Strong Conservation Laws and Equations of Motion in Covariant Field Theories*

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(Received September 11, 1952)

The relationship between the covariance of a field theory and the equations of motion is discussed in both the Lagrangian and the Hamiltonian formalism. All theories whose field equations are derivable from a variational principle and are covariant under arbitrary (curvilinear) coordinate transformations possess Bianchi identities and, hence, "strong" conservation laws. Because the strong conservation laws are ordinary divergences equal to zero, whether or not the field equations are satisfied, there exist certain skew-symmetric expressions whose divergences yield the components of the energy-momentum tensor. These superpotentials, as the skew-symmetric expressions are called, enable us to write the energy and momentum content of the field as two-dimensional surface integrals. Also, by using the superpotentials together with the field equations, one can find certain surface integrals which are independent of the surface of integration and which yield the equations of motion for the singularities enclosed by the surface. If the Einstein-Infeld approximation method is applied in the general theory of relativity, the above surface integrals reduce to integrals which are equivalent to those used by Einstein and Infeld to obtain the equations of motion for the field sources.

In the Hamiltonian formalism the covariance of the theory is revealed in the existence of a number of constraints between the momenta and the field variables. Therefore, we examine the relationship between the constraints and the Bianchi identities, which lead to the strong conservation laws. We find that the first time derivative of the constraints leads to the existence of four linear combinations of field equations which are free of the time derivatives of the canonical field variables. These four expressions are the "secondary" constraints. The second time derivative of the primary constraints leads to the Bianchi identities in terms of the canonical field variables. Thus, we are able to establish the existence of the strong conservation laws in the Hamiltonian formalism and by the same arguments as in the Lagrangian formalism establish the existence of the superpotentials. The superpotentials are written out only for the gravitational theory.

I. INTRODUCTION

IN this paper we shall discuss the relationship between the transformation properties of a covariant field theory and the equations of motion for the field sources in both the Lagrangian and Hamiltonian formalisms. All theories whose field equations are derivable from a variational principle and are covariant with respect to transformation groups involving arbitrary *functions* possess "strong" conservation laws. These strong laws are certain (ordinary) divergences which vanish whether or not the field equations are satisfied. The existence of such strong conservation laws enables us to set up certain two-dimensional surface integrals which yield the "equations of motion." These equations will be the ponderomotive laws only when the covariant transformation group under consideration is that of arbitrary (curvilinear) coordinate transformations. In any case, the type of covariant group possessed by a given field theory will determine what is conserved "strongly," and hence what kind of "motion" is determined by the field equations alone.

Maxwell's theory of electromagnetism is covariant with respect to the Lorentz group which depends only on arbitrary *parameters*. One can construct conservation laws for the Maxwell field, but these laws will hold only when the field equations are satisfied. Furthermore, the Maxwell equations do not contain the mass of the charged particles in the field; therefore, the ponderomotive laws—the Lorentz equations—represent an assumption distinct from and outside the framework of the field equations. On the other hand, the field equations of the electromagnetic theory are also covariant with respect to the gauge group, which depends on one arbitrary *function*. There exists, therefore, one strong conservation law for the Maxwell field and that is the conservation law for charge. This conservation law tells us that the charged particles producing the field must move so that the total charge is conserved. Beyond this statement, the field equations by themselves leave the motion of the particles arbitrary. Moreover, it is well known that the total charge in a given region of space may be calculated by means of a two-dimensional surface integral over a surface which encloses that region. Thus, the time rate of change of the charge enclosed by a surface may also be calculated by means of a surface integral. If the field equations are satisfied on that surface (i.e., if the charge and current distributions are zero on the surface), then the total charge enclosed is constant in time.

In a similar fashion, theories which are covariant with respect to coordinate transformations possess four strong conservation laws as they involve four arbitrary functions. These laws are the conservation laws for energy and momentum. Because these are strong laws, the energy and momentum can be calculated by means of two-dimensional surface integrals, and the equations of motion for the particles enclosed by the surface can also be calculated by means of surface integrals. The methods we use to prove this result can be applied to more general transformations to obtain corresponding conservation laws and equations of motion. However, in this paper we shall restrict our attention to coordinate transformations.

The general theory of relativity is the only significant field theory known today whose field equations deter-

* This work was supported by the ONR. This paper includes the results of a dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy at Syracuse University.
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mine the equations of motion of the particles producing the field.[1–3] As a result of the covariance of the theory of gravitation under arbitrary coordinate transformations, the field equations are nonlinear and, moreover, the field equations satisfy four differential identities (the Bianchi identities). The nonlinearity of the field equations is essential for the explicit description of the interactions between particles. If we have linear field equations, then the sum of any number of solutions is again a solution of the field equations while the world lines of the particles producing the fields are unaffected by the summation process. The role played by the Bianchi identities is very complex, but for our purposes it is sufficient that the existence of the Bianchi identities guarantees the existence of the strong conservation laws.[4] As we have noted above, the existence of these strong conservation laws permits us to determine the equations of motion from the field equations by means of surface integrals.

In their solution of the problem of motion in the gravitational theory, Einstein and Infeld also use surface integrals to obtain the equations of motion.[1–3] However, in order to set up their surface integrals they separate the gravitational metric into the metric for flat space, which is assumed to be the exact field at spatial infinity, and a part representing the deviation from flat space. With this substitution certain of the field equations consist of linear terms which may be written as the spatial divergence of an antisymmetric form, plus a form containing primarily nonlinear terms. Because of the skew-symmetry of the linear form, the two-dimensional surface integral of the nonlinear form vanishes whenever the field equations are satisfied on the surface of integration according to Stokes' theorem. These surface integrals yield the equations of motion. This derivation of the surface integrals depends on the specific structure of the gravitational field equations. There is no indication as to how these surface integrals are related to the covariance of the theory.

Bergmann has shown that any field theory whose field equations are derivable from a variational principle and whose field equations are covariant under arbitrary coordinate transformations will possess Bianchi identities and, hence, strong conservation laws.[5] Because of the existence of the strong conservation laws there exist certain skew-symmetric forms, which we shall call "superpotentials." These superpotentials enable us to find linear combinations of the field equations which may be written as the spatial divergence of an antisymmetric form, plus other terms. Both types of terms are related to the energy and momentum of the field. Thus, the surface integrals can be set up independently of any splitting-up of field variables or approximation methods. Furthermore, the fact that these surface integrals can be related to the energy-momentum tensor adds to their significance in determining the equations of motion. If we apply the Einstein-Infeld approximation, our surface integrals are equivalent to theirs.

