

it is obvious that the whole atmosphere must be considered before a complete correlation is at all possible. But it is not unreasonable to believe that the upper-atmospheric temperature should exert an appreciably large influence in this respect. Duperier has shown that much better correlation and larger regression coefficients are obtained by considering the region between 50 and 200 mb instead of that between 100 and 200 mb. His results are in agreement with the known lifetime of charged pions and the value of the mean free path for primary radiation in the atmosphere. Furthermore,

if one believes the explanation of the cosine-squared zenithal dependence of the hard intensity at low altitude, he would expect to find that the temperature coefficient would increase as telescope solid angle is decreased. This result has been found by Duperier. The use of relatively small solid angle in the experiment of Cotton and Curtis should thus yield a value of the coefficient larger than those found by Duperier and the author, although the small-angle telescope should not yield as good statistics unless appreciably longer periods of observation are used.

Molière's Theory of Multiple Scattering

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Molière's theory of multiple scattering of electrons and other charged particles is here derived in a mathematically simpler way. The differential scattering law enters the theory only through a single parameter, the screening angle χ_a , Eq. (21). The angular distribution, except for the absolute scale of angles, depends again only on a single parameter b , Eq. (22). It is shown that b depends essentially only on the thickness of the scattering foil in g/cm², and is nearly independent of Z .

The transition to single scattering is re-investigated. An asymptotic formula is obtained which agrees essentially with that of Molière, Snyder, and Scott, but which remains accurate down to smaller angles, Eq. (38).

The theory of Goudsmit and Saunderson has a close *quantitative* relation to that of Molière, and a good approximation to their distribution function can be obtained by multiplying Molière's function by $(\theta/\sin\theta)$. This relation holds until the scattering angles become so large that only very few terms in the series of Goudsmit and Saunderson need to be taken into account.

I. INTRODUCTION

AT least four different theories of the multiple scattering of electrons by atoms have been published which are mathematically closely related, and which can give exact results if carefully evaluated. They are the work of Molière,¹ Snyder, and Scott,^{2,3} Goudsmit and Saunderson,⁴ and Lewis.⁵ Of these, the first two use immediately the approximation of small scattering angles and therefore an expansion in Bessel functions (see below) (or a Fourier integral for the distribution of projected angles). Goudsmit and Saunderson develop a theory valid for any angle by means of an expansion in Legendre polynomials. Lewis starts from the Legendre expansion and then goes over to the limit of small angles, thus establishing the connection between the first three methods.

The theories of Molière and of Goudsmit and Saunderson share one important advantage, namely that they do not assume any special form for the differential

scattering cross section. In both theories it is shown that the scattering depends only on a single parameter describing the atomic screening, the critical angle χ_a , Eq. (16) of this paper. This angle can then be calculated, for instance, for the Fermi-Thomas distribution of electrons in an atom, or for any more accurate electron distribution if available. Molière has even included the *deviation* of the differential scattering from the Born approximation, Eq. (21), and Hanson, Lanzl, Lyman, and Scott⁶ have shown that this inclusion is important for explaining their own experimental results as well as those of Kulchitsky, Latyshev, and Andrievsky⁷ for heavy elements. Snyder and Scott, as well as Lewis, assume a special scattering law, *viz.*, that derived from the exponentially screened potential, $Ze^{2\gamma-1}e^{-r/a}$. Only *a posteriori* did Scott³ state that the treatment of Snyder and Scott² is mathematically identical with Molière's and can therefore also be generalized to his differential scattering law.

The theory of Goudsmit and Saunderson has, of course, the further advantage that it is valid for all

¹ G. Molière, *Z. Naturforsch.* **3a**, 78 (1948).

² H. Snyder and W. T. Scott, *Phys. Rev.* **76**, 220 (1949).

³ W. T. Scott, *Phys. Rev.* **85**, 245 (1952).

⁴ S. A. Goudsmit and J. L. Saunderson, *Phys. Rev.* **57**, 24 and **58**, 36 (1940).

⁵ H. W. Lewis, *Phys. Rev.* **78**, 526 (1950).

⁶ Hanson, Lanzl, Lyman, and Scott, *Phys. Rev.* **84**, 634 (1951).

⁷ L. A. Kulchitsky and G. D. Latyshev, *Phys. Rev.* **61**, 254 (1942); Andrievsky, Kulchitsky, and Latyshev, *J. Phys. (U.S.S.R.)* **6**, 279 (1943).

angles. On the other hand, the small-angle theories have the advantage of considerably greater transparency. This is particularly true of the theory of Molière, which remains analytical to the end.

To an accuracy of 1 percent or better, the angular distribution is given by the sum of 3 analytical terms, Eqs. (25) to (29), the first of which is the well-known Gaussian, while the second goes over into the single-scattering formula at large angles and the third is a correction. Both second and third terms are easily evaluated.

It is possible to combine the advantages of both types of theories because, up to quite large foil thicknesses and scattering angles, the Goudsmit-Saunderson result can be expressed in terms of the simpler Molière theory (Sec. 8). However, this is only possible if the total path length rather than the actual foil thickness, is used as the independent variable.

