Electron Capture by Protons Passing through Hydrogen

J. DAVID JACKSON AND HARRY SCHIFF
McGill University, Montreal, Quebec, Canada
(Received September 29, 1952)

A theoretical calculation is made of the capture of an electron by a particle of charge $Z'e$ passing a hydrogen-like atom of charge Ze. Numerical results are presented for protons passing through hydrogen gas. In contrast to earlier treatments of this problem, the complete interaction Hamiltonian is retained as the perturbation causing the electron transfer. Previous workers assumed that the interaction between the incident particle and the electron was the only perturbation causing the transition and ignored the interaction between the incident particle and the nucleus of the atom. Such a neglect is unjustified. The results of this paper show that when the complete interaction is used the agreement between theory and experiment is good for all energies greater than 25 kev ($\hbar v/e^2 \geqslant 1$) in the case of protons in hydrogen, even though the Born approximation is employed. This is a marked improvement over previous calculations which were approximately four to five times larger than experiment.

1. INTRODUCTION

THE capture of electrons by ions traversing matter has been extensively studied in the laboratory.[1] Theoretical work on the capture problem has been done by Oppenheimer,[2] Brinkman and Kramers,[3] and Massey and Smith,[4] while a survey of capture and loss phenomena with estimates for various cross sections has been given by Bohr.[5] Recently, precise experiments on the capture of electrons by protons in hydrogen gas have been performed by Keene,[6] Ribe,[7] and Whittier.[8] As a result, the capture cross section is known with good accuracy in the energy range from 2 to 150 kev.

BK,[3] with whose work we shall be most concerned in this paper, evaluated the Born approximation cross section for the capture of an electron by a particle of charge $Z'e$ passing a hydrogen-like atom of charge Ze. When applied to protons in hydrogen, their formula gives values for the cross section which are approximately four times the experimental results at 100 kev and still higher at lower energies (see Fig. 4). The discrepancy between BK's result and experiment has been largely attributed to the failure of the Born approximation in the energy range of the experiments. While it is true that $e^2/\hbar v$ equals unity at 25 kev and has only dropped to 0.41 at 150 kev, it will be shown in this paper that there is a much more important source for the discrepancy than the failure of the Born approximation. The perturbation Hamiltonian consists of two terms, the Coulomb interaction between the electron and the incident particle of charge $Z'e$, and the Coulomb interaction between the nucleus of charge Ze and the incident particle. Following Oppenheimer, BK considered only the (electron)-(incident particle) interaction, neglecting the (nucleus)-(incident particle) term. Such a neglect is well known, and completely justified, in inelastic collision problems. However, as is pointed out by Bohr,[9] the capture collision is a three-body problem, whereas ionization and excitation collisions are essentially two-body problems. Consequently, the neglect of the (nucleus)-(incident particle) interaction is unjustified.* Indeed, when it is taken into account, agreement between theory and experiment for protons in hydrogen is good for all energies above 25 kev ($\hbar v/e^2 \geqslant 1$), even though the Born approximation is still used (see Fig. 4). For energies less than 25 kev it is evident that the Born approximation will not be valid, and other methods of calculation must be employed.[10]

In Sec. 2 we briefly discuss rearrangement collisions as they pertain to the problems of this paper. In Sec. 3 the cross section for the capture of an electron from a hydrogen-like atom of charge Ze to form a hydrogen-like atom of charge $Z'e$ is set up, and detailed results are presented for the ground state captures for protons in hydrogen. Section 4 deals with the contribution to the

[1] For references to the various experiments, see H. S. W. Massey and E. H. S. Burhop, *Electronic and Ionic Impact Phenomena* (Oxford University Press, New York, 1952).
[2] J. R. Oppenheimer, Phys. Rev. **31**, 349 (1928).
[3] H. C. Brinkman and H. A. Kramers, Proc. Acad. Sci. Amsterdam **33**, 973 (1930), referred to as BK in the text.
[4] H. S. W. Massey and R. A. Smith, Proc. Roy. Soc. (London) **A142**, 142 (1933).
[5] N. Bohr, Kgl. Danske Videnskab. Selskab. Mat.-fys. Medd. **18**, No. 8 (1948).
[6] J. P. Keene, Phil. Mag. **40**, 369 (1949).
[7] F. Ribe, Phys. Rev. **83**, 1217 (1951).
[8] A. C. Whittier, Ph.D. thesis, McGill University, 1952 (to be published).
[9] Reference 5, pp. 105, 111.
* *Note added in proof.*—The situation is not really so clear-cut. It has been pointed out to us by Professor G. C. Wick that in an *exact* calculation of the capture process the (nucleus)-(incident particle) interaction will give a negligible contribution (of order m/M). This can be seen most easily by considering the nuclei to be infinitely heavy and setting the problem up as an impact parameter calculation. It is then evident that the (nucleus)-(incident particle) interaction can be removed from the Hamiltonian by an appropriate canonical transformation. Consequently it cannot effect the exact transition probability.
The good agreement with experiment obtained in the present paper implies that use of the whole perturbation Hamiltonian in an approximate calculation of the capture process greatly improves the convergence of the approximation scheme, even though it can be shown that some parts of the perturbation will give rise to negligible effects in an exact calculation. This is perhaps plausible physically in the light of Bohr's remarks.
[10] N. F. Mott and H. S. W. Massey, *Theory of Atomic Collisions* (Oxford University Press, New York, 1949), second edition, p. 140 *ff*.