The usual approach in quantizing a field theory is first to cast the classical theory into the Hamiltonian formalism. From the Poisson brackets of the field variables one obtains the commutation relations to be obeyed by the quantum-mechanical operators. That is why we have formulated the strong conservation laws and the surface integrals for the equations of motion also in terms of the canonical variables. Inasmuch as covariance in the canonical formalism expresses itself through the primary and secondary constraints, it is to be expected that the canonical constraints are related directly to the conservation laws. We shall show this relationship explicitly.

II. THE LAGRANGIAN FORMALISM

2.1 Strong Conservation Laws

In this section we shall summarize previous results (I) needed here. The field equations are assumed to be derivable from a variational principle whose Lagrangian density is a function of the field variables and their first derivatives,

$$\text{(I-2.1)} \qquad \delta I=0, \quad I=\int_V L(y_A, y_{A,\mu})d^4x. \qquad (2.1)$$

The field variables y_A are assumed to transform under arbitrary coordinate transformations according to the law

$$\text{(I-2.3)} \qquad \bar{\delta} y_A = F_{A\mu}{}^{B\nu} y_B \xi^\mu{}_{,\nu} - y_{A,\mu}\xi^\mu, \qquad (2.2)$$

where the ξ^μ are the descriptors of the transformation (for a coordinate transformation they are merely the infinitesimal changes of the coordinates) and the $F_{A\mu}{}^{B\nu}$ are certain constants in any given theory. If the field equations,

$$\text{(I-2.2)} \qquad \begin{aligned} &L^A \equiv \partial^A L - (\partial^{A\rho}L)_{,\rho}=0, \\ &\partial^A L \equiv \partial L/\partial y_A, \quad \partial^{A\rho}L \equiv \partial L/\partial y_{A,\rho}, \end{aligned} \qquad (2.3)$$

are to be covariant under the above transformations, the density function, L, must transform at most by the addition of a total divergence. In order to satisfy this condition on the Lagrangian density, the field equations must satisfy the four Bianchi identities,

$$\text{(I-3.3)} \qquad (F_{A\mu}{}^{B\nu} y_B L^A)_{,\nu} + y_{A,\mu} L^A \equiv 0. \qquad (2.4)$$

From the Bianchi identities one immediately obtains the

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[1] Einstein, Infeld, and Hoffman, Ann. Math. **39**, 66 (1938).
[2] A. Einstein and L. Infeld, Ann. Math. **41**, 455 (1940).
[3] A. Einstein and L. Infeld, Can. J. Math. **1**, 209 (1949).
[4] There is only one case where this is not so. If the infinitesimal transformation of the field variables does not involve derivatives of the "descriptors" of the transformation, then there will exist Bianchi identities but no conservation laws. This case is of no interest to us and we shall not consider it further.
[5] P. G. Bergmann, Phys. Rev. **75**, 680 (1949). This paper will be referred to as I and references to equations in this paper will be made as (I-2.1), etc.

strong conservation laws:

$$\text{(I-3.11)}\quad \begin{aligned} &T_\mu{}^\nu{}_{,\nu}\equiv 0,\\ &T_\mu{}^\nu = -F_{A\mu}{}^{B\nu}y_B L^A - \delta_\mu{}^\nu L + \partial^{A\nu} L y_{A,\mu}. \end{aligned} \tag{2.5}$$

When the field equations are satisfied, $L^A=0$, these strong laws reduce to the usual "weak" conservation laws found in all field theories derived from a variational principle:

$$\text{(I-3.8)}\quad \begin{aligned} &t_\mu{}^\nu{}_{,\nu}=0,\\ &t_\mu{}^\nu = -\delta_\mu{}^\nu L + \partial^{A\nu} L y_{A,\mu}. \end{aligned} \tag{2.6}$$

All of the above results, with minor changes, hold equally as well for the gauge transformation in the theory of electromagnetism. In that case we have

$$\bar{\delta}\varphi_\rho = \xi_{,\rho}, \tag{2.7}$$

and the "Bianchi" identity and the strong conservation law are one:

$$T^\rho{}_{,\rho}\equiv L^\rho{}_{,\rho}\equiv 0. \tag{2.8}$$

Equation (2.8) leads directly to the conservation law for charge.

2.2 Superpotentials

Although we ordinarily confine our attention to those domains where the field equations are satisfied, the existence of the strong conservation laws is of great importance. It is well known that whenever the complete divergence of a "tensor" form[6] vanishes, then the form itself may be represented as the complete divergence of an antisymmetric form whose rank is higher by one. (This corresponds to writing B as the curl of a vector potential A in the electromagnetic theory.) Therefore, we may write

$$T_\mu{}^\nu = U_\mu{}^{[\nu\sigma]}{}_{,\sigma}. \tag{2.9}$$

The square brackets around the superscripts ν and σ indicate that $U_\mu{}^{[\nu\sigma]}$ is antisymmetric in these indices. The existence of these "superpotentials," as we shall call this antisymmetric tensor-form, will enable us to obtain the surface integrals for the equations of motion.

Before we go into a discussion of the equations of motion, it is interesting to note that the superpotentials permit us to write the total energy and momentum in the field as surface integrals. As was mentioned in the introduction, this possibility corresponds to Gauss' law in electromagnetic theory. The total energy and momentum density in the field is represented by $T_\mu{}^4$. From Eq. (2.9) we find that

$$T_\mu{}^4 = U_\mu{}^{[4s]}{}_{,s},\quad (s=1,2,3). \tag{2.10}$$

[6] By a tensor-form we mean a set of quantities whose components are distinguished by indices but which may not have simple transformation properties, e.g., the ordinary divergence of a tensor.

If P_μ is the total energy and momentum in a given three-dimensional volume V with the surface S, then

$$P_\mu = \int_V T_\mu{}^4 dV = \oint U_\mu{}^{[4s]} n_s dS, \tag{2.11}$$

where the n_s are the components of the unit normal vector to the surface S. The significance of this statement is that one need not integrate across the singular regions of the field in order to calculate the energy and momentum content of the field including the singularities. Thus the total energy and momentum of the field, including the particles in the field may be calculated without the necessity of making an arbitrary separation of the total field into external and self-fields as is necessary in the theory of electromagnetism. Clearly, the time rate of change of energy and momentum contained in a volume V is also given, from Eq. (2.11), by a surface integral:

$$dP_\mu/dt = \oint U_\mu{}^{[4s]}{}_{,4} n_s dS. \tag{2.12}$$

This result is of interest in the investigation of (gravitational) radiation.

2.3 Equations of Motion

In the theory of gravitation the problem of motion was solved by means of a particular approximation method which allows certain of the field equations to be written as the sum of two terms, one of which is the spatial divergence of an antisymmetric form.[1–3] We shall now find those linear combinations of the field equations having this form, without splitting the field variables or applying an approximation procedure.

