PHYSICAL REVIEW VOLUME 89, NUMBER 2 JANUARY 15, 1953

Spin and Angular Momentum in General Relativity*

PETER G. BERGMANN AND ROBB THOMSON
Department of Physics, Syracuse University, Syracuse, New York
(Received October 6, 1952)

In the general theory of relativity, the group of coordinate transformations gives rise to four point-to-point conservation laws, which are usually identified with energy and linear momentum. In the presence of a semi-classical Dirac field, it is convenient to introduce at each point of space-time an arbitrary set of four orthonormal vectors (quadrupeds, "beine") and to consider the group of "bein" transformations, which then play the role of local, nonholonomic lorentz transformations. A search for the corresponding conservation laws leads to terms that have the form of a spin angular momentum and which, in order to be conserved, must be supplemented by terms representing the orbital angular momentum. The technique of the so-called superpotentials has enabled us to introduce, in addition to the canonical stress-energy, a "contravariant" stress-energy which contains the usual symmetric Dirac and Maxwell terms and also asymmetric, purely gravitational terms. It is this set of expressions which enters into the orbital angular momentum. The techniques presented here are applicable to more general covariant theories, provided the gravitational field is represented by a metric tensor.

1. INTRODUCTION

THE general definition of the angular momentum of a field has been discussed within the framework of Lorentz invariance of the theory by several authors.[1–3] In these and other previous papers the conservation theorem for angular momentum is a direct result of the fact that the Lagrangian is a scalar density with respect to Lorentz transformations.

We have been interested for several years in the possibilities of constructing a completely covariant quantum theory containing the gravitational field.[4–7] One of the aspects of this investigation is, of course, the role of the angular momentum in such a theory. The spin is a fundamental property of a particle in the usual Lorentz-invariant theory; it would be surprising indeed if this property were suddenly lost simply because the theory is made covariant with respect to more general coordinate transformations.

Yet it seems that in the theory of general relativity the concept of angular momentum is not as natural as it has been in special relativity. In general relativity the only identity one obtains from the general covariance of the Lagrangian is the conservation of the energy and momentum. In Lorentz-invariant theories, there are at least two separate types of transformations (each dependent on a set of *parameters*) which lead to two conservation laws. The invariance with respect to linear displacement yields the conservation of linear momentum and energy, while the invariance with respect to the homogeneous Lorentz transformation group yields the conservation of angular momentum. Now in the theory of general relativity, the most important invariance property is the invariance with respect to general (curvilinear) coordinate transformations. This is a group of transformations which depends on a set of four arbitrary *functions*, one for each coordinate. The conservation law obtained as a result of this invariance, as stated earlier, is none other than the conservation of linear momentum and energy. It is true that the theory is also invariant with respect to linear displacements, a subgroup of the group of all coordinate transformations. This invariance leads again to the conservation of linear momentum and energy. One might think that by considering other subgroups, say the Lorentz group, other conservation laws might be found. Such is not the case. Noether[8] has shown that in general, conservation laws obtained in such a way are simply restatements or even special cases of the general conservation law resulting from the invariance with respect to curvilinear transformations. Thus it seems that the angular momentum law will not be obtained by appealing to an invariance argument in the usual theory.

Our program is not to attempt to derive the angular momentum from an invariance argument, but rather to construct a quantity from the energy-momentum "tensor" itself which has the properties of the usual angular momentum. There are several types of quantities which have been used in the past as energy-momentum tensors. Rosenfeld[1] considers only the symmetric matter tensor which describes matter in the gravitational field equations. Others[2,3] start with the canonical "tensor" of a particular field and add terms which are obtained from the Lorentz covariant angular momentum law, to make the result symmetric. These

* Some of this work was performed under the sponsorship of the ONR. One of us (R. T.) gratefully acknowledges a Syracuse University Predoctoral Fellowship, whose tenure coincided with the completion of the major phase of this research. (This work is being submitted in somewhat more extended form by R. T. to the Graduate School of Syracuse University, in partial fulfillment of the requirements for the doctor's degree.)

[1] L. Rosenfeld, Acad. roy. Belg. Classe sci. Mém. **6**, 30 (1940).
[2] F. J. Belinfante, Physica **7**, 887 (1939).
[3] W. Pauli, Revs. Modern Phys. **13**, 203 (1941).
[4] P. G. Bergmann, Phys. Rev. **75**, 680 (1949).
[5] P. G. Bergmann and J. H. M. Brunings, Revs. Modern Phys. **21**, 480 (1949).
[6] Bergmann, Penfield, Schiller, and Zatkis, Phys. Rev. **80**, 81 (1950).
[7] J. L. Anderson and P. G. Bergmann, Phys. Rev. **83**, 1018 (1951).
[8] E. Noether, Nachr. kgl. Ges. Wiss. Göttingen Math.-phys. Klasse 235 (1918).

two methods lead to the same result when applied to a Lorentz-invariant Lagrangian. However, this procedure will have to be modified if the gravitational terms are included in the Lagrangian. First, the symmetric matter tensor, as is well known, is not actually conserved; its *covariant* divergence is zero, and purely gravitational terms must be added to obtain expressions whose *ordinary* divergence vanishes. The essential feature of the procedure of this paper is to find the proper energy momentum "tensor" and then to construct from it a "superpotential" for the angular momentum of the total field. The superpotential will have certain symmetry properties which guarantee the conservation of the angular momentum which is derived from it. We shall show the results of applying this method to the gravitation-electron field.

We have stated that there is no invariance property of the usual theory which leads naturally to the conservation of angular momentum. However, we can introduce into the theory of gravitation one further invariance property which on first inspection seems to point the way toward an angular momentum conservation law. This is "bein" invariance. It is possible to describe the gravitational field in terms of a system of "quadrupeds" or "beine."[9] One introduces at each point of Riemannian space four orthonormal vectors whose components are differentiable functions of the coordinates. Then the metric tensor components may be expressed in terms of the 16 components of these vectors, and the "beine" themselves act as field variables for the gravitational field. The theory is still obviously invariant with respect to coordinate transformations, since the "beine" transform as vectors, but, in addition, it is possible to transform from one system of vectors to another set by rotating the quadruped located at each point of space-time by an amount that may vary with position. Rotation, of course, usually generates angular momentum, and one might reasonably hope that such is the case here. Detailed examination shows that while this procedure does not lead to a complete conservation law, the identity obtained yields the contribution of the spin of the electron field to the total angular momentum.