cross section by captures into higher orbits, while Sec. 5 contains the comparison of theory with experiment for protons in hydrogen and a discussion of the reliability of the Born approximation in such problems.

2. REARRANGEMENT COLLISIONS

The capture of an electron by an ion passing another atom is an example of a rearrangement collision.[10,11] When such a problem is treated by perturbation theory, there is an ambiguity as to just what is to be treated as the perturbation. Consider a collision process in which systems A and B collide to form two different systems C and D. Assuming that the center of mass of the whole system is at rest, the complete Hamiltonian can be written in two ways:

$$H=T_i+H_A+H_B+V_i, \quad (1)$$

or

$$H=T_f+H_C+H_D+V_f, \quad (2)$$

where H_A, H_B, H_C, H_D are the internal Hamiltonians of the systems A, B, C, D; T_i and T_f are the kinetic energies of relative motion of systems A and B, and systems C and D, respectively; while V_i and V_f are the perturbation Hamiltonians. V_i and V_f are not equal in a rearrangement collision, and so there is an ambiguity as to which term to use in a perturbation calculation for the transition probability. These remarks are well known. The reason for mentioning them here is that in the particular problem that we are considering—that of the transfer of an electron from one hydrogen-like atom to form another hydrogen-like atom—there is *no ambiguity in the result* for the transition probability. In a first-order (Born-approximation) calculation for the transition probability V_i and V_f can be shown to lead to the same results, even though there is an apparent asymmetry because the charges are different and the capture may occur into various excited states. This result is proved in Appendix I.

The coordinate system for the capture problem is shown in Fig. 1. We consider the electron (3) of charge $-e$ and mass m initially bound around a nucleus (1) of charge Ze and mass AM, where M is the mass of a proton. As a result of the interaction of the passing particle (2) of charge $Z'e$ and mass $A'M$, the electron is captured into a bound state around particle (2). The Hamiltonian can be written initially in the form of Eq. (1) with

$$T_i=-\frac{\hbar^2}{2\mu_i}\nabla_R^2, \quad \mu_i=\frac{A'M(AM+m)}{(A+A')M+m}, \quad (3)$$

$$H_A=-\frac{\hbar^2}{2m_i}\nabla_r^2-\frac{Ze^2}{r}, \quad m_i=\frac{AMm}{AM+m}, \quad (4)$$

$$V_i=\frac{Z'Ze^2}{|\mathbf{R}+[m/(AM+m)]\mathbf{r}|}-\frac{Z'e^2}{|\mathbf{R}-[AM/(AM+m)]\mathbf{r}|}, \quad (5)$$

where H_B is irrelevant since particle (2) is assumed structureless. All terms of the Hamiltonian have been expressed in terms of the initial coordinates, $\mathbf{r}$ and $\mathbf{R}$. The first term in V_i is the (nucleus)-(incident particle) interaction, while the second is the (electron)-(incident particle) interaction $(-Z'e^2/r')$.

The Hamiltonian can be rearranged for the final configuration into the form of Eq. (2) with

$$T_f=-\frac{\hbar^2}{2\mu_f}\nabla_{R'}^2, \quad \mu_f=\frac{AM(A'M+m)}{(A+A')M+m}, \quad (6)$$

$$H_C=-\frac{\hbar^2}{2m_f}\nabla_{r'}^2-\frac{Z'e^2}{r'}, \quad m_f=\frac{A'Mm}{A'M+m}, \quad (7)$$

$$V_f=\frac{Z'Ze^2}{|\mathbf{R}'-[m/(A'M+m)]\mathbf{r}'|}-\frac{Ze^2}{|\mathbf{R}'+[A'M/(A'M+m)]\mathbf{r}'|}. \quad (8)$$

In Eq. (8) the first term is identical with the first term of Eq. (5), while the second term is just $(-Ze^2/r)$.