Snyder and Scott have calculated the distribution of projected angle, Goudsmit and Saunderson that of total scattering angle, and Molière both. We shall restrict ourselves to total scattering angle.

The aim of this paper is to give a simpler derivation of Molière's equations, to show how one is led in a straightforward way to Molière's "screening angle" χ_a , and to derive a simple asymptotic formula for the correction to single (Rutherford) scattering. In most places, the same notation as Molière's is used. His equations are quoted as M with the appropriate number.

II. DERIVATION

Molière derives his fundamental equation (M 4.4) by considering successive collisions. Like Molière, we assume that all scattering angles are small so that $\sin\theta$ may be replaced by θ , and the scattering problem is equivalent to diffusion in the plane of θ . Now let $\sigma(\chi)\chi d\chi$ be the differential scattering cross section into the angular interval $d\chi$, and $f(\theta, t)\theta d\theta$ the number of electrons in the angular interval $d\theta$ after traversing a thickness t . Then the standard transport equation is

$$\begin{aligned} \partial f(\theta, t)/\partial t = & -Nf(\theta, t) \int \sigma(\chi)\chi d\chi \\ & + N \int f(\theta', t)\sigma(\chi)d\chi, \end{aligned} \quad (1)$$

where N is the number of scattering atoms per cm^2 , $\theta' = \theta - \chi$ is the vector in the plane representing the direction of the electron before the last scattering, and $d\chi = \chi d\chi d\phi/2\pi$, where ϕ denotes the azimuth of the vector χ in the plane.

Now we make the same Fourier (Bessel) transformation as Molière, expanding

$$f(\theta, t) = \int_0^\infty \eta d\eta J_0(\eta\theta)g(\eta, t) \quad (2)$$

so that

$$g(\eta, t) = \int_0^\infty \theta d\theta J_0(\eta\theta)f(\theta, t). \quad (3)$$

Then the Fourier transformation of (1) yields, using the folding theorem,

$$\partial g(\eta, t)/\partial t = -g(\eta, t)N \int_0^\infty \sigma(\chi)\chi d\chi [1 - J_0(\eta\chi)]. \quad (4)$$

This can be integrated over t , giving

$$g(\eta, t) = e^{\Omega(\eta)t - \Omega_0 t} \quad (5)$$

in the notation of Molière, where

$$\Omega(\eta) = Nt \int_0^\infty \sigma(\chi)\chi d\chi J_0(\eta\chi), \quad (6)$$

and Ω_0 is the value of (6) for $\eta=0$, i.e., the total number of collisions. In (5) we have used the fact that $g(\eta, 0)=1$ for all η , which follows from the assumption that $f(\theta, 0)$ is a two-dimensional δ -function $\delta(\theta)$, i.e., the incident beam is exactly in the direction $\theta=0$.

Equations (2) and (5) are Molière's fundamental equations (M 4.4, 4.5). It is convenient to treat $\Omega(\eta) - \Omega_0$ together, rather than splitting them up because, as Molière himself has pointed out, the total number of collisions Ω_0 is irrelevant, and $\Omega_0 - \Omega(\eta)$ is much smaller than Ω_0 for the values of η which are important; $\Omega_0 - \Omega(\eta)$ may be called the "effective number of collisions." Inserting our results back into (2), we have

$$\begin{aligned} f(\theta, t) = & \int_0^\infty \eta d\eta J_0(\eta\theta) \\ & \times \exp\left[-Nt \int_0^\infty \sigma(\chi)\chi d\chi \{1 - J_0(\eta\chi)\}\right]. \end{aligned} \quad (7)$$

This equation is exact for *any* scattering law,⁸ provided only the angles are small compared with a radian.

III. TRANSFORMATION

The scattering from atoms is characterized by the fact that σ decreases rapidly and in simple manner, as χ^{-4} , for large χ , and is complicated only for angles of the order of

$$\chi_0 = \lambda/a = \lambda/(0.885a_0Z^{-1/2}), \quad (8)$$

⁸ The derivation of Eq. (7) in essentially the same form as in this section was shown to the author by Henry Hurwitz, Jr. in 1949, without knowledge of Molière's paper and before publication of the paper by Snyder and Scott. The derivation by Lewis (see reference 5) which starts from finite angles and spherical harmonics, could be simplified by using small angles and Bessel functions from the beginning, and would then become essentially identical with that given in this section. Lewis' result is equivalent with (7), but he uses a special scattering law before arriving at the formula corresponding to (7). Of course, Lewis' proof establishes at the same time the connection between the Goudsmit-Saunderson theory and that of Molière.

where λ is the de Broglie wavelength of the electron, a_0 the Bohr radius, and a the Fermi radius of the atom. For any reasonable foil thickness, the width of the multiple scattering distribution is very large compared with χ_0 , and this is the reason for the essential simplicity of Molière's theory.