2. CONSERVATION LAWS

For the derivation of the conservation law of angular momentum, it is assumed traditionally that there is available a symmetric energy-stress tensor that satisfies a conservation law of its own. Hence, we find in the literature a distinction made between the canonical stress tensor which is obtained directly by the requirement that the theory be invariant with respect to rigid displacement along each of the four coordinate axes, and a symmetric stress tensor $\Theta^{\mu\nu}$. The symmetric stress tensor is then made the point of departure for a derivation of the expression for the angular momentum. The symmetric tensor $\Theta^{\mu\nu}$ has the further virtue of being gauge invariant, while the canonical tensor is not. The canonical tensor is written

$$t_\mu{}^\nu = \frac{\partial L}{\partial \zeta_{,\nu}} \zeta_{,\mu} - \delta_\mu{}^\nu L, \tag{2.1}$$

where ζ is short for the totality of all field variables, and L is the Lagrangian density.

In a theory that is generally covariant, the expression (2.1) can be introduced in a natural manner, but it is impossible to introduce, by a general procedure, a symmetric tensor $\Theta^{\mu\nu}$ whose divergence vanishes. What we shall do in this paper is to derive a quasi-symmetric expression consisting of two parts: one that depends exclusively on gravitational terms and which is neither covariant nor symmetric; the other, "the matter tensor," which is covariant, gauge invariant, spin invariant, and symmetric. This result is independent of the special nature of the theory in so far as it does not concern the gravitational field. But we cannot obtain our results without assuming explicitly the existence of a symmetric gravitational tensor $g_{\mu\nu}$ which, in the absence of other fields, satisfies Einstein's field equations $G^{\mu\nu}=0$. We shall, therefore, assume that the Lagrangian of our theory has the form

$$L = \mathfrak{G}' + \mathfrak{M}, \quad \mathfrak{G}' = -(c^4/16\pi\kappa)(-g)^{\frac{1}{2}}R, \tag{2.2}$$

where R is the curvature scalar, c the velocity of light and κ the gravitational constant.

In what follows, we shall establish a relationship between the canonical and the quasi-symmetric expressions by means of the so-called strong conservation laws.[4] These laws are actually identities and hold whether the field equations are satisfied or not, simply because of the covariance of the theory with respect to general coordinate transformations. They will, therefore, hold both for the Lagrangian of the whole theory (2.2) and also for the gravitational term $\mathfrak{G}'$ alone. We shall make use of the latter fact. The field equations,

$$\delta L/\delta \rho = 0, \tag{2.3}$$

contain two types of terms, originating in the two terms of the Lagrangian (2.2),

$$\delta L/\delta\rho = \delta\mathfrak{G}'/\delta g_{\mu\nu} + \delta\mathfrak{M}/\delta\rho, \tag{2.4}$$

which we shall designate for short as the gravitational and the matter terms, respectively. If we write down the strong conservation laws for the gravitational terms only, we obtain

$$T_{G\mu}{}^\nu{}_{,\nu} = 0, \quad T_{G\mu}{}^\nu = t_{G\mu}{}^\nu - F_{G\mu}{}^\nu \delta\mathfrak{G}/\delta\zeta, \tag{2.5}$$

where t_G is the canonical stress "tensor" obtained from the gravitational term of the Lagrangian alone. In obtaining this expression, we must observe one precaution, though. The expression $\mathfrak{G}'$ contains second

[9] L. Eisenhart, *An Introduction to Differential Geometry* (Princeton University Press, Princeton, 1947).

derivatives of the field variables, while the canonical stress "tensor" is defined ordinarily only for Lagrangians depending on zeroth and first derivatives of the field variables. To meet this difficulty we introduce, as is customary, the gravitational Lagrangian $\mathfrak{G}$ which differs from $\mathfrak{G}'$ only by a divergence but is free of second derivatives,

$$\mathfrak{G}=\mathfrak{G}'-Q^{\rho}{}_{,\rho}. \tag{2.6}$$

The variational derivative of the last term on the right vanishes, and we have, therefore,

$$\delta\mathfrak{G}/\delta\rho=\delta\mathfrak{G}'/\delta\rho. \tag{2.7}$$

The canonical stress tensor t_G in Eq. (2.5) is then understood as having been formed from the Lagrangian $\mathfrak{G}$. The $F_{G\mu}{}^{\nu}$ in Eq. (2.5) is defined by means of the transformation properties of the field variables. In an infinitesimal transformation of the field variables, we set

$$\delta\zeta=F_{\nu}{}^{\mu}\xi^{\nu}{}_{,\mu}, \tag{2.8}$$

where the ξ^{ν} are the infinitesimal variations of the coordinates. The subscript G in (2.5) obviously refers to the gravitational field variables.

We may write

$$\begin{aligned}T_{G\mu}{}^{\nu}&=t_{G\mu}{}^{\nu}-F_{G\mu}{}^{\nu}\delta G/\delta\zeta\\&=t_{G\mu}{}^{\nu}+2g_{\lambda\mu}\delta G/\delta g_{\lambda\nu}\\&=t_{G\mu}{}^{\nu}-(-g)^{\frac{1}{2}}G_{\mu}{}^{\nu}.\end{aligned} \tag{2.9}$$

$G_{\mu}{}^{\nu}$ stands for the Einstein tensor, with one index raised,

$$G_{\mu}{}^{\nu}=R_{\mu}{}^{\nu}-\tfrac{1}{2}\delta_{\mu}{}^{\nu}R. \tag{2.10}$$

Because of the set of field equations ($\delta L/\delta g_{\mu\nu}=0$), Eq. (2.9) may also be written in the form

$$T_{G\mu}{}^{\nu}=t_{G\mu}{}^{\nu}+\mathcal{T}_{\mu}{}^{\nu}, \tag{2.11}$$

where $\mathcal{T}_{\mu}{}^{\nu}$ is the matter tensor density defined (as usual) as

$$\mathcal{T}^{\mu\nu}=-2\delta\mathfrak{M}/\delta g_{\mu\nu}. \tag{2.12}$$

It is now clear from Eq. (2.11) that $T_{G\nu}{}^{\mu}$ may be interpreted as the total energy-momentum of all fields present, and is the combination of the energy-momentum tensor density of the matter fields with the canonical energy-momentum of the gravitational field.