We note that BK considered only the second term of Eq. (5) [or Eq. (8)] as the perturbation producing the capture process. We shall keep both terms of the interaction, but will still employ the Born approximation. In treating the problem of protons in hydrogen, we have ignored the identity of the protons. This introduces a negligible error in the results because the cross section for capture is so peaked in the forward direction (the mean angle is of order m/M) that the protons are, in practice, distinguishable.[12]

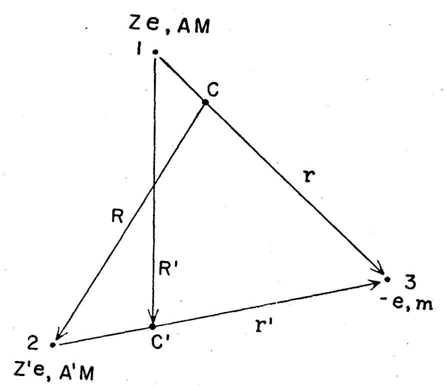

FIG. 1. Coordinate system. The points marked C and C' are the centers of mass of the initial and final atoms.

[11] L. I. Schiff, *Quantum Mechanics* (McGraw-Hill Book Company, Inc., New York, 1949), p. 230 *ff*.

[12] Reference 10, p. 290.

3. CALCULATION OF THE CROSS SECTION

The Born-approximation cross section for the capture of the electron into a final state f is given by[10]

$$\frac{d\sigma_f}{d\Omega}=\left(\frac{\mu_f}{2\pi\hbar^2}\right)^2\left(\frac{v'}{v}\right)|I_f|^2, \tag{9}$$

with

$$I_f=\int \exp(-i\mathbf{k}'\cdot\mathbf{R}')\psi_f^*(\mathbf{r}')V\exp(i\mathbf{k}\cdot\mathbf{R})\psi_0(\mathbf{r})d\mathbf{r}d\mathbf{R}, \tag{10}$$

where ψ_0 is the ground state wave function for the electron in atom Z; ψ_f is the wave function for the electron in the state f of the atom Z'; V is either V_i or V_f; $\hbar\mathbf{k}=\mu_i\mathbf{v}$, $\hbar\mathbf{k}'=\mu_f\mathbf{v}'$, where $\mathbf{v}$ is the velocity of the incident particle relative to the atom Z at rest, and $\mathbf{v}'$ is the corresponding outgoing velocity of atom Z' relative to the stripped nucleus Z. The conservation of energy requirement is

$$\tfrac{1}{2}\mu_i v^2-\epsilon=\tfrac{1}{2}\mu_f v'^2-\epsilon', \tag{11}$$

where ϵ, ϵ' are the binding energies of the electron in atoms Z, Z'.

It is convenient, following BK, to use $\mathbf{r}$ and $\mathbf{r}'$ as independent coordinates in the evaluation of the integral in Eq. (10). By reference to Fig. 1, it can be seen that I_f can be written

$$I_f=\int \exp(-i\mathbf{C}\cdot\mathbf{r}')\psi_f^*(\mathbf{r}')V(\mathbf{r},\mathbf{r}')\times\exp(i\mathbf{B}\cdot\mathbf{r})\psi_0(\mathbf{r})d\mathbf{r}d\mathbf{r}', \tag{12}$$

where

$$\mathbf{C}=\mathbf{k}-\frac{A'M}{A'M+m}\mathbf{k}',\quad \mathbf{B}=\frac{AM}{AM+m}\mathbf{k}-\mathbf{k}'. \tag{13}$$

The interaction [Eq. (5) or (8)] becomes

$$V_i=Z'Ze^2/|\mathbf{r}-\mathbf{r}'|-Z'e^2/r', \tag{5'}$$

$$V_f=Z'Ze^2/|\mathbf{r}-\mathbf{r}'|-Ze^2/r. \tag{8'}$$

The integration in Eq. (12) involving the second term in Eq. (5′) or (8′) is straightforward (see Appendix I), and leads to the well-known BK result for capture into the ground state and similar expressions for capture into excited states.[13] The first term of the interaction is somewhat more difficult to handle. The integral in question is

$$I_f'=Z'Ze^2\int \exp(-i\mathbf{C}\cdot\mathbf{r}')\psi_f^*(\mathbf{r}')\frac{1}{|\mathbf{r}-\mathbf{r}'|}\times\exp(i\mathbf{B}\cdot\mathbf{r})\psi_0(\mathbf{r})d\mathbf{r}d\mathbf{r}.$$

[13] M. N. Saha and D. Basu, Indian J. Phys. **19**, 121 (1945).

We introduce the Fourier transform of $1/|\mathbf{r}-\mathbf{r}'|$ to get

$$I_f'=\frac{Z'Ze^2}{2\pi^2}\int\frac{d\mathbf{k}}{k^2}\int\exp[-i(\mathbf{C}-\mathbf{k})\cdot\mathbf{r}']\times\psi_f^*(\mathbf{r}')d\mathbf{r}'\int\exp[i(\mathbf{B}-\mathbf{k})\cdot\mathbf{r}]\psi_0(\mathbf{r})d\mathbf{r}.$$