Following Molière, we set

$$Nt\sigma(\chi)\chi d\chi = 2\chi_c^2 \chi d\chi q(\chi)/\chi^4, \quad (9)$$

where q is the ratio of actual to Rutherford scattering, and

$$\chi_c^2 = 4\pi N t e^4 Z(Z+1)z^2/(pv)^2; \quad (10)$$

p is the momentum and v the velocity of the scattered particle of charge z . The factor $Z+1$ instead of Z is to take into account the scattering by the atomic electrons, as first suggested by Kulchitsky and Latyshev.⁷ The physical meaning of χ_c is that the total probability of single scattering through an angle greater than χ_c is exactly one. This angle was already used by Williams⁹ in his theory of multiple scattering. The ratio $q(\chi)$ is 1 for large χ and decreases to zero at $\chi=0$, the main drop occurring in the neighborhood of χ_0 .

Inserting (9) into (6) and (5), we get

$$\begin{aligned} -\log g(\eta, t) &= \Omega_0 - \Omega(\eta) \\ &= 2\chi_c^2 \int_0^\infty \chi^{-3} d\chi [1 - J_0(\chi\eta)] q(\chi). \end{aligned} \quad (11)$$

The important values of η will be of order $1/\chi_c$ or less.¹⁰ Since q becomes appreciably different from 1 only for values of χ of order χ_0 , and since χ_0 is much less than χ_c (of the order of $1/100$), it is possible to split the integral at some angle k such that

$$\chi_0 \ll k \ll 1/\eta \sim \chi_c. \quad (11a)$$

Then, for the part of the integral from k to infinity, $q(\chi)$ can be replaced by unity and the integral evaluated analytically. For the part from 0 to k , on the other hand, the argument of the Bessel function is small and we may write

$$1 - J_0(\chi\eta) = \frac{1}{4}\chi^2\eta^2, \quad (12)$$

which will make it possible to reduce the integral to a universal one, independently of η .

The analytical integral may be written:

$$\begin{aligned} \int_k^\infty d\chi \chi^{-3} [1 - J_0(\chi\eta)] &= \eta^2 \int_{k\eta}^\infty dt t^{-3} [1 - J_0(t)] \\ &\equiv \frac{1}{4}\eta^2 I_1(k\eta). \end{aligned} \quad (13)$$

⁹ E. J. Williams, Phys. Rev. **58**, 306 (1940).

¹⁰ This is most easily seen from Eq. (17) according to which $g(\eta, t) = \exp[\Omega_0 - \Omega(\eta)]$ becomes extremely small when $\chi, \eta \gg 1$. The use of (17) to justify (11a) may seem circular because (17) itself is based on the assumption (11a); but the only part of (11a) actually used in deriving (17) is that $\eta\chi_0 \ll 1$. Our method will therefore be in error if, and only if, $\eta\chi_0$ is of the order 1, in which case (17) gives $g \sim \exp[-\frac{1}{2}(\chi_c/\chi_0)^2]$ which is exceedingly small (see the end of this section).

The integral I_1 , defined by (13), can be integrated by parts and gives, since the lower limit $x = k\eta$ is small:

$$\begin{aligned} I_1(x) &= 4 \int_x^\infty \frac{dt}{t^3} [1 - J_0(t)] = \frac{2}{x^2} [1 - J_0(x)] + \frac{J_1(x)}{x} \\ &\quad + \int_x^\infty \frac{dt}{t} J_0(t) = 1 - \ln x + \ln 2 - C + O(x^2), \end{aligned} \quad (14)$$

where $C = 0.577 \dots$ is Euler's constant.

The lower range of integration gives, using (12),

$$\begin{aligned} \int_0^k d\chi \chi^{-3} q(\chi) [1 - J_0(\chi\eta)] &= \frac{1}{4}\eta^2 \int_0^k q(\chi) d\chi/\chi \\ &\equiv \frac{1}{4}\eta^2 I_2(k). \end{aligned} \quad (15)$$

Molière now defines the characteristic screening angle χ_a by the equation¹¹

$$-\ln \chi_a = \lim_{k \rightarrow \infty} \left[\int_0^k q(\chi) d\chi/\chi + \frac{1}{2} - \ln k \right]. \quad (16)$$

Then, taking together (11) and (13) to (16), we get

$$\Omega_0 - \Omega(\eta) = \frac{1}{2}(\chi_c\eta)^2 [-\ln(\chi_a\eta) + \frac{1}{2} + \ln 2 - C], \quad (17)$$

Putting then $\chi_c\eta = y$, we get

$$\Omega_0 - \Omega(\eta) = \frac{1}{4}y^2 [b - \ln(\frac{1}{4}y^2)], \quad (18)$$

where

$$b = \ln(\chi_c/\chi_a)^2 + 1 - 2C \equiv \ln(\chi_c/\chi_a')^2. \quad (19)$$

Then, setting

$$\theta/\chi_c = \lambda, \quad (19a)$$

we get Molière's transformed equation

$$f(\theta)\theta d\theta = \lambda d\lambda \int_0^\infty y dy J_0(\lambda y) \exp[\frac{1}{4}y^2(-b + \ln \frac{1}{4}y^2)], \quad (20)$$

which is very much simpler in form than (7).