Because of Eq. (2.5), it is possible to introduce a set of "superpotentials" related to the $T_{G\mu}{}^{\nu}$ in the following way:

$$T_{G\nu}{}^{\mu}=-U_G{}^{[\lambda\mu]}{}_{\nu,\lambda}. \tag{2.13}$$

The brackets indicate antisymmetry in the superscripts. The actual expression for the gravitational superpotentials was obtained by Freud.[10] In the case of a field whose Lagrangian is a scalar density, such as the matter fields, it has the form[11]:

$$U_M{}^{[\lambda\mu]}{}_{\nu}=-(\partial\mathfrak{M}/\partial\rho_{,\lambda})F_{M\nu}{}^{\mu}. \tag{2.14}$$

We shall have no occasion to use the explicit expression for the gravitational superpotentials. It should perhaps be emphasized that there are superpotentials for each of the terms introduced into the Lagrangian. Here we have introduced only the gravitational superpotentials. Later, when a definite expression for the matter tensor is needed, superpotentials associated with the other terms in the Lagrangian will be introduced.

In order to obtain a conservation law for a "contravariant" energy-momentum tensor, we note that it is the antisymmetry property of the superpotentials which leads to a conservation law. We can, therefore, raise and lower the index not involved in the antisymmetry with the help of the metric tensor to obtain the new expression

$$U_G{}^{[\lambda\mu]\nu}=U_G{}^{[\lambda\mu]}{}_{\rho}g^{\rho\nu}. \tag{2.15}$$

The energy-momentum derived from this set of modified superpotentials will also be conserved and will be taken to be the "contravariant" form of the energy-momentum. The conservation law now reads

$$\begin{aligned}\{\mathcal{T}^{\mu\nu}+t_{G\sigma}{}^{\mu}g^{\sigma\nu}+U_G{}^{[\lambda\mu]}{}_{\sigma}g^{\sigma\nu}{}_{,\lambda}\}_{,\mu}\\=\{\mathcal{T}^{\mu\nu}+\tau_G{}^{\mu\nu}\}_{,\mu}=U_G{}^{[\lambda\mu]\nu}{}_{,\lambda\mu}=0.\end{aligned} \tag{2.16}$$

Here, we have grouped two terms together as $\tau_G{}^{\mu\nu}$; this expression represents the gravitational terms in the "contravariant" energy-momentum. Note, however, that $\tau_G{}^{\mu\nu}$ is not symmetric in its superscripts (though $\mathcal{T}^{\mu\nu}$ is) and that

$$\{\mathcal{T}^{\mu\nu}+\tau_G{}^{\mu\nu}\}_{,\mu}=0,\quad \{\mathcal{T}^{\mu\nu}+\tau_G{}^{\mu\nu}\}_{,\nu}\neq 0. \tag{2.17}$$

In special relativity, the energy-momentum may be symmetrized so that both lines of Eq. (2.17) turn into equalities. Here the energy-momentum will be left unsymmetric as will be discussed below.

Using the $U_G{}^{[\lambda\mu]\sigma}$, we can now construct a superpotential of a quantity which will reduce to the angular momentum of special relativity in the limit of a flat metric. We define

$$W^{[\lambda\mu][\nu\sigma]}=U_G{}^{[\lambda\mu]\nu}x^{\sigma}-U_G{}^{[\lambda\mu]\sigma}x^{\nu} \tag{2.18}$$

and define the angular momentum as

$$\Omega^{\mu[\nu\sigma]}=W^{[\lambda\mu][\nu\sigma]}{}_{,\lambda}. \tag{2.19}$$

Then, as before, because of the antisymmetry of $W^{[\lambda\mu][\nu\sigma]}$ in $[\lambda\mu]$, we have

$$\Omega^{\mu[\nu\sigma]}{}_{,\mu}=0. \tag{2.20}$$

[10] Ph. Freud, Ann. Math. 40, 417 (1939).

[11] R. Schiller, Ph.D. dissertation, Syracuse University (1952) (to be published).

In terms of the energy-momentum

$$\Omega^{\mu[\nu\sigma]} = (\mathcal{T}^{\mu\nu}x^\sigma - \mathcal{T}^{\mu\sigma}x^\nu) + (\tau_G{}^{\mu\nu}x^\sigma - \tau_G{}^{\mu\sigma}x^\nu) + (U_G{}^{[\sigma\mu]\nu} - U_G{}^{[\nu\mu]\sigma}). \quad (2.21)$$

The extra terms involving U_G are necessary because of the unavoidable asymmetry of the gravitational energy-momentum $\tau_G{}^{\mu\nu}$. The only reason for requiring that any stress tensor in classical mechanics be symmetric is to insure that there will be no net torque on the material in the absence of external forces. Here we make up for the fact that the energy-momentum is not symmetric by redefining the torque in (2.21). For the limiting case of zero gravitational field, U_G goes to zero, and the angular momentum assumes the same form as in Lorentz covariant theories.

3. ANGULAR MOMENTUM OF THE ELECTRON FIELD

In this section we will consider the matter tensor density $\mathcal{T}^{\mu\nu}$ by itself and derive an expression for the angular momentum of a Dirac field. Originally $\mathcal{T}^{\mu\nu}$ is defined as $-2(\delta\mathfrak{M}/\delta g_{\mu\nu})$. But as $\mathcal{T}^{\mu\nu}$ represents the contribution of the nongravitational fields, it is preferable to express $\mathcal{T}^{\mu\nu}$ in terms of the "matter" field variables (including the electromagnetic field). This expression for $\mathcal{T}^{\mu\nu}$ can be obtained in a similar way to Rosenfeld.[1] First, however, we shall summarize briefly the results of introducing, in a covariant way, a spinor field into a curvilinear metric.

The Lagrangian for the original Dirac electron may be generalized quite easily so that it is a scalar density with respect to general coordinate transformations. It then becomes

$$\mathfrak{M} = \tfrac{1}{2}\hbar c(-g)^{\frac{1}{2}}(i\psi^\dagger\gamma^\mu\psi_{;\mu} - i\psi^\dagger{}_{;\mu}\gamma^\mu\psi + 2\epsilon\psi^\dagger\psi). \quad (3.1)$$

The γ^μ are matrices satisfying the generalized anticommutation law

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}, \quad (3.2)$$

and $\psi_{;\mu}$ is defined to be

$$\psi_{;\mu} = \psi_{,\mu} + \Gamma_\mu\psi, \qquad \psi^\dagger{}_{;\mu} = \psi^\dagger{}_{,\mu} - \psi^\dagger\Gamma_\mu, \quad \Gamma_\mu + \Gamma_\mu{}^\dagger = 0, \quad (3.3)$$

where the comma is ordinary partial differentiation and the spinors, Γ_μ, are the coefficients of connection of the spinor field. They satisfy the equation

$$\gamma^\mu{}_{;\nu} \equiv \gamma^\mu{}_{,\nu} + \left\{ \begin{matrix} \mu \\ \lambda\nu \end{matrix} \right\} \gamma^\nu + \Gamma_\nu\gamma^\mu - \gamma^\mu\Gamma_\nu = 0, \quad (3.4)$$

and the $\left\{ \begin{matrix} \mu \\ \lambda\nu \end{matrix} \right\}$ are the usual Christoffel symbols.