We define the Fourier transforms of the wave functions ψ_0 and ψ_f:

$$\phi_0(\mathbf{K})=\int\exp(i\mathbf{K}\cdot\mathbf{r})\psi_0(\mathbf{r})d\mathbf{r},\qquad \phi_f(\mathbf{K}')=\int\exp(i\mathbf{K}'\cdot\mathbf{r}')\psi_f(\mathbf{r}')d\mathbf{r}'. \tag{14}$$

Then the integral I_f' can be written as a single integral in $\mathbf{k}$ space:

$$I_f'=\frac{Z'Ze^2}{2\pi^2}\int\phi_0(\mathbf{B}-\mathbf{k})\phi_f^*(\mathbf{C}-\mathbf{k})\frac{d\mathbf{k}}{k^2}. \tag{15}$$

For the ground state of a hydrogen-like atom of charge Ze the Fourier transform is

$$\phi_0(\mathbf{K})=8(\pi)^{1/2}(Z/a_0)^{5/2}[(Z/a_0)^2+K^2]^{-2}, \tag{16}$$

while the excited states have similar, but more involved, transforms.

The evaluation of I_f' in Eq. (15) is outlined in Appendix II, with the ground state capture for $Z'=Z=1$ being used as an illustration. The calculation for arbitrary Z' and Z, even for the relatively simple case of capture into the ground state, is too involved algebraically to warrant discussion here.

As was mentioned earlier, the angular distribution is peaked very sharply in the forward direction. The mean angle of the distribution in the center-of-mass system is of the order of $\theta_0=(A+A')m/2AA'M$. It should be mentioned that the only place that the masses of the heavy particles enter is in the angular distribution. $|I_f|^2$ can be written as a function of Z', Z, $\hbar v/e^2$, and $y=(\theta/\theta_0)^2$. When the differential cross section Eq. (9) is integrated over angles, the solid angle $d\Omega$ is approximately $\pi d(\theta)^2$ for the angles of interest and the factor μ_f^2 is equal to $(m/2\theta_0)^2$, so that $\mu_f^2 d\Omega$ is proportional to dy. Thus the total cross section is independent of the values of A and A'. However, the cross section does depend in detail on the values of Z' and Z.

For protons in hydrogen ($Z'=Z=1$) the partial cross section for captures into the ground state is

$$\sigma_1=\sigma_{BK}\left[\frac{1}{192}\left(127+\frac{14}{E}+\frac{2}{E^2}\right)-\frac{1}{96}\frac{\tan^{-1}E^{\frac{1}{2}}}{E^{\frac{1}{2}}}\left(83+\frac{15}{E}+\frac{2}{E^2}\right)+\frac{1}{96E}(\tan^{-1}E^{\frac{1}{2}})^2\left(31+\frac{8}{E}+\frac{1}{E^2}\right)\right], \tag{17}$$

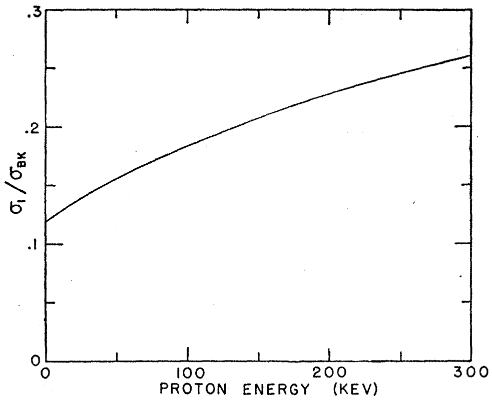

FIG. 2. Ratio of the ground-state capture cross section σ_1 to the Brinkman-Kramers cross section σ_{BK} for protons in hydrogen ($Z'=Z=1$) as a function of the incident proton energy.

where

$$\sigma_{BK}=\frac{112.5\times10^{-17}\text{ cm}^2}{E(1+E)^5} \tag{18}$$

is the BK cross section for capture into the ground state, and $E=(\hbar v/2e^2)^2$ is the proton energy in units of 100 kev. The ratio (σ_1/σ_{BK}) is shown as a function of incident proton energy in Fig. 2. The limiting values of the ratio are 0.117 at zero energy and 0.661 at very high energies (the high energy limit is approached very slowly, e.g., at 1 Mev the ratio is only 0.369). Figure 2 shows that the inclusion of the incident particle-nucleus interaction has a decisive influence on the cross section, even at high energies. This term produces destructive interference with the other term of the interaction to reduce the cross section from the BK value by a factor of from 5 to 3 over the energy range from 0 to 1 Mev. Both σ_{BK} and σ_1 are plotted in Fig. 4 over the energy range where there is experimental data.