From Eqs. (2.5), (2.6), and (2.9) we find that the following linear combinations of field equations may be written in terms of the superpotentials:

$$F_{A\mu}{}^{B\nu}y_B L^A \equiv U_\mu{}^{[\sigma\nu]}{}_{,\sigma} + t_\mu{}^\nu = 0. \tag{2.13}$$

Let us set ν equal to s, $(1,\cdots,3)$, and in the potentials separate the derivative with respect to x^4 from the spatial derivatives:

$$F_{A\mu}{}^{Bs}y_B L^A \equiv U_\mu{}^{[rs]}{}_{,r} + U_\mu{}^{[4s]}{}_{,4} + t_\mu{}^s = 0. \tag{2.14}$$

It is evident that these linear combinations of field equations have just the required property. If we now take the divergence of Eq. (2.14), the first term vanishes identically because of the anti-symmetry of r and s in $U_\mu{}^{[sr]}$, and because of the symmetry of r and s in the derivatives. Thus the divergence reduces to

$$(F_{A\mu}{}^{Bs}y_B L^A)_{,s} \equiv (U_\mu{}^{[4s]}{}_{,4} + t_\mu{}^s)_{,s} = 0. \tag{2.15}$$

Equation (2.15) tells us that the closed-surface integral over $(U_\mu{}^{[4s]}{}_{,4} + t_\mu{}^s)$ will be independent of the surface as long as the field equations are satisfied at least on the surface of integration.

Therefore, we form the two-dimensional surface integral over Eq. (2.14):

$$\oint F_{A\mu}{}^{Bs} y_B L^A n_s dS \equiv \oint U_\mu{}^{[rs]}{}_{,r} n_s dS + \oint (U_\mu{}^{[4s]}{}_{,4} + t_\mu{}^s) n_s dS = 0. \quad (2.16)$$

Here, the first integral vanishes identically because of what Einstein calls "the lemma."[1–3] The essential argument is that the spatial divergence of any antisymmetric form, whose antisymmetric indices are spatial, may always be represented as the curl of a vector, the remaining indices being disregarded. The integral of a curl over a closed surface always vanishes. Thus, from Eq. (2.16) we are left with

$$\oint F_{A\mu}{}^{Bs} y_B L^A n_s dS \equiv \oint (U_\mu{}^{[4s]}{}_{,4} + t_\mu{}^s) n_s dS = 0. \quad (2.17)$$

If the surface does not enclose any singularities, then this result is trivial. Applying Gauss' theorem, and remembering Eq. (2.15), the surface integrals give an identically vanishing result. When singularities are enclosed by the surface, however, Gauss' theorem cannot be applied to yield this trivial result. The surface integrals need no longer vanish identically. Equation (2.15) tells us that no surface integral of the form (2.17) can depend on the shape of the surface as long as the field equations are satisfied on that surface. If two different surfaces enclose the same singularities, then Gauss' theorem applied to the region between the surfaces will give an identically vanishing result. Therefore, the surface integrals (2.17) can be functions only of the coordinates and the time derivatives of the coordinates of any singularities enclosed by the surface. Hence, these surface integrals must yield the equations of motion for the singularities enclosed.

It is important to remember that the surface integrals, Eq. (2.17), would have no meaning were it not for the existence of the strong conservation laws. A surface integral taken over a linear combination of the full field equations would vanish on a cap, as well as on a closed surface, as long as the field equations are satisfied on the surface of integration. By means of the strong conservation laws, and the consequent skew-symmetric superpotentials, we have shown that the closed surface integral over a quantity which is not a linear combination of the full field equations vanishes. Moreover, the existence of the superpotentials enabled us to prove that these closed surface integrals are independent of the surface of integration in the sense of the previous paragraph.

Indeed, one can obtain these surface integrals directly from the statement of the strong conservation laws by making use of the existence of the superpotentials and the condition that the field equations be satisfied on the surface of integration. Separating the derivative with respect to x^4 in the strong conservation laws, Eq. (2.5), we have:

$$T_\mu{}^4{}_{,4} + T_\mu{}^s{}_{,s} \equiv 0. \quad (2.18)$$

Taking the volume integral of the above equation and applying Gauss' theorem to the second term, we find that,

$$\int_V T_\mu{}^4{}_{,4} dV + \oint T_\mu{}^s n_s dS \equiv 0. \quad (2.19)$$

We can now substitute for T^4 from Eq. (2.10) and thus obtain an expression involving only a surface integral:

$$\oint (U_\mu{}^{[4s]}{}_{,4} + T_\mu{}^s) n_s dS \equiv 0. \quad (2.20)$$

If the field equations are satisfied on the surface of integration, then Eq. (2.20) reduces to (2.17). The only advantage of the original derivation of the surface integrals is in proving that these integrals really give the equations of motion and for comparison with the method of Einstein and Infeld.

The surface integrals can be generalized to the case where matter is represented by a tensor field P^A rather than by singularities in the field. The P^A become the right-hand sides of the field equations; that is,

$$L^A = P^A. \quad (2.21)$$

In this case, Eqs. (2.15) and (2.17) are replaced by

$$(U_\mu{}^{[4s]}{}_{,4} + t_\mu{}^s)_{,s} = (F_{A\mu}{}^{Bs} y_B P^A)_{,s}, \quad (2.15')$$

and

$$\oint (U_\mu{}^{[4s]}{}_{,4} + t_\mu{}^s) n_s dS = \oint F_{A\mu}{}^{Bs} y_B P^A n_s dS. \quad (2.17')$$

Clearly these integrals are no longer independent of the surface. If the distribution of matter is not zero on the surface, then changing the surface will in turn change the amount of matter enclosed by the surface. In this case, the equations of motion cannot be independent of the surface. These surface integrals depend on the distribution of matter through the field variables themselves as well as through the P^A explicitly. The dependence of the field variables on the distribution of matter is determined by the field equations, Eq. (2.21), and thus by the distribution of matter throughout all space. The explicit dependence of the equations of motion on the P^A, however, is determined solely by the distribution of matter on the surface. This situation is to be expected as the change in the energy and momentum contained within the surface will depend on the flux through the surface. The analogous situation occurs in the theory of electromagnetism with respect to the conservation law for charge.

It is easy to see that no new information can be obtained from Eq. (2.13) by setting ν equal to 4. The same surface integrals result after taking a time derivative and substituting from the strong conservation laws for those terms which are not a spatial divergence. Thus, once having satisfied the surface integrals (2.17), these additional relations are empty.