These equations may also be written in terms of the bein (or quadruped) gravitational variables. The Lagrangian will then be

$$\mathfrak{M} = \tfrac{1}{2}\hbar c(-g)^{\frac{1}{2}}(i\psi^\dagger\gamma^s\psi_{;s} - i\psi^\dagger{}_{;s}\gamma^s\psi + 2\epsilon\psi^\dagger\psi), \quad (3.5)$$

where

$$\gamma^s = h^s{}_\mu\gamma^\mu, \quad \psi_{;s} \equiv \psi_{,\mu}h_s{}^\mu + \Gamma_s\psi. \quad (3.6)$$

In all that follows bein indices will be written with Latin letters, and coordinate indices will appear with Greek letters. The bein vectors $h^\mu{}_s$, are defined so that

$$g_{\mu\nu} = h^m{}_\mu h^n{}_\nu \eta_{mn}. \quad (3.7)$$

(Greek indices are raised or lowered by $g_{\mu\nu}$ or $g^{\mu\nu}$ and Latin indices by η_{mn} or η^{mn}.) η_{mn} is the Minkowski flat-metric tensor. The practical importance of the bein is that it is possible to find a particular representation of the ψ field such that for all bein and coordinate systems, the γ^s are constants, and are given by the original Dirac-Pauli matrices

$$\gamma^s\gamma^k + \gamma^k\gamma^s = 2\eta^{ks}. \quad (3.8)$$

In the bein notation it is easier to find a closed expression for the spin connection coefficients in terms of the bein gravitational field variables. (We need this expression because the Γ_l appears in the Lagrangian, and we need to know how it depends on the field variables.)

The analog to Eq. (3.4) now becomes

$$\left\{ \begin{matrix} s \\ kl \end{matrix} \right\} \gamma^k + \Gamma_l\gamma^s - \gamma^s\Gamma_l = 0, \quad (3.9)$$

which is a linear expression for the Γ_l and does not contain the partial derivatives of γ^s (which vanish). Since the γ^s generate a hypercomplex number system (sedenions), any spinor of rank 2 may be represented by a linear combination of the γ^s and their various products. It turns out that the only nonvanishing terms are of the zeroth and second degrees. The bein Christoffel symbol in Eq. (3.9) is determined as follows:

$$h^m{}_{\mu;n} \equiv h^m{}_{\mu,n} - \left\{ \begin{matrix} \lambda \\ \mu\sigma \end{matrix} \right\} h^\sigma{}_n h^m{}_\lambda + \left\{ \begin{matrix} m \\ kn \end{matrix} \right\} h^k{}_\mu = 0,$$
$$\left\{ \begin{matrix} m \\ ln \end{matrix} \right\} = -h^m{}_{\mu,n}h^\mu{}_l + \left\{ \begin{matrix} \lambda \\ \mu\sigma \end{matrix} \right\} h^\sigma{}_n h^m{}_\lambda h^\mu{}_l, \quad (3.10)$$

where the $\left\{ \begin{matrix} \lambda \\ \mu\sigma \end{matrix} \right\}$ are the usual coordinate Christoffel symbols. The solution of Eq. (3.9) becomes then

$$\Gamma_l = \tfrac{1}{4}\left\{ \begin{matrix} m \\ nl \end{matrix} \right\} \gamma^n\gamma_m + i\phi_l, \quad (3.11)$$

where ϕ_l are four real numbers and equal to $-i/4$ trace$\{\Gamma_l\}$. They are not determined by Eq. (3.9) and are usually interpreted as representing the electromagnetic potentials. The ϕ_l transform as vectors with respect to coordinate transformations and are invariant under bein transformations. In this paper we shall set all trace terms equal to zero as we are not concerned with the electromagnetic field.

We shall now resume our undertaking to express the matter energy-stress tensor density in terms of the bein variables. After a short calculation, we have

$$\frac{\delta\mathfrak{G}}{\delta h^m{}_\mu}=\frac{\delta\mathfrak{G}}{\delta g_{\sigma\lambda}}\frac{\partial g_{\sigma\lambda}}{\partial h^m{}_\mu}=\frac{\delta G}{\delta g_{\mu\lambda}}h^l{}_\lambda\eta_{ml}+\frac{\delta\mathfrak{G}}{\delta g_{\lambda\mu}}h^k{}_\lambda\eta_{km}, \quad (3.12)$$

where the second equation follows from Eq. (3.7).

$$\frac{\delta\mathfrak{G}}{\delta h^m{}_\mu}h^m{}_\nu=\frac{\delta\mathfrak{G}}{\delta g_{\mu\lambda}}g_{\lambda\nu}+\frac{\delta\mathfrak{G}}{\delta g_{\sigma\mu}}g_{\sigma\nu}=\mathcal{T}_\nu{}^\mu. \quad (3.13)$$

Since $(\delta/\delta h^m{}_\mu)\{\mathfrak{G}+\mathfrak{M}\}=0$, then

$$\mathcal{T}_\nu{}^\mu=-\frac{\delta\mathfrak{M}}{\delta h^m{}_\mu}h_\nu{}^m=\frac{\delta\mathfrak{M}}{\delta h}F_{(h)\nu}{}^\mu. \quad (3.14)$$

