})$		
	Scalar-Tensor-Pseudoscalar	Scalar-Vector-Tensor	Scalar-Vector-Pseudovector	Vector Pseudoscalar	Vector-Pseudovector-Pseudoscalar
"Spin-Field"	$(\beta/m^2c)(\theta\theta^P\boldsymbol{\sigma}\cdot\mathbf{P})$	$(i/m^2c)(\theta\boldsymbol{\sigma}\cdot\mathbf{P}\times\boldsymbol{\theta})$	$-(i\beta/m^2c)(\theta S_4\boldsymbol{\sigma}\cdot\boldsymbol{\theta})$	$\dfrac{\beta i\hbar}{4m^2c}\left\{\boldsymbol{\sigma}\cdot\left(-i\nabla\theta_4+\dfrac{1}{c}\dfrac{\partial\boldsymbol{\theta}}{\partial t}\right)\theta^P-\boldsymbol{\sigma}\cdot\boldsymbol{\theta}\dfrac{1}{c}\dfrac{\partial\theta^P}{\partial t}\right\}$	$-(i\beta/2m^2c)(\theta^P\boldsymbol{\sigma}\cdot\mathbf{S}\times\boldsymbol{\theta})$

TABLE III. Contributions of relativistic correction terms in Eq. (10) to hydrogenic energy levels.

Energy level correction	Term responsible
Kinetic energy correction	
$-\frac{Z^4Rch\alpha^2}{n^4}\left(\frac{n}{l+\frac{1}{2}}-\frac{3}{4}\right)$	$-\frac{p^4}{8m^3c^2}$
Darwin correction for S states	
$\frac{Z^4Rch\alpha^2}{n^3}$	$-\frac{e\hbar^2}{8m^2c^2}\nabla\cdot\mathbf{E}$
Thomas correction for states of $l\neq 0$	
$\left(\frac{Z^4Rch\alpha^2}{n^3}\right)\left(\frac{1}{(2l+1)(l+1)}\right)$ for $j=l+\frac{1}{2}$	$-\frac{e\hbar}{4m^2c^2}\boldsymbol{\sigma}\cdot\mathbf{E}\times\mathbf{p}$
$-\left(\frac{Z^4Rch\alpha^2}{n^3}\right)\left(\frac{1}{l(2l+1)}\right)$ for $j=l-\frac{1}{2}$	
$(8e_1/e)\times$Darwin correction	
$8\frac{e_1}{e}\left(\frac{Z^4Rch\alpha^2}{n^3}\right)$	$-e_1\left(\frac{\hbar}{mc}\right)^2\nabla\cdot\mathbf{E}$
$2(\mu_e-1)\times$Darwin correction	
$2(\mu_e-1)\left(\frac{Z^4Rch\alpha^2}{n^3}\right)$	$-(\mu_e-1)\frac{e\hbar^2}{4m^2c^2}\nabla\cdot\mathbf{E}$
$2(\mu_e-1)\times$Thomas correction	
$2(\mu_e-1)\left(\frac{Z^4Rch\alpha^2}{n^3}\right)\left(\frac{1}{(2l+1)(l+1)}\right)$ for $j=l+\frac{1}{2}$	$-(\mu_e-1)\frac{e\hbar}{2m^2c^2}\boldsymbol{\sigma}\cdot\mathbf{E}\times\mathbf{p}$
$-2(\mu_e-1)\left(\frac{Z^4Rch\alpha^2}{n^3}\right)\left(\frac{1}{l(2l+1)}\right)$ for $j=l-\frac{1}{2}$	

The quantum-electrodynamic factors are given by[9,10]

$$(\mu_e-1)=\alpha/2\pi, \tag{11}$$

$$e_1=e\frac{\alpha}{3\pi}\left\{\ln\frac{mc^2}{k_0(n,0)}-\ln 2-\frac{1}{5}+\frac{11}{24}+3\pi Z\alpha\left[1+\frac{11}{128}-\frac{1}{2}\ln 2+\frac{5}{192}\right]\right\}, \tag{12}$$

where $k_0(n,l)$ is the average atomic excitation energy for the level nl.

The contribution of the anomalous electron magnetic moment to the displacement of states of any l is easily calculated by using Table III and Eq. (11). The result is

$$E(n,l)=-\frac{2(\mu_e-1)Z^4Rch\alpha^2}{n^3}\left\{\frac{1+k}{(l+1)(2l+1)}\right\}, \tag{13}$$

where $k=-l-1$ for $j=l+\frac{1}{2}$ and $k=l$ for $j=l-\frac{1}{2}$. This expression agrees with that obtained by Breit[11] from an approximate evaluation. Upon inserting the 1951 constants of Bearden and Watts[12] into Eq. (13), the separation of the $2S$, $2P_{\frac{1}{2}}$ states of hydrogen due to the anomalous magnetic moment of the electron is 67.77 Mc/sec. Also the doublet separation of the $2P_{\frac{3}{2}}$, $2P_{\frac{1}{2}}$ states is increased by 25.41 Mc/sec from the value predicted by Dirac theory. These numerical values are in good agreement with the latest values quoted by Lamb.[13]