4. CONTRIBUTION FROM CAPTURES INTO EXCITED STATES

The captures into higher orbits of the atom Z' will contribute to the cross section and increase its value over the ground-state result. Oppenheimer[2] showed that at high velocities ($\hbar v/e^2\gg1$) only the higher s states contribute, and that the ratio $(\sigma_n/\sigma_1)=n^{-3}$, where n is the principal quantum number. Thus, at high velocities, the complete capture cross section σ_c is

$$(\sigma_c)_{BK}\simeq\sigma_{BK}\sum_1^\infty n^{-3}=1.202\sigma_{BK}.$$

For protons in hydrogen we have evaluated the partial cross sections for capture into the $2s$ and $2p$ states, using the complete interaction, Eq. (5′) or (8′). The methods of Sec. 3 and Appendix II were used, along with numerical integration of some of the more involved expressions which result. In Fig. 3 we have plotted the ratios of these partial cross sections to the ground-state cross section σ_1, Eq. (17), for the energy range from 0 to 200 kev. Also plotted are the corresponding ratios in the approximation of BK. The remarkable thing is the similarity in the shape and magnitude of the ratio curves from the present calculations and the BK calculations, even though the absolute magnitudes differ by a factor of five.

We wish to exploit the similarity of the ratio curves in Fig. 3 in order to estimate the partial cross sections for the capture into states with $n\geqslant3$. It is found that for the higher partial cross sections in the BK approximation the ratios to the ground-state cross section have roughly the same energy dependence as the $n=2$ ratios but are down in magnitude by the factor n^{-3}. From the evidence for $n=2$ shown in Fig. 3, it is reasonable to assume that a similar behavior occurs for the (small) higher partial cross sections calculated with the complete interaction, Eq. (5′) or (8′). Thus we can write approximately that $\sigma_n\simeq(8/n^3)\sigma_2$ for $n\geqslant3$. As a result, the complete-capture cross section σ_c can be written to a good approximation as

$$\sigma_c=\sum_1^\infty\sigma_n\simeq\sigma_1\left[1+\frac{8\sigma_2}{\sigma_1}\left(\sum_1^\infty n^{-3}-1\right)\right],$$

or

$$\sigma_c=\sigma_1(1+1.616\sigma_2/\sigma_1). \tag{19}$$

The ratio (σ_c/σ_1) of the complete-capture cross section to the ground-state cross section given by Eq. (19) is plotted as the solid curve in Fig. 3. We see that for protons in hydrogen the captures into excited states account for about one-third of the total cross section in the energy region 25 to 100 kev, and about 17 percent at very high velocities as found by Oppenheimer.

Although we have not calculated in detail the partial cross sections for captures into excited states for values of Z' and Z other than unity, it is evident that the

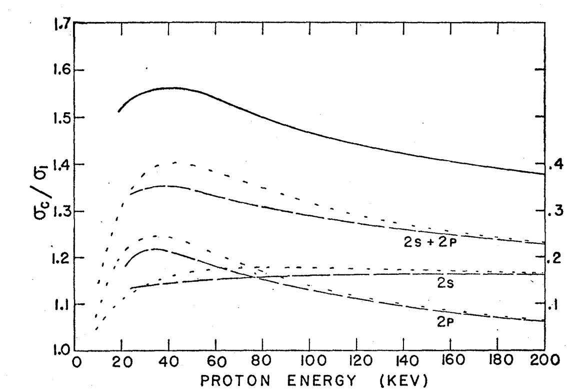

FIG. 3. (a) Ratios of the capture cross sections for the $2s$ and $2p$ states to the ground-state capture cross section for protons in hydrogen as functions of the incident proton energy (right-hand ordinate scale). Present results ——————. Brinkman-Kramers results - - - - -. (b) Ratio of the total capture cross section σ_c to the ground-state cross section σ_1 for protons in hydrogen as a function of incident proton energy (left-hand ordinate scale)—solid curve at the top.