Having set the electron field in the general framework which we use, we now return to the direct discussion of its contribution to the angular momentum of the theory. In the discussion leading up to Eq. (2.5), we mentioned that each part of the Lagrangian had a set of identities related to the invariance of the Lagrangian density function with respect to coordinate transformations. Equations (2.5) and (2.11) summarize the identities obtained for the gravitational part of the total Lagrangian. Since the electron Dirac Lagrangian function given in Eq. (3.5) is also a scalar density for such transformations, the process which led to (2.5) and (2.11) will lead to a new set of identities pertaining to the electron field. Writing, thus, Eq. (2.5) and (2.11) for the matter fields, we have

$$U_M{}^{[\lambda\mu]}{}_{\nu,\lambda}=-t_{M\nu}{}^\mu+F_{M\nu}{}^\mu\delta\mathfrak{M}/\delta\zeta, \quad (3.15)$$

which then becomes

$$\mathcal{T}_\nu{}^\mu=-\frac{\delta\mathfrak{M}}{\delta\psi}F_{(\psi)\nu}{}^\mu+\frac{\partial\mathfrak{M}}{\partial h^k{}_{\sigma,\mu}}h^k{}_{\sigma,\nu}+\frac{\partial\mathfrak{M}}{\partial\psi_{,\mu}}\psi_{,\nu}$$
$$+\psi^\dagger{}_{,\nu}\frac{\partial\mathfrak{M}}{\partial\psi^\dagger{}_{,\mu}}-\mathfrak{M}\delta_\nu{}^\mu+U_M{}^{[\lambda\mu]}{}_{\nu,\lambda}. \quad (3.16)$$

The subscripts in parentheses on the F's show that each field variable has its own set of F's, the type of F being indicated by the letter. $U_M{}^{[\lambda\mu]}{}_\nu$ is the same expression obtained before in Eqs. (2.13) and (2.14), but with the Dirac Lagrangian understood. It must also be remembered that there are two distinct sets of terms in $U_M{}^{[\lambda\mu]}{}_\nu$—those containing derivatives with respect to the gravitational variables, and those with derivatives with respect to the matter fields. By a similar argument to that which leads to (3.13), it may be shown that the $t_{G\nu}{}^\mu$ used in Eq. (2.5) is not changed when it is expressed in terms of the bein variables. Hence we still have the equation

$$\{\mathcal{T}_\nu{}^\mu+t_{G\nu}{}^\mu\}_{,\mu}=0, \quad (3.17)$$

where the $\mathcal{T}_\nu{}^\mu$ is now the generalized bein expression for the energy-momentum tensor, (3.16).

It is probably pertinent to remark here that the tensor $\mathcal{T}^{\mu\nu}$ defined by Eqs. (3.14) and (3.16) is symmetric by virtue of the fact that the matter field equations are satisfied. This is a direct result of the invariance of the Lagrangian with respect to the choice of any particular bein system. Different bein systems are related to one another by orthogonal bein transformations.

From here on, we shall change our viewpoint slightly. When heretofore we have used Greek indices for coordinate components, it was not implied that there existed an underlying orthogonal be in system. In all subsequent formulas, even when coordinate indices are used on spinors, it will be understood that some orthogonal quadruped system has been chosen to which all spinors are implicitly referred. In other words, from now on, we shall take

$$\gamma^\mu=h^\mu{}_m\gamma^m, \quad (3.18)$$

and assume, as a matter of course, that the γ^m are the constant Dirac matrices chosen in (3.8). The γ^m are no longer field variables. Of course, the usual notation for ordinary vectors and tensors is still the same as before:

$$A^\mu=h^\mu{}_mA^m. \quad (3.19)$$

It is desirable to eliminate those terms in Eq. (3.16) which contain the partial derivatives of the electron Lagrangian with respect to the gravitational variables. They are hard to evaluate as they stand, and it is possible to eliminate them by the use of certain identities. In order to take advantage of this method, however, it is necessary to put the expression for $\mathcal{T}^{\mu\nu}$ into a form in which its tensor character is explicit. Rosenfeld[1] has done this. His expression is

$$\mathcal{T}_\nu{}^\mu=\frac{\partial\mathfrak{M}}{\partial\psi_{;\mu}}\psi_{;\nu}+\psi^\dagger{}_{;\nu}\frac{\partial\mathfrak{M}}{\partial\psi_{;\mu}}-\mathfrak{M}\delta_\nu{}^\mu+U_M{}^{[\lambda\mu]}{}_{\nu;\lambda}. \quad (3.20)$$

The terms in question now are

$$\frac{\partial\mathfrak{M}}{\partial h_{,\lambda}}F_{(h)\nu}{}^\mu=\frac{\partial\mathfrak{M}}{\partial h^\alpha{}_{a,\lambda}}\delta_\nu{}^\alpha h^\mu{}_a. \quad (3.21)$$

We will need to work with a particular antisymmetric expression of the U's:

$$Z^{\lambda[\mu\nu]}=U_M{}^{[\lambda\mu]\nu}-U_M{}^{[\lambda\nu]\mu}$$
$$=\frac{\partial\mathfrak{M}}{\partial h^\alpha{}_{a,\lambda}}\{h^\mu{}_ag^{\alpha\nu}-h^\nu{}_ag^{\alpha\mu}\}$$
$$+\frac{\partial\mathfrak{M}}{\partial\psi_{,\lambda}}\{F_{(\psi)\sigma}{}^\mu g^{\sigma\nu}-F_{(\psi)\sigma}{}^\nu g^{\sigma\mu}\}. \quad (3.22)$$

The program is to eliminate $\partial\mathfrak{M}/\partial h_{,\lambda}$ from this expression also, and we do this by means of an identity due to the bein invariance of the electron Lagrangian. We speak now of the transformation

$$\delta h^{\alpha}{}_{a}=\epsilon^{b}{}_{a}h^{\alpha}{}_{b}, \tag{3.23}$$

where the $\epsilon^{b}{}_{a}$ are the Lorentz transformation coefficients which satisfy the relations

$$\epsilon^{b}{}_{a}\epsilon^{a}{}_{c}=\delta_{c}{}^{b},\quad \epsilon^{ba}+\epsilon^{ab}=0,\quad \epsilon^{ba}=\epsilon^{b}{}_{C}\eta^{ca}. \tag{3.24}$$

The possibility is left open here for the "amount of rotation" of the bein vectors to be a function of position in the space. Under this transformation the variation of the matter part of the action becomes

$$\delta I=\int\left\{\left[\frac{\delta\mathfrak{M}}{\delta h^{\alpha}{}_{a}}\delta h^{\alpha}{}_{a}+\frac{\delta\mathfrak{M}}{\delta\psi}\delta\psi+\delta\psi^{\dagger}\frac{\delta\mathfrak{M}}{\delta\psi^{\dagger}}\right]+\left[\frac{\partial\mathfrak{M}}{\partial h^{\alpha}{}_{a,\beta}}\delta h^{\alpha}{}_{a}+\frac{\partial\mathfrak{M}}{\partial\psi_{,\beta}}\delta\psi+\delta\psi^{\dagger}\frac{\partial\mathfrak{M}}{\partial\psi^{\dagger}{}_{,\beta}}\right]_{,\beta}\right\}d^{4}x=0. \tag{3.25}$$