2.4 Theory of Gravitation

In the theory of gravitation, Eq. (2.14) becomes

$$-2(-g)^{\frac{1}{2}}G_{\mu}{}^{s}\equiv U_{\mu}{}^{[rs]}{}_{,r}+U_{\mu}{}^{[4s]}{}_{,4}+t_{\mu}{}^{s}=0, \quad (2.22)$$

where[7,8]

$$U_{\mu}{}^{[\nu\lambda]}=\tfrac{1}{2}(-g)^{\frac{1}{2}}\{\delta_{\mu}{}^{\nu}(g^{\lambda\tau}g^{\rho\sigma}-g^{\lambda\sigma}g^{\rho\tau}) -\delta_{\mu}{}^{\lambda}(g^{\nu\tau}g^{\rho\sigma}-g^{\nu\sigma}g^{\rho\tau}) +\delta_{\mu}{}^{\rho}(g^{\nu\tau}g^{\lambda\sigma}-g^{\lambda\tau}g^{\nu\sigma})\}g_{\rho\sigma,\tau}. \quad (2.23)$$

Following Einstein and Infeld, we write the gravitational metric as the sum of the metric for flat space and a term representing the deviation from flat space:

$$g_{\mu\nu}=\eta_{\mu\nu}+h_{\mu\nu}, \qquad \eta_{44}=1, \quad \eta_{mn}=-\delta_{mn}, \quad \eta_{4m}=0. \quad (2.24)$$

It is convenient to introduce a linear combination of the $h_{\mu\nu}$ as the field variables,

$$\gamma_{\mu\nu}=h_{\mu\nu}-\tfrac{1}{2}\eta_{\mu\nu}\eta^{\rho\sigma}h_{\rho\sigma}. \quad (2.25)$$

Substituting Eqs. (2.24) and (2.25) into (2.23) we find that $U_{\mu}{}^{[rs]}{}_{,r}$ becomes

$$U_{\mu}{}^{[sr]}{}_{,r}=(\gamma_{\mu r,s}-\gamma_{\mu s,r}+\delta_{\mu r}\gamma_{sm,m}-\delta_{\mu s}\gamma_{rm,m})_{,r} +(\delta_{\mu s}\gamma_{4r,4}-\delta_{\mu r}\gamma_{4s,4})_{,r}+\text{nonlinear terms}. \quad (2.26)$$

The first group of linear terms on the right-hand side of Eq. (2.26) are just those terms which Einstein and Infeld separate from the field equations in order to obtain their surface integrals. The second group is not separated out because in their approximation procedure these terms behave like the nonlinear terms. Therefore, if we apply the Einstein-Infeld approximation procedure, in each step of the approximation we obtain the same surface integrals modulo terms which vanish because of "the lemma." The reason we do not obtain the identical surface integrals right away is that Einstein and Infeld only separate out linear terms, whereas we separate out all the terms in the field equations which are the spatial divergences of skew-symmetric expressions. Thus, one sees that these surface integrals, which result from the covariance of the theory, really give the equations of motion for the field sources.

[7] P. Freud, Ann. Math. **40**, 417 (1939).
[8] H. Zatskis, Phys. Rev. **81**, 1023 (1951).

III. THE HAMILTONIAN FORMALISM

3.1 Primary Constraints

In order to show the relationship between the constraints and the strong conservation laws in the Hamiltonian formalism, we shall require an explicit expression for the Hamiltonian so that we may calculate Poisson brackets of the constraints and the Hamiltonian. Such relations are meaningless without definite expressions for the Hamiltonian and the constraints. Therefore, in what follows we shall assume that our Lagrangian is an integral over a density function L which is homogeneous quadratic in the first derivatives of the field variables:

$$L=\Lambda^{A\rho B\sigma}(y_C)y_{A,\rho}y_{B,\sigma}. \quad (3.1)$$

The covariance of the theory under arbitrary coordinate transformations leads to the Bianchi identities, Eq. (2.4), which place certain restrictions on the coefficients $\Lambda^{A\rho B\sigma}$. After substituting the above expression for the Lagrangian into Eq. (2.4) and separately considering the coefficients of various differential orders, we find that

$$\begin{aligned}
&\text{(a)}\ u_{\mu A}{}^{\rho}L^{AC\sigma\tau}+u_{\mu A}{}^{\sigma}L^{AC\tau\rho}+u_{\mu A}{}^{\tau}L^{AC\rho\sigma}\equiv 0,\\
&\text{(b)}\ \delta_{\mu}{}^{\sigma}L^{AC\rho\tau}+\partial^{C}(u_{\mu B}{}^{\sigma}L^{AB\rho\tau})-u_{\mu B}{}^{\tau}\Gamma^{\rho\sigma B,AC} -u_{\mu B}{}^{\rho}\Gamma^{\tau\sigma B,AC}\equiv 0,\\
&\text{(c)}\ \delta_{\mu}{}^{\nu}\Gamma^{\rho\sigma C,AB}+\delta_{\mu}{}^{\sigma}\Gamma^{\nu\rho B,CA}+\delta_{\mu}{}^{\rho}\Gamma^{\sigma\nu A,BC} +\partial^{C}(u_{\mu D}{}^{\nu}\Gamma^{\rho\sigma D,AB})+\partial^{B}(u_{\mu D}{}^{\sigma}\Gamma^{\nu\rho D,CA}) +\partial^{A}(u_{\mu D}{}^{\rho}\Gamma^{\sigma\nu D,BC})\equiv 0.
\end{aligned} \quad (3.2)$$

$$u_{\mu A}{}^{\nu}\equiv F_{A\mu}{}^{B\nu}y_B,$$
$$L^{AB\rho\sigma}\equiv\Lambda^{A\rho B\sigma}+\Lambda^{A\sigma B\rho},$$
$$\Gamma^{\rho\sigma B,AC}\equiv\partial^{B}\Lambda^{A\rho C\sigma}-\partial^{A}\Lambda^{B\rho C\sigma}-\partial^{C}\Lambda^{A\rho B\sigma}.$$

These three sets of relations will be frequently used in the following calculations.