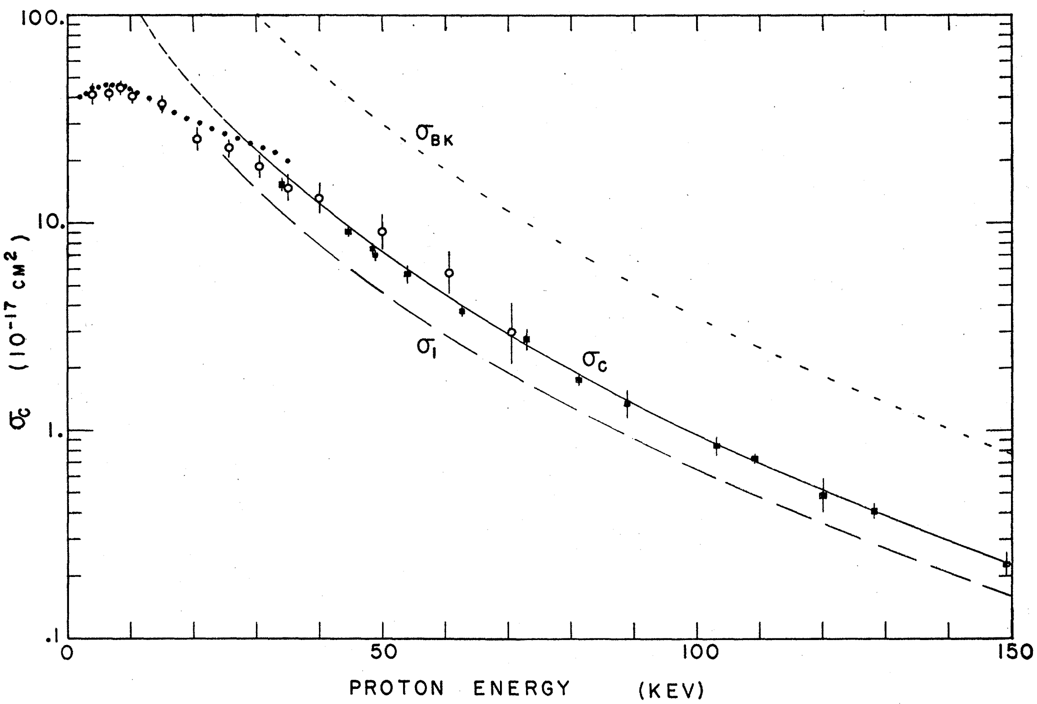

FIG. 4. Electron-capture cross section σ_c per atom for protons in hydrogen gas as a function of incident proton energy. The solid curve labeled σ_c is the theoretical result [Eq. (19)] for the total capture cross section; the dashed curve labeled σ_1 is the partial cross section for capture into the ground state [Eq. (17)], while the dotted curve labeled σ_{BK} is the corresponding cross section due to Brinkman and Kramers [Eq. (18)]. The experimental data are those of Keene (●), Ribe (■), and Whittier (○).

relative magnitude of these cross sections will not necessarily be similar to that found for $Z'=Z=1$. For example, for alpha-particles in hydrogen ($Z'=2$, $Z=1$), a case in which we have evaluated the ground-state cross section, it is expected that σ_2 may be comparable with, or even larger than, σ_1, at least at low velocities ($\hbar v/e^2 \sim 1$), since there is no difference in the binding energies ϵ and ϵ' [Eq. (11)] and a resonance phenomenon occurs.

5. COMPARISON WITH EXPERIMENT; REMARKS ON THE USE OF THE BORN APPROXIMATION

For protons in hydrogen gas the total cross section σ_c [Eq. (19)] is shown plotted in Fig. 4 along with the most recent experimental data of Keene,[6] Ribe,[7] and Whittier.[8] For energies below 25 kev, where $\hbar v/e^2$ is less than unity, the curve is shown dotted to indicate that the Born approximation certainly cannot be used at such low velocities. At energies above 50 kev the agreement between theory and experiment is excellent, and even down to 25 kev is adequate.

It is perhaps somewhat surprising that the Born approximation leads to such good agreement even for relatively low velocities. This may be merely another example of the well-known empirical rule that the Born approximation is, at least in atomic physics, often much better than it has any right to be. There is another qualitative rationalization for its reliability in this particular problem. The capture of an electron by a proton in hydrogen is the transfer of an electron from one neutral atom to form another neutral atom. The force exerted by a neutral atom on an incident charged particle is of relatively short range. This means that the incident (or outgoing) wave describing the relative motion will be distorted only at close distances, so that the approximation involved in using plane waves will not be too great. This is substantiated by an impact parameter calculation made by BK which showed that most of the captures occur for distances of the order of, or greater than, $a_0(e^2/\hbar v)$; when $(e^2/\hbar v) \ll 1$, the Born approximation is valid, while for $(e^2/\hbar v) \sim 1$ the captures occur at distances of the order of a_0 where the neutral atom's potential is weak. This argument would apply with equal validity to any singly ionized atom capturing an electron in any substance. It would indicate that the Born approximation would be less successful in the case of more highly ionized atoms, at least in the velocity region $\hbar v/e^2 \gtrsim 1$.

It should be noted that we have ignored molecular effects in these calculations. It is not clear to what extent the fact that the incident proton actually captures the electron from a hydrogen molecule instead of an isolated hydrogen atom will modify the present results. This aspect of the problem, as well as other examples of electron capture, are being studied at present.

The authors wish to thank Dr. Whittier for allowing them to quote his measurements of σ_c before publication. One of the authors (H.S.) was the recipient of a fellowship from the National Research Council of Canada during the year 1951–52, and wishes to acknowledge this support.