This equation gives rise to two independent identities: that due to the volume integral, and that due to the divergence or surface part. This is because the descriptors of the transformation may be taken zero on the surface, thus making the volume integrand zero; but since the combination is also zero, then the surface contribution must also be zero. In order to obtain the identities themselves, we need the expression for $\delta\psi$ and $\delta\psi^{\dagger}$. Assume

$$\delta\psi=\sigma\psi;\quad \delta\psi^{\dagger}=-\psi^{\dagger}\sigma, \tag{3.26}$$

where σ is some spinor of second rank. Then we must have

$$\delta\gamma^{l}=0=\epsilon^{l}{}_{m}\gamma^{m}+\sigma\gamma^{l}-\gamma^{l}\sigma; \tag{3.27}$$

from which, by assuming a solution quadratic in the γ^{l},

$$\sigma=\tfrac{1}{4}\epsilon_{ij}\gamma^{i}\gamma^{j}. \tag{3.28}$$

Incidentally, here we have the same situation we had in the expression for the Γ_{l}. That is, there is a trace term omitted which we have set equal to zero. As it stands now, Eq. (3.28) contributes no traces to Γ_{l}, and hence the electromagnetic fields once set equal to zero remain zero. This also shows how the electromagnetic interaction terms may be incorporated into the spinor fields, and at the same time remain independent of the electron field.[12] If we put the expressions for the variations into Eq. (3.25), and remember the antisymmetry of the ϵ_{ij}, see (3.24), then the arbitrariness of the rotation coefficients allows us to write the two identities

$$\begin{aligned}&\frac{\delta\mathfrak{M}}{\delta h^{\alpha}{}_{m}}\{h^{\alpha b}\delta_{m}{}^{a}-h^{\alpha a}\delta_{m}{}^{b}\}+\frac{1}{4}\frac{\delta\mathfrak{M}}{\delta\psi}[\gamma^{a}\gamma^{b}-\gamma^{b}\gamma^{a}]\psi+\frac{1}{4}\psi^{\dagger}[\gamma^{b}\gamma^{a}-\gamma^{a}\gamma^{b}]\frac{\delta\mathfrak{M}}{\delta\psi^{\dagger}}=0,\\&\frac{\partial\mathfrak{M}}{\partial h^{\alpha}{}_{m,\lambda}}\{h^{\alpha b}\delta_{m}{}^{a}-h^{\alpha a}\delta_{m}{}^{b}\}+\frac{1}{4}\frac{\partial\mathfrak{M}}{\partial\psi_{,\lambda}}[\gamma^{a}\gamma^{b}-\gamma^{b}\gamma^{a}]\psi+\frac{1}{4}\psi^{\dagger}[\gamma^{b}\gamma^{a}-\gamma^{a}\gamma^{b}]\frac{\partial\mathfrak{M}}{\partial\psi^{\dagger}{}_{,\lambda}}=0.\end{aligned} \tag{3.29}$$

After multiplying the second of these identities by $h^{\mu}{}_{a}h^{\nu}{}_{b}$, and using Eq. (3.7), we see that

$$\frac{\partial\mathfrak{M}}{\partial h^{\alpha}{}_{K,\lambda}}\{h^{\mu}{}_{K}g^{\alpha\nu}-h^{\nu}{}_{K}g^{\alpha\mu}\}+\frac{1}{4}\frac{\partial\mathfrak{M}}{\partial\psi_{,\lambda}}[\gamma^{\mu}\gamma^{\nu}-\gamma^{\nu}\gamma^{\mu}]\psi+\frac{1}{4}\psi^{\dagger}[\gamma^{\nu}\gamma^{\mu}-\gamma^{\mu}\gamma^{\nu}]\frac{\partial\mathfrak{M}}{\partial\psi^{\dagger}{}_{,\lambda}}=0. \tag{3.30}$$

It is now possible with the help of Eq. (3.30) to eliminate $\partial\mathfrak{M}/\partial h_{,\lambda}$ in Eq. (3.22), and that equation now becomes

$$Z^{\lambda[\mu\nu]}=-\frac{1}{4}\frac{\partial\mathfrak{M}}{\partial\psi_{,\lambda}}[\gamma^{\mu}\gamma^{\nu}-\gamma^{\nu}\gamma^{\mu}]\psi+\frac{1}{4}\psi^{\dagger}[\gamma^{\mu}\gamma^{\nu}-\gamma^{\nu}\gamma^{\mu}]\frac{\partial\mathfrak{M}}{\partial\psi^{\dagger}{}_{,\lambda}}. \tag{3.31}$$

Here $F_{(\psi)\nu}{}^{\mu}=0$ because the infinitesimal coordinate transformation does not change ψ; only a bein transformation affects the ψ field. The $U_{M}{}^{[\lambda\mu]\nu}$ thus assumes a particularly simple expression

$$U_{M}{}^{[\lambda\mu]\nu}=\tfrac{1}{2}\{Z^{\lambda[\mu\nu]}-Z^{\nu[\lambda\mu]}+Z^{\mu[\nu\lambda]}\}. \tag{3.32}$$

The expression $U_{M}{}^{[\lambda\mu]\nu}$ itself now contains no partial derivatives of $\mathfrak{M}$ with respect to the gravitational variables, and is quite easy to evaluate.