The contribution of e_1 to the displacement of the S states is also easily obtained by using Table III and Eq. (12):

$$E(n,0)=\frac{8Z^4Rch\alpha^3}{3\pi n^3}\left\{\ln\frac{mc^2}{k_0(n,0)}-\ln 2-\frac{1}{5}+\frac{11}{24}+3\pi Z\alpha\left[1+\frac{11}{128}-\frac{1}{2}\ln 2+\frac{5}{192}\right]\right\}. \tag{14}$$

[9] J. Schwinger, Phys. Rev. **73**, 416 (1948).
[10] Karplus, Klein, and Schwinger, Phys. Rev. **86**, 288 (1952); see also, Bethe, Brown, and Stehn, Phys. Rev. **77**, 370 (1950); J. B. French and V. E. Weisskopf, Phys. Rev. **75**, 1240 (1949).
[11] G. Breit, Phys. Rev. **72**, 984 (1947).
[12] J. A. Bearden and H. M. Watts, Phys. Rev. **81**, 73 (1951).
[13] W. E. Lamb, Jr., Phys. Rev. **85**, 259 (1952).

This agrees with the expression obtained by other authors.[10] Using the numerical value for $k_0(2, 0)$ as calculated by Bethe *et al.*,[10] the $2S$ state of hydrogen is shifted by 994.82 Mc/sec. Table III does not, however, predict the known additional displacement of the $2P_{\frac{1}{2}}$ state of hydrogen by 4.00 Mc/sec.

It is hoped that in future publications we may be able to demonstrate the utility of Tables I and II in beta-decay and meson theories.

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The Diffraction of Strong Shock Waves*

WAYLAND GRIFFITH AND DAVID E. BRICKL
Palmer Physical Laboratory, Princeton University, Princeton, New Jersey
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The pseudostationary flow field resulting from diffraction of a strong shock in air over a convex corner has been investigated with the shock tube and interferometer. Only when the shock has turned nearly 180°, is it observed to become vanishingly weak. A potential flow theory for part of the field is inadequate to predict the extent of a Prandtl-Meyer expansion around the corner. Viscous forces produce a boundary layer which causes the flow to separate at the corner with significant changes in the density field as a result.

INTRODUCTION

THE diffraction of a plane shock wave by a convex corner is one example of the general problem of interacting shock and rarefaction waves. Since no unit of length is given for the flow, the pattern remains similar to itself in time, i.e., is pseudostationary, and may be described by the variables x/t and y/t. Like the complementary problem of Mach reflection, the nonlinear nature of the fluid mechanical equations present such formidable difficulties that no complete solution of the problem has been found. Jones, Martin, and Thornhill,[1] in a recent paper, have shown that a part of the flow field may be readily obtained when the incident shock is strong enough for the flow behind it to be supersonic. Previously, Lighthill[2] linearized the flow equations by retaining only terms of first order in the angle of the corner and calculated the pressure on the wall and the shape of the diffracted shock. This solution applies to shocks of any strength as long as the angle is sufficiently small. Previous experiments in this shock tube[3] established the validity of Lighthill's solution. Keller[4] has computed the density field for an acoustic wave rounding a 90° corner which is in good agreement with experimental data obtained for very weak shocks by White. The purpose of this paper will be to present the results of experiments with strong shocks (such that the flow behind them is supersonic) and any corner angle for comparison with the theory of reference 1 and as a guide to further theoretical work.

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[1] Jones, Martin, and Thornhill, Proc. Roy. Soc. (London) **A209**, 238 (1951).
[2] M. J. Lighthill, Proc. Roy. Soc. (London) **A198**, 454 (1949).
[3] Fletcher, Taub, and Bleakney, Revs. Modern Phys. **23**, 271 (1951).
[4] Mathematics Research Group Rept. EM-43, New York University, unpublished.

THEORETICAL CONSIDERATIONS

Consider a shock wave of velocity V advancing into still air in which the velocity of sound is a_1. According to one-dimensional shock theory, the flow velocity u_2 behind the shock is

$$u_2/a_1 = 2(M - 1/M)/(\gamma+1), \tag{1}$$

where M is the shock Mach number V/a_1, and the local velocity of sound a_2 is

$$a_2/a_1 = \{[2\gamma M^2 - \gamma + 1][2/M^2 + \gamma - 1]\}^{\frac{1}{2}}/(\gamma+1). \tag{2}$$

The flow behind the shock becomes supersonic when $u_2/a_2 = 1$ or for air with $\gamma = 1.4$ when $M = 2.068$.

When the incident shock passes the corner, a rarefaction wave advances back through the moving air with the velocity a_2, but the air itself is carried along with the velocity u_2. As a result, only the region below OAE in Fig. 1 will know of the existence of the corner. The strength of the diffracted shock EF varies in some unknown way so that the boundary conditions are difficult to establish.

A part of the flow can still be found, however. Jones, Martin, and Thornhill show that a Prandtl-Meyer expansion at the corner may be terminated in either one of two ways: (a) a uniform flow parallel to the wall, or (b) uniform flow parallel to a line on the other side of which some other flow maintains equal pressure, i.e., a slip stream. They predict that transition to the second case will occur when the Prandtl-Meyer flow expands to the ambient pressure p_1. The region below the slip stream would then be at rest at pressure p_1.

The radial extent of the Prandtl-Meyer flow may be found from the characteristics in the x/t, y/t system. One set consists of course of the radial lines through O. The other family must be found numerically. The