APPENDIX I

We wish to show that in the Born approximation the initial and final perturbations, V_i and V_f, lead to the same results for the capture cross section. Since the first terms in V_i [Eq. (5′)] and V_f [Eq. (8′)] are equal, all that is necessary is to prove that the following integrals (the two alternative forms of the BK matrix element) are equal:

$$I_{BK}=\int \exp(-i\mathbf{C}\cdot\mathbf{r}')\psi_f^*(-Z'e^2/r')d\mathbf{r}' \times\int \exp(i\mathbf{B}\cdot\mathbf{r})\psi_0 d\mathbf{r}, \quad (\mathrm{I}.1)$$

$$I_{BK}'=\int \exp(-i\mathbf{C}\cdot\mathbf{r}')\psi_f^* d\mathbf{r}'\int \exp(i\mathbf{B}\cdot\mathbf{r}) \times\psi_0(-Ze^2/r)d\mathbf{r}. \quad (\mathrm{I}.2)$$

Using the internal Hamiltonian H_C [Eq. (7)] with eigenvalue ϵ' [see Eq. (11)], we can write I_{BK} in the form

$$I_{BK}=\int \exp(-i\mathbf{C}\cdot\mathbf{r}')\left(\frac{\hbar^2}{2m_f}\nabla_{r'}^2-\epsilon'\right)\psi_f^* d\mathbf{r} \times\int \exp(i\mathbf{B}\cdot\mathbf{r})\psi_0 d\mathbf{r}.$$

Integration by parts twice leads to the result

$$I_{BK}=-\left(\frac{\hbar^2C^2}{2m_f}+\epsilon'\right)\int \exp(-i\mathbf{C}\cdot\mathbf{r}')\psi_f^* d\mathbf{r}' \times\int \exp(i\mathbf{B}\cdot\mathbf{r})\psi_0 d\mathbf{r}.$$

Or, introducing the Fourier transforms Eq. (14),

$$I_{BK}=-\left(\frac{\hbar^2C^2}{2m_f}+\epsilon'\right)\phi_0(\mathbf{B})\phi_f^*(\mathbf{C}). \quad (\mathrm{I}.3)$$

A similar treatment of I_{BK}' leads to the form

$$I_{BK}'=-\left(\frac{\hbar^2B^2}{2m_i}+\epsilon\right)\phi_0(\mathbf{B})\phi_f^*(\mathbf{C}). \quad (\mathrm{I}.4)$$

By means of the conservation of energy Eq. (11), it is easy to show that

$$(\hbar^2C^2/2m_f)+\epsilon'=(\hbar^2B^2/2m_i)+\epsilon,$$

so that $I_{BK}=I_{BK}'$, as required.

APPENDIX II

We will outline the evaluation of I_f' [Eq. (15)], using the capture into the ground state for $Z'=Z=1$ as an example. The method can be readily extended to arbitrary Z' and Z and captures into excited states. For the special case we have, using Eq. (16),

$$I_0'=\frac{32e^2}{\pi a_0^5}\int (a_0^{-2}+C^2+k^2-2\mathbf{k}\cdot\mathbf{C})^{-2} \times(a_0^{-2}+B^2+k^2-2\mathbf{k}\cdot\mathbf{B})^{-2}\frac{d\mathbf{k}}{k^2}. \quad (\mathrm{II}.1)$$

By introducing an auxiliary integral of the type used by Feynman[14] in another connection,

$$(ab)^{-2}=\int_0^1 \frac{6x(1-x)dx}{[ax+b(1-x)]^4},$$

we can transform (II.1) into the form

$$I_0'=\frac{32e^2}{\pi a_0^5}\int_0^1 6x(1-x)dx\int \frac{d\mathbf{k}}{k^2(k^2+\Delta-2\mathbf{k}\cdot\mathbf{q})^4}, \quad (\mathrm{II}.2)$$

where

$$\Delta=a_0^{-2}+xC^2+(1-x)B^2, \quad \text{and} \quad \mathbf{q}=x\mathbf{C}+(1-x)\mathbf{B}.$$

Differentiation of the integral,

$$\int \frac{d\mathbf{k}}{k^2(k^2+\Delta-2\mathbf{k}\cdot\mathbf{q})^2}=\pi^2(\Delta)^{-1}(\Delta-q^2)^{-\frac{1}{2}},$$

twice with respect to Δ, gives 6 times the integral over $\mathbf{k}$ space in (II.2). Consequently, I_0' becomes

$$I_0'=\frac{32\pi e^2}{a_0^5}\int_0^1 x(1-x)\left[\frac{2}{\Delta^3(\Delta-q^2)^{1/2}} +\frac{1}{\Delta^2(\Delta-q^2)^{3/2}}+\frac{\frac{3}{4}}{\Delta(\Delta-q^2)^{5/2}}\right]dx. \quad (\mathrm{II}.3)$$

Now $C^2=B^2$ in this case, so that Δ does not depend on x (this is true for the ground state capture for arbitrary Z' and Z) but does depend upon angle. $(\Delta-q^2)$ can be written as $(\alpha x^2+\beta x+\gamma)$ where α, β, γ depend upon Z', Z, and v, but not on angles. Thus (II.3) can be written as

$$I_0'=(32\pi e^2/a_0^5)(2\lambda_0\Delta^{-3}+\lambda_1\Delta^{-2}+\tfrac{3}{4}\lambda_2\Delta^{-1}), \quad (\mathrm{II}.4)$$

where

$$\lambda_n=\int_0^1 x(1-x)(\Delta-q^2)^{-n-\frac{1}{2}}dx =\int_0^1 x(1-x)(\alpha x^2+\beta x+\gamma)^{-n-\frac{1}{2}}dx,$$