At this point, it remains only to carry out the operations indicated in the definition of $Z^{\lambda[\mu\nu]}$ with the electron Lagrangian. With reference to (3.5), after a short calculation, we have

$$\begin{aligned}Z^{\lambda[\mu\nu]}=&-\tfrac{1}{8}ic\hbar(-g)^{\frac{1}{2}}\psi^{\dagger}\gamma^{\lambda}[\gamma^{\mu}\gamma^{\nu}-\gamma^{\nu}\gamma^{\mu}]\psi\\&-\tfrac{1}{8}ic\hbar(-g)^{\frac{1}{2}}\psi^{\dagger}[\gamma^{\mu}\gamma^{\nu}-\gamma^{\nu}\gamma^{\mu}]\gamma^{\lambda}\psi,\end{aligned} \tag{3.33}$$

and

$$\begin{aligned}U_{M}{}^{[\lambda\mu]}{}_{\nu}=\tfrac{1}{8}ic\hbar(-g)^{\frac{1}{2}}\psi^{\dagger}\{&2\gamma^{\dagger}\delta_{\nu}{}^{\mu}-2\gamma^{\mu}\delta_{\nu}{}^{\lambda}\\&-[\gamma^{\lambda}\gamma^{\mu}-\gamma^{\mu}\gamma^{\lambda}]\gamma_{\nu}\}\psi.\end{aligned} \tag{3.34}$$

Because of the terms in $U_{M}{}^{[\lambda\mu]}{}_{\nu}$ in the expression for the matter energy-momentum tensor density, that quantity

[12] See for a fuller discussion of this point E. Schrödinger, Preuss. Akad. Wiss. Berlin, Ber. 11–12, 105 (1932).

has become somewhat complicated: it takes the form

$$\mathcal{T}_\nu{}^\mu=\tfrac{1}{2}ic\hbar(-g)^{\frac{1}{2}}\{\psi^\dagger\gamma^\mu\psi_{;\nu}-\psi^\dagger{}_{;\nu}\gamma^\mu\psi\}-\tfrac{1}{8}ic\hbar(-g)^{\frac{1}{2}}\psi^\dagger{}_{;\lambda}\{[\gamma^\lambda\gamma^\mu-\gamma^\mu\gamma^\lambda]\gamma_\nu+2\gamma^\mu\delta_\nu{}^\lambda-2\gamma^\lambda\delta_\nu{}^\mu\}\psi$$
$$-\tfrac{1}{8}ic\hbar(-g)^{\frac{1}{2}}\psi^\dagger\{[\gamma^\lambda\gamma^\mu-\gamma^\mu\gamma^\lambda]\gamma_\nu+2\gamma^\mu\delta_\nu{}^\lambda-2\gamma^\lambda\delta_\nu{}^\mu\}\psi_{;\lambda}. \quad (3.35)$$

It should be emphasized that for our use the complete expression is necessary for the matter tensor. It will be remembered that in the special relativity theories the U part is added only to make the energy-momentum symmetric, and that if it is deleted entirely, the integrated energy and momentum is unchanged. In those theories the covariant divergence becomes an ordinary divergence and may be integrated out. In this generally covariant theory, however. Gauss' law may not be used in the same fashion for this term, and the entire expression, (3.35), must always be used.

We should like now to write down the expressions representing the angular momentum of the electron-gravitation field. In this connection we must bear in mind that the only unique aspect of the energy-momentum or the angular momentum is the total, or integrated, value taken over the whole three-dimensional space at some particular time t_0. This is because one may add to the four-dimensional density expression any divergence-less expression, and it would still be "conserved." In addition, it is easy to demonstrate that the integrated form of all such expressions gives identical values. But first we shall obtain the integrated expressions themselves. In order to obtain them, we integrate Eq. (2.20) over some four-dimensional region. We specialize the region to be a cylinder between two time surfaces, with the sides of the cylinder taken to spatial infinity, where the electron field is assumed zero and the metric flat. Designating the region by V_4, we have

$$\int \Omega^{\mu[\nu\lambda]}{}_{,\mu}d^4x=0, \quad (3.36)$$

which because of Gauss' law, can be turned into

$$J^{\nu\lambda}=\int_{t_1}\Omega^{4[\nu\lambda]}d^3x=\int_{t_2}\Omega^{4[\nu\lambda]}d^3x. \quad (3.37)$$

$J^{\nu\lambda}$ is now the total angular momentum of the field, and is an invariant with respect to arbitrary coordinate transformations carried out within the region V_4, with the sole restriction that in any transformation, the new coordinates go over into the old ones on the boundary of V_4. Now it can be seen that if we add to $\Omega^{4[\nu\lambda]}$ some expression whose divergence vanishes $\chi^{4[\nu\lambda]}$, then it may be represented as the divergence of an antisymmetric quantity (a curl),

$$\chi^{4[\nu\lambda]}=V^{[4\sigma][\nu\lambda]}{}_{,\sigma}. \quad (3.38)$$

When $\chi^{4[\nu\lambda]}$ is integrated over the infinite three-space, then

$$\int V^{[4\sigma][\nu\lambda]}{}_{,\sigma}d^3x=\oint_{\text{2-surface}} V^{[4\sigma'][\nu\lambda]}dS_{\sigma'} + \int V^{[44][\nu\lambda]}{}_{,4}d^3x, \quad (3.39)$$
$$\sigma'=1,2,3.$$

The second term on the right is zero by the symmetry assumed in the $V^{[\alpha\beta][\gamma\delta]}$ function. Hence, for all those functions $V^{[4\sigma'][\nu\lambda]}$ which fall off faster than $1/R^2$ (assumption of no radiation), the integrated value of the angular momentum is uniquely defined.

From Eq. (2.21), the matter contribution to $J^{\nu\lambda}$ may be written

$$J_M{}^{\nu\lambda}=\int\{\mathcal{T}^{4\nu}x^\lambda-\mathcal{T}^{4\lambda}x^\nu\}d^3x \quad (3.40)$$

or

$$J_M{}^{\nu\lambda}=\int\bigg\{\frac{\partial\mathfrak{M}}{\partial\psi_{,4}}\psi^{;\nu}x^\lambda+\psi^{\dagger;\nu}\frac{\partial\mathfrak{M}}{\partial\psi_{,4}}x^\lambda-\frac{\partial\mathfrak{M}}{\partial\psi_{,4}}\psi^{;\lambda}x^\nu$$
$$-\psi^{\dagger;\lambda}\frac{\partial\mathfrak{M}}{\partial\psi_{,4}}x^\nu+U_M{}^{[\sigma4]\nu}{}_{;\sigma}x^\lambda-U^{[\sigma4]\lambda}{}_{;\sigma}x^\nu\bigg\}d^3x. \quad (3.41)$$