The derivation was based on the inequality (11a) and will, therefore, fail if η is of order $1/\chi_0$, or y of order $\chi_c/\chi_0 \sim e^{3b}$. Indeed, the exponent in (20) has a minimum when $y = y_1 = 2e^{\frac{1}{2}(b-1)}$; thereafter it increases and becomes in fact positive infinite as y goes to infinity. This increase is spurious and due to the approximation (11a); therefore, the integral should only be extended to $y = y_1$. How little difference this makes can be seen from the fact that the exponential, for $y = y_1$, has the value $\exp(-\frac{1}{4}y_1^2) = \exp(-e^{b-1})$. Since $e^b \sim (\chi_c/\chi_a)^2$ is about the number of collisions Ω_0 , the formula (20) is correct to the relative order $e^{-\Omega_0/e}$. For foils of moderate thickness, the number of collisions is 1000 to 100 000 so

¹¹ The term $\frac{1}{2}$ would not need to be in the definition. It is included by Molière in order that χ_a be exactly λ/a for an exponentially screened potential, $V(r) = (Ze^2/r)e^{-r/a}$.

that the error would be only e^{-400} to $e^{-40\,000}$! Actually, the error in (17) for values of γ of order 1 is more important; it gives corrections to $f(\theta)$ of order $1/\Omega_0$ with a small numerical factor.^{12*}

IV. THE SCREENING ANGLE

Perhaps the most important result of Molière's theory is that the scattering is described by a single parameter, the screening angle χ_a' . The angular distribution depends only on the ratio of the "unit probability angle" χ_c , Eq. (9), which describes the foil thickness, to the screening angle χ_a' which describes the scattering atom. The distribution function $f(\theta)$ is entirely independent of the shape of the differential cross section $d\sigma$ provided only $d\sigma$ goes over into the Rutherford law for large angles. In most other^{2,3,5} derivations of exact theories of multiple scattering, an explicit assumption was made about the differential cross section, namely the Born-approximation cross section for an exponentially screened potential. This potential is not a good approximation to the actual atomic potential, and the only rigorous published proofs that the shape of the potential is immaterial for the multiple scattering, are those of Molière and of Goudsmit and Saunderson.

For the actual determination of the screening angle χ_a , Molière uses his own calculation¹³ of the *single* scattering by a Thomas-Fermi potential which does not make use of the Born approximation, the solution being accomplished by means of the WKB method. An exact formula for the differential cross section in terms of an integral is given in Molière's paper,¹³ Eq. (4, 6), but his final evaluation of integrals over the Fermi function is numerical and only approximate; it yields

$$\chi_a^2 = \chi_0^2(1.13 + 3.76\alpha^2), \quad \chi_a'^2 = 1.167\chi_a^2, \quad (21)$$

with α the usual parameter,

$$\alpha = zZe^2/\hbar v. \quad (21a)$$

The term in α^2 represents the deviation from the Born approximation. We now insert (21) and the definitions (8) and (10) of χ_0 and χ_c into the definition of b , Eq.

¹² Exactly the same formulas as in this section were already obtained by Molière. The only difference is that he used in the proof a rather complicated series of Hankel functions, whereas we used the simple Bessel function J_0 so that every step in our derivation can be easily followed. This simplification made it possible to show in a more logical way why just the quantity χ_a , Eq. (16), enters the theory. Finally, it also makes possible a more precise estimate of the errors.

* *Note added in proof:*—We have not taken into account any spin or relativity corrections, except for the use of the relativistically correct denominator $(\beta v)^2$ in (10). These corrections are appreciable only at numerically large single-scattering angles χ , and therefore will not materially affect the small-angle multiple scattering treated in this section. In the region where single scattering predominates, it should still be a good approximation to consider the quantity R which will be calculated in Section VI, as the ratio of actual to correct single scattering.

¹³ G. Molière, Z. Naturforsch. 2a, 133 (1947).

TABLE I. Last factor in Eq. (22) as a function of atomic number.

Z	1*	6	10	20	30	40	50	70	92
$\frac{(Z+1)Z^{\frac{1}{2}}}{A(1+3.34\alpha^2)}$	1.00	1.05	1.15	1.32	1.27	1.20	1.00	0.90	0.70

* Deuterium; for hydrogen the factor is 2.00.

(19); this gives

$$e^b = \frac{\chi_c^2}{\chi_a'^2} = \frac{\chi_c^2}{1.167\chi_a^2} = 4\pi N_0 t \left(\frac{\hbar}{mc}\right)^2 \frac{(Z+1)Z^{\frac{1}{2}}z^2 0.885^2}{\beta^2 A 1.167(1.13 + 3.76\alpha^2)} = \frac{6680t}{\beta^2} \frac{(Z+1)Z^{\frac{1}{2}}z^2}{A(1+3.34\alpha^2)}, \quad (22)$$

where t is measured in g/cm^2 , $\beta = v/c$, A is the atomic weight, and $N_0 = 6.02 \times 10^{23}$ is Avogadro's number. The relations $p\lambda = \hbar$ and $a_0 = \hbar^2/me^2$ have been used. It is interesting that the Z dependent last factor in (22) never deviates much from 1; its values for $z=1$ and $v=c$ are given in Table I. This means that the effective "number of collisions" per g/cm^2 is nearly the same for all elements.

Hanson, Lanzl, Lyman, and Scott⁶ have pointed out that the Thomas-Fermi potential is not suitable for substances of low atomic number, such as Be, and that in fact their experiments indicate a somewhat larger χ_a for this substance than that required by the Thomas-Fermi atom. If the potential of the atom is known, it is easy to insert it in (16) and thus to derive χ_a .