One can consider the transition to the Hamiltonian formalism as a Legendre transformation.[9] Therefore, the following two equations are valid whether or not the field equations are satisfied:

$$\pi^A=\partial L/\partial y_{A,4}, \quad (3.3)$$

$$y_{A,4}=\partial H/\partial\pi^A. \quad (3.4)$$

Equation (3.3) is merely the definition of the momenta canonically conjugate to the field variables y_A; on the

[9] The function $L(q_k,\dot{q}_k)$ is considered as the generator of the transformation to the coordinates q_k and p_k by defining p_k as $\partial L/\partial\dot{q}_k$. One then asks for that function $H(q_k,p_k)$ which generates the inverse transformation $\dot{q}_k\partial H/\partial p_k$. By considering the total differential dL, or dH, one finds that $H=\Sigma_k\, p_k\dot{q}_k-L$ which is the usual definition of the Hamiltonian.

other hand, Eq. (3.4) is one-half of the Hamiltonian field equations. In the Lagrangian formalism, when we consider situations in which the field equations are not satisfied, we set the right-hand sides of the field equations equal to expressions P^A, which represent the distribution of matter. When we pass to the Hamiltonian formalism, the number of field equations is doubled. The question arises whether all of them may be considered eligible for nonvanishing right-hand sides. Setting up the Hamiltonian formalism through a Legendre transformation shows that only half of the field equations may acquire right-hand sides:

$$-(\dot{\pi}^A+\delta H/\delta y_A)=P^A. \tag{3.5}$$

Setting $\rho=\sigma=\tau=4$ in Eq. (3.2a), one finds that the $u_{\mu A}{}^4$ are null vectors of Λ^{A4B4}. Because of the existence of these null vectors, there are four linear combinations of field equations in the Lagrangian formalism that are free of second time derivatives of the field variables. This situation is reflected in the Hamiltonian formalism in the existence of the primary constraints which are four linear combinations of the momenta and field variables not involving the $y_{A,4}$. Multiplying Eq. (3.3) by the $u_{\mu A}{}^4$ we have

$$g_\mu \equiv u_{\mu A}{}^4\bar{\pi}^A=0, \quad \bar{\pi}^A=\pi^A-2\Lambda^{A4Bs}y_{B,s}. \tag{3.6}$$

Thus, although the π^A are uniquely defined in terms of the field variables and their derivatives, the $y_{A,4}$ are not uniquely determined by the canonical field variables. They are determined only up to a linear combination of the null vectors $u_{\mu A}{}^4$. This lack of uniqueness in the $y_{A,4}$ is reflected in the corresponding lack of uniqueness of the Hamiltonian. All generally covariant theories possess this property. The Hamiltonian density (if we do not introduce parameters) is,[10]

$$H=\tfrac{1}{4}E_{AB}\bar{\pi}^A\bar{\pi}^B-\Lambda^{ArBs}y_{A,r}y_{B,s}+w^\rho g_\rho, \tag{3.7}$$

where the w^ρ are four arbitrary functions of the field variables. E_{AB} is the quasi-inverse of Λ^{A4B4} and is defined by the relationship

$$E_{AC}\Lambda^{C4D4}E_{DB}=E_{AB}. \tag{3.8}$$

Therefore,

$$E_{AC}\Lambda^{C4B4}=\delta_A{}^B-v^{B\mu}u_{\mu A}, \tag{3.9}$$

where

$$E_{AB}v^{B\mu}=0,$$

and

$$u_{\mu A}v^{A\nu}=\delta_\mu{}^\nu.$$

Since the E_{AB} is only a quasi-inverse, it is not quite uniquely determined by the defining relation above. Therefore, the last term in Eq. (3.7) is redundant. However, in practice one chooses a particular form for the quasi-inverse and any further changes in the Hamiltonian are then represented as linear combinations of primary constraints as indicated in Eq. (3.7). The null vectors $v^{A\mu}$ exhibit a corresponding lack of uniqueness, although for a particular choice of inverse the null vectors are uniquely defined by the relations (3.9). In the following, we assume that a particular choice of E_{AB} has been made. Therefore, there is no redundancy in Eq. (3.7) and the null vectors are unique.

3.2 Secondary Constraints

Since we are looking for the strong conservation laws, which are satisfied even when the field equations are not, we must use the field equations with right-hand sides. In order to set these up consistently, we shall redefine the Lagrangian density as

$$L^*=L-y_AP^A, \tag{3.10}$$

where L is the usual density function. In the variational principle the P^A are considered given and are not to be varied. The transformation law for the y_A is unaffected by this change and that of the P^A is,

$$\bar{\delta}P^B=-F_{A\mu}{}^{B\nu}P^A\xi^\mu{}_{,\nu}-(P^B\xi^\mu)_{,\mu}. \tag{3.11}$$

If one carries out the argument normally leading to the Bianchi identities, one finds that the requirement that $\bar{\delta}L^*$ be a divergence leads to the usual identities without P^A appearing at all. The terms involving the P^A either cancel outright or go together to form a total divergence. Clearly, the definition of the momenta, π^A, is not altered and, therefore, the primary constraints are the same as in the usual case. However, the Hamiltonian density is changed by the addition of the term y_AP^A:

$$H^*=H+y_AP^A. \tag{3.12}$$

In this Hamiltonian the P^A are not dynamical variables, and thus this theory does not by itself lead to differential equations for the P^A. It is clear that the new Hamiltonian density H^* is determined, as is the old one, up to a linear combination of the primary constraints.

Following Anderson and Bergmann[11] we introduce the function space whose "points" are arbitrary sets of functions $y_A(x^1, \cdots, x^4)$ and call it the "configuration" space of the Lagrangian theory, Σ_c. If we double the dimensionality of this space by adjoining to it the second set of arbitrary functions $\pi^A(x^1, \cdots, x^4)$, we get the phase space of the corresponding canonical formalism, Σ. However, because ofthe existence of the primary constraints, not all points of Σ correspond to possible situations in the Lagrangian theory. We must restrict ourselves to that sub-space Σ_l of Σ where the primary constraints, Eq. (3.6) are satisfied. The fact that the primary constraints are satisfied in Σ_l for all time places certain restrictions on the field equations which are called "secondary" constraints. The question arises as to how many such secondary constraints exist. In

[10] R. Penfield, Phys. Rev. 84, 737 (1951).

[11] J. Anderson and P. G. Bergmann, Phys. Rev. 83, 1018 (1951). This paper will be referred to as II and reference to equations in this paper will be made as (II-3.2), etc.

general, the answer depends on the transformation law for the field variables. For the case which we consider there are only four secondary constraints. We shall delay the proof of this statement until the next section. Here we shall calculate the secondary constraints.

Consider a three-dimensional integral over the primary constraints g_μ with arbitrary weight functions σ^μ. Obviously, this integral vanishes in the space Σ_l,

$$\mathcal{G}=\int \sigma^\mu g_\mu d^3x \overset{l}{=} 0, \tag{3.13}$$

and, conversely, if it vanishes for all possible σ^μ, then automatically the primary constraints are required to be satisfied. (The symbol l above an equal sign means that the relationship is valid in Σ_l.) Since the primary constraints are satisfied in Σ_l for all time, it follows that the time derivatives of the g_μ must also vanish in Σ_l. Thus the time derivatives of $\mathcal{G}$ must likewise vanish in Σ_l. Because of the field equations, Eqs. (3.4) and (3.5), we have:

$$\frac{d\mathcal{G}}{dt}=\frac{\partial\mathcal{G}}{\partial t}+(\mathcal{G}, \mathcal{H}^*) \overset{l}{=} 0, \tag{3.14}$$

where

$$\mathcal{H}^*=\int H^* d^3x.$$

$(\mathcal{G}, \mathcal{H}^*)$ is the Poisson bracket of the two functionals $\mathcal{G}$ and $\mathcal{H}^*$.[12]