---

[14] R. P. Feynman, Phys. Rev. **76**, 769 (1949), Eqs. (14a) and (15a) in the Appendix.

and

$$\Delta = a_0^{-2} + B^2 \simeq a_0^{-2}[1+(\hbar v/2e^2)^2(1+y)] + O(m/M),$$

where

$$y = [2A'AM\theta/(A'm+Am)]^2 = (\theta/\theta_0)^2.$$

We note that $I_{BK} \sim \Delta^{-3}$ [see Eq. (I.3) and Eq. (16)], so that the differential cross section, Eq. (9), is proportional to Δ^{-2} and higher reciprocal powers of Δ. Consequently, the angular distribution is largely confined to angles less θ_0, as discussed in Sec. 3.

When the integral I_0' is combined with I_{BK} in Eq. (9) and integrated over angles, the resulting total cross section can be expressed in the form of Eq. (17). For arbitrary Z' and Z and/or captures into excited states the general procedure is the same. However the algebraic complexity grows enormously, and it is advantageous in some instances to evaluate certain expressions (such as the integrals λ_n) numerically rather than analytically.

PHYSICAL REVIEW VOLUME 89, NUMBER 2 JANUARY 15, 1953

The $C^{12}(p,p)C^{12}$ Differential Cross Section*

H. L. JACKSON, A. I. GALONSKY, F. J. EPPLING, R. W. HILL, E. GOLDBERG,† AND J. R. CAMERON†
University of Wisconsin, Madison, Wisconsin
(Received July 28, 1952)

The $C^{12}(p,p)C^{12}$ differential cross section has been observed at four scattering angles by using differentially pumped gas targets of propane and ethylene. The scattering angles employed were 106.4, 127.8, 148.9, and 169.2 degrees in the center-of-mass system, and the energy range covered extended from 0.4 to 4.3 Mev. These measurements show the angular behavior of the previously discovered scattering anomalies at 0.46 and 1.7 Mev and give values of the absolute cross section accurate to within five percent. A careful search in three- and six-kev steps failed to reveal any indication of any hitherto unknown scattering resonances within the energy range surveyed.

I. INTRODUCTION

IN a previous article[1] we reported the results of a partial wave analysis of the differential cross section for elastically scattered protons from ordinary carbon obtained by Goldhaber and Williamson.[2] Although that analysis yielded definite values for the momenta and parities of the excited states of N^{13}, it also led to values of the resonant energies and widths which differed somewhat from those obtained from the proton capture data.[3] In addition, the experimental and calculated scattering cross sections could not be brought into agreement below one Mev. In the hope of removing these discrepancies, we have measured the $C^{12}(p,p)C^{12}$ differential cross section with increased accuracy at four scattering angles and have analyzed the new data by the same method. This paper describes the experiment and presents the data obtained. The following paper will deal with the analysis.

II. APPARATUS

Unless the absolute value of the scattering cross section is known to within a few percent, the phase shift analysis is extremely difficult and the results uncertain. In view of this fact, one of the major considerations in planning the experiment was the type of target to be used and the technique for measuring its thickness. Solid targets of the required purity and uniformity of thickness are difficult to prepare and are liable to additional carbon deposition during bombardment. The final decision was to employ a gas target, since its thickness depends only upon the dimensions of the counter slit system, the scattering angle, and the pressure and temperature of the scattering gas. All these quantities are readily measurable to a degree of accuracy somewhat higher than required for the projected experiment.

The gas actually used for most of the experiment was propane, although ethylene was used for some of the data at low bombarding energies, because it gives rise to less small angle scattering than propane at the same pressure. Being compounds of hydrogen and carbon only, these gases behave like pure carbon targets at scattering angles greater than 90 degrees. They have the further advantage of giving satisfactorily high yields of scattered protons at feasible chamber pressures and incident beam intensities.

Figure 1 is a cross-sectional view of the apparatus as seen from above. Its four principal components are the differential pumping column A, the scattering chamber B, the collector cup assembly C, and the two proportional counters together with their collimating slit systems D and D'. In operation the incident beam from

* Work supported by the AEC and the Wisconsin Alumni Research Foundation.
† AEC Predoctoral Fellow.
[1] H. L. Jackson and A. I. Galonsky, Phys. Rev. 84, 401 (1951).
[2] G. Goldhaber and R. M. Williamson, Phys. Rev. 82, 495 (1951).
[3] W. A. Fowler and C. C. Lauritsen, Phys. Rev. 76, 314 (1949); D. M. Van Patter, Phys. Rev. 76, 1264 (1949); and J. D. Seagrave, Phys. Rev. 84, 1219 (1951).