V. EVALUATION

Molière has given a satisfactory evaluation of (20) for all angles. He defines the new parameter B by the transcendental equation

$$B - \ln B = b, \quad (23)$$

and the variable ϑ by

$$\vartheta = \lambda B^{-\frac{1}{2}} = \theta/(\chi_c B^{\frac{1}{2}}). \quad (24)$$

It will also be useful to use

$$x = \vartheta^2. \quad (24a)$$

The value of B is usually between 5 and 20. The integration variable is now changed to

$$u = B^{\frac{1}{2}}\vartheta, \quad (24b)$$

and the distribution function is expanded in a power series in $1/B$, which gives

$$f(\theta)\theta d\theta = \vartheta d\vartheta [f^{(0)}(\vartheta) + B^{-1}f^{(1)}(\vartheta) + B^{-2}f^{(2)}(\vartheta) + \dots], \quad (25)$$

where

$$f^{(n)}(\vartheta) = n!^{-1} \int_0^\infty u du J_0(u\vartheta) \times \exp(-\frac{1}{4}u^2) \left[\frac{1}{4}u^2 \ln(\frac{1}{4}u^2)\right]^n. \quad (26)$$

TABLE II. Comparison of the asymptotic formulas for scattering with the exact theory.

ϑ	$f^{(0)}$	$f^{(1)}$	$f^{(2)}$	$\frac{1}{2}\vartheta^4 f^{(1)}$	$\frac{1}{2}\vartheta^4 f^{(2)}$
0	2	0.8456	2.4929	0	0
0.2	1.9216	0.7038	2.0694	0.0006	0.0008
0.4	1.7214	0.3437	1.0488	0.0044	0.0034
0.6	1.4094	-0.0777	-0.0044	-0.0050	-0.0003
0.8	1.0546	-0.3981	-0.6068	-0.0815	-0.1246
1	0.7338	-0.5285	-0.6359	-0.2642	-0.318
1.2	0.4738	-0.4770	-0.3086	-0.4946	-0.320
1.4	0.2817	-0.3183	0.0525	-0.6113	0.101
1.6	0.1546	-0.1396	0.2423	-0.4573	0.669
1.8	0.0783	-0.0006	0.2386	-0.0032	1.251
2	0.0366	+0.0782	0.1316	0.6258	1.053
2.2	0.01581	0.1054	0.0196	1.2347	0.475
2.4	0.00630	0.1008	-0.0467	1.6718	-0.775
2.6	0.00232	0.08262	-0.0649	1.8877	-1.483
2.8	0.00079	0.06247	-0.0546	1.9200	-1.676
3	0.000250	0.04550	-0.03568	1.8429	-1.448
3.2	7.3×10^{-5}	0.03288	-0.01923	1.7240	-1.008
3.4	1.9×10^{-5}	0.02402	-0.00847	1.6050	-0.566
3.6	4.7×10^{-6}	0.01791	-0.00264	1.5038	-0.222
3.8	1.1×10^{-6}	0.01366	0.00005	1.4237	0.005
	1000 $f^{(0)}$	1000 $f^{(1)}$	1000 $f^{(2)}$		
4	2.3×10^{-4}	10.638	1.0741	1.3617	0.1375
4.5	3×10^{-6}	6.140	1.2294	1.2588	0.2521
5	2×10^{-8}	3.831	0.8326	1.1972	0.2602
5.5	2×10^{-10}	2.527	0.5368	1.1563	0.2456
6	5×10^{-13}	1.739	0.3495	1.1275	0.2264
7	1×10^{-18}	0.9080	0.1584	1.0901	0.1901
8	3×10^{-25}	0.5211	0.0783	1.0679	0.1604
9	1×10^{-32}	0.3208	0.0417	1.0523	0.1369
10	1×10^{-40}	0.2084	0.0237	1.0419	0.1186

For the first two functions $f^{(n)}$, he gives simple analytical formulas,

$$f^{(0)} = 2e^{-x}, \tag{27}$$

$$f^{(1)} = 2e^{-x}(x-1)[\overline{\text{Ei}}(x) - \ln x] - 2(1-2e^{-x}), \tag{28}$$

where $\overline{\text{Ei}}$ is defined in Jahnke-Emde.¹⁴

The next partial function $f^{(2)}$ is given by Molière in terms of a definite integral, viz.

$$\begin{aligned} \frac{1}{4}e^x f^{(2)} = & [\Psi^2(2) + \Psi'(2)](x^2 - 4x + 2) \\ & + \int_0^1 t^{-3} dt [\ln t / (1-t) - \Psi(2)] [(1-t)^2 e^{xt} - 1 \\ & - (x-2)t - (\frac{1}{2}x^2 - 2x + 1)t^2], \end{aligned} \tag{29}$$

where $\Psi(n) = d \ln \Gamma(n+1) / dn$, as defined by Jahnke and Emde. We have not found any way to evaluate the integral analytically, but a power series was obtained by Dr. Max Goldstein of the Los Alamos Scientific

¹⁴ E. Jahnke and F. Emde, *Table of Functions* (Dover Publications, New York, 1945), pp. 1 and 2.