Now $U_M{}^{[\sigma\mu]\nu}$ is a tensor density, so

$$U_M{}^{[\sigma4]\nu}{}_{;\sigma}=\left\{\begin{matrix}\sigma\\ \sigma\beta\end{matrix}\right\}U_M{}^{[\beta4]\nu}+\left\{\begin{matrix}\nu\\ \sigma\beta\end{matrix}\right\}U_M{}^{[\sigma4]\beta}$$
$$+U^{[\sigma4]\nu}{}_{,\sigma}-\left\{\begin{matrix}\sigma\\ \sigma\beta\end{matrix}\right\}U_M{}^{[\beta4]\nu} \quad (3.42)$$

giving

$$J_M{}^{\nu\lambda}=\int\bigg\{\frac{\partial\mathfrak{M}}{\partial\psi_{,4}}\psi^{;\nu}x^\lambda-\frac{\partial\mathfrak{M}}{\partial\psi_{,4}}\psi^{;\lambda}x^\nu+\psi^{\dagger;\nu}\frac{\partial\mathfrak{M}}{\partial\psi^\dagger{}_{,4}}x^\lambda$$
$$-\psi^{\dagger;\lambda}\frac{\partial\mathfrak{M}}{\partial\psi^\dagger{}_{,4}}x^\nu+\left\{\begin{matrix}\nu\\ \sigma\beta\end{matrix}\right\}U_M{}^{[\sigma4]\beta}x^\lambda$$
$$-\left\{\begin{matrix}\lambda\\ \sigma\beta\end{matrix}\right\}U_M{}^{[\sigma4]\beta}x^\nu-U_M{}^{[\lambda4]\nu}+U_M{}^{[\nu4]\lambda}$$
$$-(U_M{}^{[\sigma4]\nu}x^\lambda-U_M{}^{[\sigma4]\lambda}x^\nu)_{,\sigma}\bigg\}d^3x. \quad (3.43)$$

The spatial divergence term in the integrand may be converted into a two-surface integral, but the original region V_4 was chosen so that the contribution is zero. (The $U_M{}^{[\sigma4]\lambda}$ contains electron field functions, which will be assumed to fall off faster than $1/R^2$.) The 4-component is zero again because of the antisymmetry of the U function. Thus we have, finally,

$$J_M{}^{\nu\lambda}=\int\Big\{\frac{\partial\mathfrak{M}}{\partial\psi_{,4}}\psi^{;\nu}x^\lambda-\frac{\partial\mathfrak{M}}{\partial\psi_{,4}}\psi^{;\lambda}x^\nu+\psi^{\dagger;\nu}\frac{\partial\mathfrak{M}}{\partial\psi^\dagger{}_{,4}}x^\lambda-\psi^{\dagger;\lambda}\frac{\partial\mathfrak{M}}{\partial\psi^\dagger{}_{,4}}x^\nu+U_m{}^{[\sigma4]\beta}\left(\begin{Bmatrix}\nu\\ \sigma\beta\end{Bmatrix}x^\lambda-\begin{Bmatrix}\lambda\\ \sigma\beta\end{Bmatrix}x^\nu\right)+U_M{}^{[\lambda4]\nu}-U_M{}^{[\nu4]\lambda}\Big\}d^3x. \quad (3.44)$$

By using the expression (3.34) then the last equation becomes

$$J_M{}^{\nu\lambda}=\int\Big\{\frac{\partial\mathfrak{M}}{\partial\psi_{,4}}\psi^{;\nu}x^\lambda-\frac{\partial\mathfrak{M}}{\partial\psi_{,4}}\psi^{;\lambda}x^\nu+\psi^{\dagger;\nu}\frac{\partial\mathfrak{M}}{\partial\psi^\dagger{}_{,4}}x^\lambda-\psi^{\dagger;\lambda}\frac{\partial\mathfrak{M}}{\partial\psi^\dagger{}_{,4}}x^\nu+U_M{}^{[\sigma4]\beta}\left(\begin{Bmatrix}\nu\\ \sigma\beta\end{Bmatrix}x^\lambda-\begin{Bmatrix}\lambda\\ \sigma\beta\end{Bmatrix}x^\nu\right)+\tfrac{1}{4}ic\hbar(-g)^{\frac{1}{2}}\psi^\dagger\gamma^4(\gamma^\nu\gamma^\lambda-\gamma^\lambda\gamma^\nu)\psi\Big\}d^3x. \quad (3.45)$$

To $J_M{}^{\nu\lambda}$ must be added the gravitational contribution:

$$J_G{}^{\nu\lambda}=\int\{\tau_G{}^{4\nu}x^\lambda-\tau_G{}^{4\lambda}x^\nu+U_G{}^{[\lambda4]\nu}-U_G{}^{[\nu4]\lambda}\}d^3x. \quad (3.46)$$

It is interesting to compare (3.45) with the quantity used in the special relativity theories. In the limiting case where the gravitational field is zero, and the space is flat, (3.45) degenerates precisely to the Lorentz-invariant angular momentum in general use.[3]

The expression of (3.45) in terms of the bein components has a particular significance.

$$J_M{}^{nl}=\int\Big\{\frac{\partial\mathfrak{M}}{\partial\psi_{,4}}\psi^{;n}x^l-\frac{\partial\mathfrak{M}}{\partial\psi_{,4}}\psi^{;l}x^n+\psi^{\dagger;n}\frac{\partial\mathfrak{M}}{\partial\psi^\dagger{}_{,4}}x^l-\psi^{\dagger;l}\frac{\partial\mathfrak{M}}{\partial\psi_{;4}}x^n+\left(\begin{Bmatrix}\nu\\ \sigma\beta\end{Bmatrix}x^\lambda-\begin{Bmatrix}\lambda\\ \sigma\beta\end{Bmatrix}x^\nu\right)h^l{}_\lambda h^n{}_\nu U_M{}^{[\sigma4]\beta}+\tfrac{1}{4}ic\hbar(-g)^{\frac{1}{2}}\psi^\dagger\gamma^4[\gamma^n\gamma^l-\gamma^l\gamma^n]\psi\Big\}d^3x. \quad (3.47)$$

Equation (3.47) breaks into three parts. The first four terms are the usual orbital momentum terms, and the last term is the direct analog to the spin, provided we use the bein representation. This last term contributes the usual value of $\frac{1}{2}\hbar$ to the total angular momentum, since the γ^l matrices involved are simply the constant Dirac matrices. But the third term represents a correction to the usual expression of the *orbital* momentum (since it is seen to be proportional to the coordinates) which is due to the curvature of the space.

4. CONCLUSION

In conclusion, then, we have found a generally covariant representation for the angular momentum of a physical field which includes a matter field in addition to the gravitational field. A semiclassical discussion of the Dirac electron field shows that the intrinsic angular momentum of the electron has the expected value of $\frac{1}{2}\hbar$. The orbital momentum is a direct generalization of that of the usual theory, with some added terms that may be traced to the nonflatness of the space in the vicinity of the electron.