From Eq. (3.12) it is evident that if $\mathfrak{F}(y_A, \pi^A)$ is an arbitrary functional, then

$$(\mathfrak{F}, \mathcal{H}^*)=(\mathfrak{F}, \mathcal{H})-\int \frac{\delta \mathfrak{F}}{\delta \pi^A} P^A d^3x. \tag{3.15}$$

Using the above relation, Eq. (3.14) becomes

$$\int \left\{ \dot{\sigma}^\mu g_\mu + \sigma^\mu \left[(g_\mu, H) - \frac{\partial g_\mu}{\partial \pi^A} P^A \right] \right\} d^3x \overset{l}{=} 0. \tag{3.16}$$

Since the weight functions σ^μ are arbitrary, the coefficients of the differentiated and the undifferentiated weight functions must vanish separately. Clearly, the coefficient of $\dot{\sigma}^\mu$ vanishes in Σ_l. In order to identify the terms multiplied by σ^μ, we must evaluate the Poisson bracket (g_μ, H), which we shall denote by L_μ. We have:

$$\begin{aligned} L_\mu \equiv & (u_{\mu A}{}^s \pi^A)_{,s} + \delta_\mu{}^4 H + \delta_\mu{}^s y_{A,s} \pi^A \\ & + \delta_\mu{}^4 \Lambda^{AmBn} y_{A,m} y_{B,n} - 2\delta_\mu{}^m \Lambda^{A4Bn} y_{A,m} y_{B,n} \\ & + u_{\mu C}(\Gamma^{nmC,AB} y_{A,n} y_{B,m} - 2\Lambda^{AnCm} y_{A,mn}) \\ & + f_\mu(g_\rho). \end{aligned} \tag{3.17}$$

[12] For the definition of the Poisson brackets of functionals see P. G. Bergmann and J. H. M. Brunings, Revs. Modern Phys. 21, 480 (1949).

In L_μ, above, $f_\mu(g_\rho)$ represents many factors of g_ρ and $g_{\rho,s}$. We have chosen to write them in this way because in Σ_l these terms make no new contribution and in taking further time derivatives of $\mathcal{G}$ these terms will never contribute anything new. Therefore, the coefficient of σ^μ in Eq. (3.16) is,

$$(g_\mu, H^*) \equiv L_\mu - u_{\mu A}{}^4 P^A \overset{l}{=} 0. \tag{3.18}$$

By a direct comparison with the field equations, Eq. (3.5), we find that

$$L_\mu \equiv -u_{\mu A}{}^4(\delta H/\delta y_A + \dot{\pi}^A) + \dot{g}_\mu. \tag{3.19}$$

From the above, it is evident that there exist four linearly independent combinations of the field equations which are free of the $\dot{\pi}^A$ and $\dot{y}_A$. These four equations correspond to the four linear combinations of field equations in the Lagrangian formalism which are free of second time derivatives of the field variables. Because the L_μ are four expressions involving the momenta and the field variables, but no time derivatives, the relationships (3.18) are called "secondary" constraints.

In the theory of the electromagnetism a similar situation arises. There the primary constraint is $\pi^4=0$, and the secondary constraint is $\pi^s{}_{,s}+c\rho=0$, where ρ is the charge density. π^s is equal to the negative of the electric field strength divided by $4\pi c$, and therefore the secondary constraint corresponds to the Lagrangian field equation $\operatorname{div}\mathbf{E}=4\pi\rho$.

3.3 The Bianchi Identities

Before proceeding to take further time derivatives, we shall stop to examine how far this process need be carried. According to II the number of secondary constraints in any theory is P times the number of arbitrary functions in the transformation law for the field variables, where P is the highest order of differentiation occurring in the transformation law. Therefore, we should expect to find only four secondary constraints in a coordinate covariant theory such as we are considering. Our case differs from the situations considered in II in that we are dealing with the possibility that the field equations may not be satisfied. However, the argument is not altered by this fact and, therefore, we shall not repeat the whole argument. Instead we shall merely indicate where changes need to be made in order to include the possibility that the field equations need not be satisfied. We shall continue to restrict our discussion to coordinate transformations, although the extension to the more general transformations considered in II is trivial. In order to facilitate comparison with II, we shall place the number of the equation in II which corresponds to the equations we write to the left of our equations.

In II it was shown that the generating density function for an infinitesimal invariant transformation can be brought into the particularly simple form,

(II-5.4) $$C = {}^1A_\mu \xi^\mu - {}^0A_\mu \xi^\mu. \quad (3.20)$$

If the Lagrangian, and therefore the Hamiltonian, explicitly describes the possibility that the field equations may not be satisfied, then the coefficients ${}^nA_\mu$ may depend on the P^A, which are a measure of the amount by which the field equations are not satisfied, as well as on the canonical field variables. The group character of the transformation law is not altered by the introduction of the P^A into the theory. Therefore, in this case as well, the ${}^nA_\mu$ form a function group.

The investigation of the Poisson bracket relations of the ${}^nA_\mu$ with the Hamiltonian proceeds in the same fashion as in II. Because of the covariance of the theory under arbitrary coordinate transformations, the Hamiltonian can change as a function of its arguments only by adding a linear combination of primary constraints. Furthermore, this change is given by the total time derivative of the generating functional of the transformation.[13]

Therefore, we have

(II-5.8) $$(\mathcal{C}, \mathcal{H}^*) + \frac{\partial \mathcal{C}}{\partial t} \equiv \int \delta' w^\rho g_\rho d^3x. \quad (3.21)$$

As a result it follows that

(II-5.12) $$g_\mu \equiv {}^1A_\mu, \quad (3.22)$$

and[14]

(II-5.13) $$\frac{\partial\, {}^1A_\mu}{\partial t} + ({}^1A_\mu, H^*) = {}^0A_\mu \equiv a_\mu{}^\nu g_\nu,$$
$$(g_\mu, H^*) - {}^0A_\mu \equiv a_\mu{}^\nu g_\nu, \quad (3.23)$$

$$\frac{\partial\, {}^0A_\mu}{\partial t} + ({}^0A_\mu, H^*) \equiv b_\mu{}^\nu g_\nu. \quad (3.24)$$

The $a_\mu{}^\nu$ and $b_\mu{}^\nu$ are certain functions of the field variables which are determined by the transformation properties of the Hamiltonian. The only difference between the above relations and the corresponding relations in II is the occurrence of the partial derivative with respect to t on the left-hand side of the above equations. This additional term results from the fact that the ${}^nA_\mu$ can depend on the time through the P^A as well as through the field variables.