Laboratory, viz.

$$\begin{aligned} \text{Integral} = & \sum_{n=0}^{\infty} \frac{1}{n+1} [\Psi(n) + C - \Psi(2)] \\ & \times \left[\frac{x^{n+3}}{(n+3)!} - \frac{2x^{n+2}}{(n+2)!} + \frac{x^{n+1}}{(n+1)!} \right]. \end{aligned} \tag{29a}$$

Dr. Goldstein was able to use this series to evaluate $f^{(2)}$ for ϑ up to 10, or x up to 100. Various transformations of (29a) were found but none proved more practical than the series itself.

In Table II, we give the functions $f^{(0)}$, $f^{(1)}$, and $f^{(2)}$ more accurately and over a wider interval of ϑ than Molière.¹⁵ We also give $\frac{1}{2}\vartheta^4 f^{(i)}$, $i=1$ and 2 , because these functions determine the ratio to Rutherford scattering at large angles, Eq. (32), and are easier to interpolate than the f 's themselves. As will be shown, the functions $f^{(0)}$ to $f^{(2)}$ are sufficient to determine the distribution function to about 1 percent or better for any angle.

For small angles, i.e., ϑ less than about 2, the Gaussian $f^{(0)}$ is the dominant term. In this region, $f^{(1)}$ is in general less than $f^{(0)}$, so that the correction to the Gaussian is of order $1/B$, i.e., of the order of 10 percent. Hanson *et al.*,⁶ have pointed out that a better approximation than $f^{(0)}$ in this region is given by a Gaussian of slightly smaller width: The angle at which the intensity has dropped to $1/e$ of the maximum, is

$$\theta_w = \chi_c(B - 1.2)^{\frac{1}{2}} \tag{30}$$

rather than $\theta_w = \chi_c B^{\frac{1}{2}}$, and a Gaussian of width (30) is the better approximation mentioned.

The formula for the width of the multiple scattering peak can be understood simply as follows: The width can in principle be found by calculating the average of χ^2 from the single-scattering law, but since $\sigma(\chi)$ is proportional to χ^{-4} , the integral $\int \chi^2 \sigma(\chi) \chi d\chi$ diverges logarithmically at large χ . Now a reasonable way to cut off this divergence is to extend the integral to $\chi = \theta_w$, the width of the multiple scattering peak itself. This leads to the transcendental equation (23) for the width parameter, B .

For larger angles, $\vartheta > 2$, the function $f^{(1)}$ in (28) becomes larger than $f^{(0)}$. Indeed, for very large ϑ , (28) goes over into the single scattering law $f^{(1)} = 2\vartheta^{-4}$, while $f^{(0)}$ decreases exponentially. Now the fortunate point is that $f^{(2)}$ and the higher $f^{(n)}$ behave for large ϑ as ϑ^{-2n-2} and the series (25) therefore converges even faster at large ϑ than at moderate ones. Therefore, $f^{(0)} + B^{-1}f^{(1)}$ will be a good approximate representation of the distribution at any angle. If an accuracy of 1

¹⁵ In general, agreement with Molière is good, the maximum error being about 3 units of the last significant figure carried by him. The only major error in Molière's table is $f^{(2)}(\vartheta)$ for $\vartheta = 3.5$ for which he gives +0.0052, while the correct value is -0.0051. This mistake had caused us considerable trouble in trying to obtain smooth distributions.

percent is desired, and if B is of order 10, the function $f^{(2)}$ must be included. This is sufficient, even at large angles where $f^{(1)}$ is the dominant term, because $f^{(2)}$ is smaller than $f^{(1)}$ in this region, as can be seen both from Table II and from the asymptotic formula (34), and $f^{(3)}$ is presumably still smaller.

VI. ASYMPTOTIC FORMULA

In the limit of large angles, the distribution function tends toward the Rutherford single-scattering law. According to (9), (19a), and (24), this is

$$f_R(\theta)\theta d\theta = 2d\lambda/\lambda^3 = (2/B)d\vartheta/\vartheta^3. \tag{31}$$

Therefore the ratio of actual to Rutherford scattering is, using (25),

$$R = f/f_R = \frac{1}{2}\vartheta^4(f^{(1)} + B^{-1}f^{(2)} + \dots), \tag{32}$$

neglecting the Gaussian term $Bf^{(0)}$. The relative magnitude of this term can be seen from Table I.

For $f^{(1)}$, Molière (M 9.3a) gives the asymptotic expression

$$R_1 \equiv \frac{1}{2}\vartheta^4 f^{(1)} = (1 - 5\vartheta^{-2})^{-4/5}, \tag{33}$$

whose expansion in inverse powers of ϑ agrees with that of the exact expression (28) up to and including the term of order ϑ^{-4} and agrees reasonably well with it even in higher orders. Similarly, the asymptotic behavior of $f^{(2)}$ is

$$R_2 \equiv \frac{1}{2}\vartheta^4 f^{(2)} = 8\vartheta^{-2}(\ln\vartheta + C - \frac{3}{2})/1 - 9\vartheta^{-2} - 24\vartheta^{-4}. \tag{34}$$

In (34), $C - \frac{3}{2}$ may be replaced by $\ln 0.4$ which differs from it by only 0.0065. Equation (34) is about 6 percent too high at $\vartheta=6$, 1.2 percent high at $\vartheta=8$, and 0.2 percent high at $\vartheta=10$. The error of Eq. (33) is much smaller; in parts per thousand, it is

For $\vartheta =$	3	3.2	3.6	4	5	6	8	10
Error =	+40	-8	-17	-9	-1.4	-0.8	-0.5	-0.1

The most obvious way to obtain an asymptotic formula for R is to expand (33) and (34) in inverse powers of ϑ and neglect all terms of order ϑ^{-4} and higher. This procedure, however, is obviously very inaccurate since the higher terms in the series expansion of (33) have very large coefficients. A much better convergent series is obtained by taking the *reciprocal*,

$$R_1^{-1} = (1 - 5\vartheta^{-2})^{4/5} = 1 - 4\vartheta^{-2} - 2\vartheta^{-4} - \dots \tag{35}$$

Because of the denominator occurring in its first term, (34) gives a simple result when combined with (33), namely (including order ϑ^{-4}),

$$R^{-1} = 1 - 4\vartheta^{-2}[1 + 2B^{-1} \ln(0.4\vartheta)] + 2\vartheta^{-4}(12B^{-1} - 1). \tag{36}$$

Here $R = R_1 + (R_2/B)$ is the ratio of actual to Rutherford scattering, so that R^{-1} is the ratio of Rutherford to actual. A term of order $B^{-2}\vartheta^{-4}$, arising from $f^{(3)}$, has been neglected.

A further simplification of (36) can be effected by noting that in most practical cases B is of the order of

10, so that the last term in (36) is very small. We may then write

$$R^{-1} = 1 - 4\vartheta^{-2}[1 + 2B^{-1} \ln(2\vartheta/5)]. \tag{37}$$

Here we may re-insert the value of ϑ and B from (23), (24), (19) and get for the ratio of *Rutherford to actual* scattering without further approximation,

$$1/R = 1 - 8(\chi_c^2/\theta^2) \ln(2\theta/5\chi_a'). \tag{38}$$

In Table III, we have compared the asymptotic formulas (36) and (37) with the exact value of R for two values of B and various values of ϑ . The ratio of the first to the second term in Molière's series (25), $Bf^{(0)}/f^{(1)}$, is also listed: this ratio is small in the asymptotic region. It is seen from the table that for $B=14$, the simpler asymptotic expression (37) is excellent, down to $\vartheta=2.4$, i.e., right down to the point where the Gaussian begins to dominate. For $B=7.3$, the agreement is not quite so good, and the more complicated expression (36) is somewhat better than the simple (37), although a larger correction than the last term of (36) would improve the agreement. Anyway, also for $B=7.3$ the agreement is good down to $\vartheta=3$. The fact that $f^{(1)}$ and $f^{(2)}$ combine in such a simple way to give the asymptotic formula (38) may seem somewhat mysterious from the derivation given here. A more natural derivation is given in Appendix A.

The main difference between (38) and previous asymptotic formulas is that (38) gives $1/R$ rather than R , and that the asymptotic series for $1/R$ obviously converges much better than that for R . Otherwise, in agreement with the theories of Molière and of Snyder and Scott, (38) has a logarithmic dependence of the correction term on ϑ , in addition to the $1/\vartheta^2$ dependence. Other theories, e.g., that of Butler,¹⁶ failed to get the logarithmic dependence but had $\ln(\chi_c/\chi_a)$ instead.

As was pointed out before, χ_c^2/θ^2 is the probability of having a single scattering through an angle greater than θ in the foil. The correction term in (38) is roughly 50 times greater than this probability. This shows that the approach to single scattering is extremely slow as has been pointed out in the earlier papers.

VII. COMPARISON WITH EXPERIMENT

Hanson, Lanzl, Lyman, and Scott⁶ have shown that Molière's theory is in almost perfect agreement with their experiments, up to angles ϑ (Sec. 5) of about 2 or

TABLE III. Comparison of asymptotic formulas with exact value.

ϑ	4	3.6	3.2	3	2.8	2.6	2.4	2.2
$B=14$								
Exact	1.373	1.49 ₂	1.70 ₆	1.87 ₆	2.14	2.52	3.08	3.84
Eq. (37)	1.362	1.48 ₀	1.68	1.84	2.07 ₆	2.47 ₆	3.22	5.30
$Bf^{(0)}/f^{(1)}$	0.0003	0.004	0.031	0.077	0.18	0.39	0.88	2.10
$B=7.3$								
Exact	1.378	1.477	1.610	1.71 ₆	1.87	2.07		
Eq. (36)	1.384	1.49 ₆	1.68	1.82	2.01 ₆	2.29		
Eq. (37)	1.393	1.513	1.71 ₆	1.87	2.10 ₆	2.45		
$Bf^{(0)}/f^{(1)}$	0.0001	0.005	0.015	0.04	0.09	0.20		

¹⁶ S. T. Butler, Proc. Phys. Soc. (London) A63, 599 (1950).