[13] A canonical transformation produces the following changes in value:
$$\delta q_k = \partial C/\partial p_k, \quad \delta p_k = -\partial C/\partial q_k, \quad \delta H = \partial C/\partial t.$$
Therefore,
$$\delta' H = H(q_k, p_k) - H(q_k', p_k') + \partial C/\partial t = (C, H) + \partial C/\partial t.$$

[14] The Poisson bracket $({}^nA_\mu, H^*)$ is to be calculated as follows:
$$\left(\int \sigma^\mu\, {}^nA_\mu d^3x, \mathcal{H}^*\right) = \int \sigma^\mu ({}^nA_\mu, H^*) d^3x,$$
where the σ^μ are arbitrary weight functions.

Comparing Eq. (3.23) with (3.18) we find that, modulo the primary constraints, the ${}^0A_\mu$ are equal to the secondary constraints. In fact, by considering the transformation law for the y_A, one can show that

$${}^0A_\mu \equiv L'_\mu - u_{\mu A}{}^4 P^A, \quad (3.25)$$

where

$$L'_\mu \equiv L_\mu - f_\mu(g^\rho).$$

Equation (3.24) tells us, therefore, that the total time derivative of the secondary constraints does not lead outside the function group composed of the g_μ and ${}^0A_\mu$. Hence, the four secondary constraints are the only conditions imposed on the field equations by the existence of the primary constraints.

From the above it is evident that the second time derivative of the primary constraints should vanish identically modulo the primary and secondary constraints. This condition yields the Bianchi identities in the canonical formalism. Let us consider

$$\mathfrak{a} = \int \sigma^\mu\, {}^0A_\mu d^3x = \int \sigma^\mu (L'_\mu - u_{\mu A}{}^4 P^A) d^3x \overset{l}{=} 0. \quad (3.26)$$

Equation (3.16) shows that we need only consider $d\mathfrak{a}/dt$ in order to obtain all the interesting information from $d^2\mathcal{G}/dt^2$:

$$\frac{d\mathfrak{a}}{dt} = \int \left\{ \dot\sigma^\mu\, {}^0A_\mu + \sigma^\mu \left[({}^0A_\mu, H^*) + \frac{\partial\, {}^0A_\mu}{\partial t} \right] \right\} d^3x \overset{l}{=} 0. \quad (3.27)$$

Again we consider the coefficients of σ^μ and $\dot\sigma^\mu$ separately. The coefficient of $\dot\sigma^\mu$ is merely the secondary constraint and, therefore, gives no new information. From the previous argument concerning the function group, the coefficient of σ^μ is equal to a linear combination of primary constraints and, hence, vanishes in Σ_l. Let us calculate this term explicitly:

$$(L'_\mu, H^*) \equiv \partial^A L'_\mu \frac{\partial H}{\partial \pi^A} + \partial^{As} L'_\mu \left(\frac{\partial H}{\partial \pi^A}\right)_{,s} + \partial^{Ars} L'_\mu \left(\frac{\partial H}{\partial \pi^A}\right)_{,rs} - \left(u_{\mu A}{}^s \frac{\delta H^*}{\delta y_A}\right)_{,s} - \left(\delta_\mu{}^4 \frac{\partial H}{\partial \pi^A} + \delta_\mu{}^s y_{A,s}\right)\frac{\delta H^*}{\delta y_A}$$
$$\equiv B_\mu - (u_{\mu A}{}^s P^A)_{,s} - \left(\delta_\mu{}^4 \frac{\partial H}{\partial \pi^A} + \delta_\mu{}^s y_{A,s}\right) P^A. \quad (3.28)$$

Therefore, we obtain

$$\partial\, {}^0A_\mu/\partial t + ({}^0A_\mu, H^*) \equiv B_\mu - (u_{\mu A}{}^\nu P^A)_{,\nu} - (\delta_\mu{}^4 \partial H/\partial \pi^A + \delta_\mu{}^s y_{A,s}) P^A \overset{l}{\equiv} 0. \quad (3.29)$$

The symbol l above an identity sign means that the relationship is an identity modulo the primary constraints (but not involving time derivatives of primary constraints). The secondary constraints are satisfied only as long as we remain in Σ_l. However, the relation (3.29) are satisfied, in the above sense, throughout Σ; i.e., Eq. (3.24) is valid throughout Σ. Furthermore, since Eq. (3.29) must be satisfied identically in Σ, whether or not P^A is zero, the terms containing P^A and those not containing P^A must vanish separately. Hence, we have

$$(u_{\mu A}{}^{\nu}P^A)_{,\nu}-(\delta_\mu{}^4\partial H/\partial\pi^A+\delta_\mu{}^s y_{A,s})P^A\overset{l}{\equiv}0, \tag{3.30}$$

and

$$B_\mu\overset{l}{\equiv}0. \tag{3.31}$$

Comparing Eq. (3.30) with the Bianchi identities, Eq. (2.4), we see that this identity must be satisfied by the P^A, if they are to be the right-hand sides of covariant field equations.

The B_μ can also be written in the same form and are, therefore, the Bianchi identities in terms of the canonical variables. By using the relations given in Eq. (3.2) one can show that

$$\frac{\partial L'_\mu}{\partial\pi^A}\dot\pi^A+\frac{\partial L'_\mu}{\partial\pi^A{}_{,s}}\dot\pi^A{}_{,s}-(u_{\mu A}{}^s\dot\pi^A)_{,s}-\left(\delta_\mu{}^4\frac{\partial H}{\partial\pi^A}+\delta_\mu{}^s y_{A,s}\right)\dot\pi^A\equiv0. \tag{3.32}$$

Adding the above identity to B_μ we have

$$B_\mu\equiv L'_{\mu,4}-\left[u_{\mu A}{}^s\left(\frac{\delta H}{\delta y_A}+\dot\pi^A\right)\right]_{,s}-\left(\delta_\mu{}^4\frac{\partial H}{\partial\pi^A}+\delta_\mu{}^s y_{A,s}\right)\left(\frac{\delta H}{\delta y_A}+\dot\pi^A\right)\overset{l}{\equiv}0. \tag{3.33}$$

If we remember Eq. (3.19), we see that the above is just what one would expect the Bianchi identities to become in the Hamiltonian formalism.

3.4 Strong Conservation Laws

As has been pointed out previously, the strong conservation laws are certain ordinary divergences which vanish whether or not the field equations are satisfied. Furthermore, they are a result of the existence of the Bianchi identities. Having already obtained the Bianchi identities in the canonical formalism we are prepared to derive from them the strong conservation laws.

The first two terms of B_μ, Eq. (3.33) can easily be put into the form of a divergence:

$$L'_{\mu,4}-\left[u_{\mu A}{}^s\left(\frac{\delta H}{\delta y_A}+\dot\pi^A\right)\right]_{,s}\equiv\left\{\delta_4{}^\nu L'_\mu-\delta_s{}^\nu u_{\mu A}{}^s\left(\frac{\delta H}{\delta y_A}+\dot\pi^A\right)\right\}_{,\nu}. \tag{3.34}$$

The remaining two terms require a little manipulation, but without difficulty one can show that,

$$\delta_\mu{}^s y_{A,s}\left(\frac{\delta H}{\delta y_A}+\dot\pi^A\right)\equiv\left\{\delta_s{}^\nu\left[\delta_\mu{}^s\left(H-\pi^A\frac{\partial H}{\partial\pi^A}\right)-\delta_\mu{}^r y_{A,r}\partial^{As}H\right]+\delta_\mu{}^s\delta_4{}^\nu y_{A,s}\pi^A\right\}_{,\nu}, \tag{3.35}$$

$$\delta_\mu{}^4\frac{\partial H}{\partial\pi^A}\left(\frac{\delta H}{\delta y_A}+\dot\pi^A\right)\equiv\left\{\delta_\mu{}^4\delta_4{}^\nu H-\delta_\mu{}^4\delta_s{}^\nu\frac{\partial H}{\partial\pi^A}\partial^{As}H\right\}_{,\nu}. \tag{3.36}$$

Substituting from the above three relations, Eq. (3.33) becomes

$$T_\mu{}^\nu{}_{,\nu}\overset{l}{\equiv}0,$$
$$T_\mu{}^s=u_{\mu A}{}^s\left(\frac{\delta H}{\delta y_A}+\dot\pi^A\right)+\delta_\mu{}^s\left(H-\pi^A\frac{\partial H}{\partial\pi^A}\right), \tag{3.37}$$
$$T_\mu{}^4=-L'_\mu+\delta_\mu{}^4H+\delta_\mu{}^r y_{A,r}\pi^A.$$

The above is just the statement of the strong conservation laws.

One should not expect the Bianchi identities and hence the strong conservation laws to hold in the total phase space Σ. The covariance of the Lagrangian theory, which is responsible for both the Bianchi identities and the conservation laws, is reflected in the canonical formalism by the existence of the primary constraints. Since the constraints are satisfied only in Σ_l, it would be unrealistic to expect to find identities and conservation laws, resulting from the covariance of the theory, which are valid in regions of Σ where the constraints are not satisfied.

3.5 Surface Integrals

The arguments leading up to the surface integrals for the equations of motion are the same here as in the Lagrangian case, Sec. II. Formally the surface integrals are the same. Unfortunately, however, although the superpotentials may be found in principle, even the restriction to a quadratic Lagrangian does not permit their explicit calculation. Bergmann and Schiller, by making a specific assumption about the transformation properties of the Lagrangian density, are able to derive expressions for the superpotentials.[15] In their formulation the superpotentials are easily carried over into the Hamiltonian formalism. However, in any case the determination of the Bianchi identities follows from similar arguments to those presented in this paper.

In a particular theory, such as the theory of gravitation, the calculation of the superpotentials should not lead to any serious difficulties. From Eqs. (3.19) and (2.13) with $\nu=4$, one sees that modulo the primary constraints

$$L_\mu'=U_\mu{}^{[s4]}{}_{,s}+t_\mu{}^4. \quad (3.38)$$

Therefore, in the theory of gravitation one can show that,[16]

$$U_\mu{}^{[s4]}=u_{\mu A}{}^s\bar{\pi}^A-\tfrac{2}{3}(u_{\mu A}{}^4L^{ABns}-u_{\mu A}{}^sL^{ABn4})y_{B,n}. \quad (3.39)$$

If we substitute for the π^A from Eq. (3.3), the above expression for $U_\mu{}^{[s4]}$ agrees with that obtained from Eq. (2.23). The antisymmetry in s and 4 can be shown explicitly by making use of Eq. (3.9) and dropping extra factors containing g_μ:

$$U_\mu{}^{[s4]}=\tfrac{1}{3}(u_{\mu A}{}^sL^{AC44}-u_{\mu A}{}^4L^{AC4s})E_{CB}\bar{\pi}^B +\tfrac{2}{3}(u_{\mu A}{}^sL^{ABn4}-u_{\mu A}{}^4L^{ABns})y_{B,n}. \quad (3.40)$$

Thus, for the theory of gravitation we can write out the surface integrals explicitly.

IV. DISCUSSION

As a result of the existence of strong conservation laws in a generally covariant field theory, there exist superpotentials whose divergences yield the components of the energy-momentum tensor, $T_\mu{}^\nu$. We have shown that by means of these superpotentials the total energy and momentum contained in a three-dimensional region of space can be calculated by a two-dimensional surface integral taken over the boundary of that region. Furthermore, we were able to set up certain two-dimensional surface integrals which were independent of the surface of integration and which yielded the equations of motion for any singularities (particles) enclosed by the surface. In the canonical formalism the fact that the constraints together with the Hamiltonian form a function group enabled us to obtain the Bianchi identities and, hence, the strong conservation laws in terms of the canonical variables. Thus, the superpotentials and the surface integrals can be found in the Hamiltonian formalism.

Up to now the investigation of gravitational radiation has been unsuccessful because of the necessary restriction to small particle velocities (compared with the velocity of light) in the Einstein-Infeld approximation method.[17,18] If this restriction to slow motion is removed, then the field equations cannot be integrated beyond the first order without knowledge of the motion of the particles in advance. However, by using the Hamiltonian formalism, one can build up a solution of the field equations as a Taylor series with respect to time. Since the constraints and Hamiltonian form a function group, one need only choose the initial conditions such that the primary and secondary constraints are satisfied. Having done so, one then proceeds as though there were no constraints. Thus, the problem of investigating gravitational radiation is reduced to the choice of initial conditions and the evaluation of certain surface integrals.

Since the energy and momentum of the field can be calculated by means of surface integrals, there is no need to separate the self-field of a particle from the total field. Indeed, because of the nonlinearity of the field equation one cannot make this separation. It appears conceivable that when the gravitational field is quantized the usual renormalization of the mass and charge may not be necessary. The role played by the equations of motion in a quantized theory is still open to question. There seems to be no way of interpreting equations of motion within the framework of quantum mechanics as it exists today. However, this question will have to be investigated further.

In conclusion, I wish to express my appreciation to Dr. Peter G. Bergmann for many valuable discussions.

[15] P. G. Bergmann and R. Schiller, Phys. Rev. **89**, 4 (1952).
[16] The definition of $\Lambda^{A\rho B\sigma}$ in the theory of gravitation is given by Bergmann, Penfield, Schiller, and Zatzkis, Phys. Rev. **80**, 81 (1950).
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