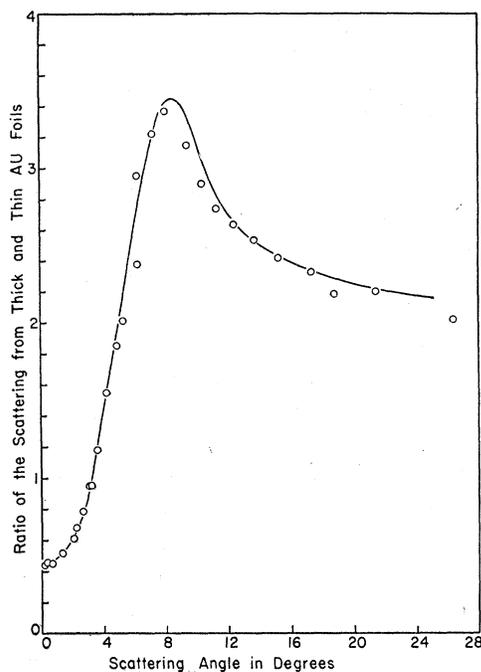


FIG. 1. Comparison of the experiments of Hanson *et al.* with the theory. Abscissa is scattering angle; ordinate is the ratio of the scattering by two gold foils differing by a factor 2 in thickness.

3 (their Fig. 3). They showed that similarly good agreement exists with the earlier experiments of Andrievsky, Kulchitsky, and Latyshev.⁷

At larger angles, the analysis was made somewhat difficult by the absence of accurate tables of Molière's functions for these angles. Hanson *et al.*, interpolated between Molière's tables and his asymptotic formula, which do not fit accurately together. This is especially apparent in their Fig. 4 which gives the ratio of the scattering by two gold foils having a ratio 2 in thickness: Molière's asymptotic formula gives smaller scattering ratios than observed.

In Fig. 1, we have compared the experiments with our more accurate Table II, which we have already shown to agree with our asymptotic formula (38) in Table III. The agreement is seen to be excellent. At large angles, our theoretical curve now lies slightly above the measurements, whereas Hanson's lay below. The position of the maximum now agrees exactly with experiment while Hanson's was shifted somewhat toward larger angles.

VIII. COMPARISON WITH THE THEORY OF GOUDSMIT AND SAUNDERSON

Goudsmit and Saunderson⁴ have shown that for any angle, small or large, the angular distribution is given *exactly* by

$$f(t, \theta) = \sum (l + \frac{1}{2}) P_l(\theta) \times \exp \left\{ -Nt \int \sigma(\chi) \sin \chi d\chi [1 - P_l(\chi)] \right\}, \quad (39)$$

where $f(t, \theta) \sin \theta d\theta$ is the number of electrons between θ and $\theta + d\theta$.

It should be noted that the quantity l means actually the distance travelled along the path of the electron. No attempt is being made in this paper to take into account the "detour factor," i.e., the difference between the distance travelled and the foil thickness, produced by the crooked path. This problem has been considered by Wang and Guth.¹⁷

Lewis⁵ has pointed out that expression (39) goes over into the small-angle expression of Molière, Snyder, and Scott if we replace $\sin \theta$ by θ and P_l by the well-known formula¹⁸

$$P_l(\theta) = J_0(l + \frac{1}{2})\theta, \quad (40)$$

which is valid for small θ regardless of whether l is small or large. To obtain (7), $l + \frac{1}{2}$ is replaced by η and the sum over l by an integral over η .

The approximation made by Molière therefore consists of 3 parts, *viz.* (a) the replacement of the sum over l by an integral, (b) approximations made in the exponential in (39), and (c) the use of the approximate formula (40) in the factor $P_l(\theta)$. Concerning (a), we may use the Euler summation formula,

$$\sum_{l=0}^{\infty} g(l + \frac{1}{2}) = \int_0^{\infty} g(\eta) d\eta + \frac{1}{24} g'(0) + \dots \quad (41)$$

Now in our case, according to (20),

$$g(\eta) = \eta J_0(\eta\theta) \exp \left[\frac{1}{4} \eta^2 (-b + \ln \frac{1}{4} \eta^2 \chi_e^2) \right], \quad (42)$$

and therefore

$$g'(0) = 1, \quad (42a)$$

from which

$$f_{GS}(t, \theta) = f_M(t, \theta) + 1/24 + \dots \quad (43)$$

(GS = Goudsmit and Saunderson, M = Molière). In the Gaussian region, $f_M \approx 1/\chi_e^2$ so that the correction term $1/24$ is very small as long as the critical angle χ_e is small compared with a radian. In the single-scattering region where θ is large, higher-order corrections in the Euler formula become important; the correction $1/24$ should certainly not be used when f_M is of the order of $1/24$ or less, and it should generally be regarded as an estimate of error rather than as a useful correction.

Now we investigate the errors introduced in the *exponent* of (39). First of all, it is now necessary to use the exact Rutherford formula,

$$Nt\sigma(\chi) \sin \chi d\chi = 2\chi_e^2 q(\chi) \frac{\sin \chi d\chi}{4(1 - \cos \chi)^2}, \quad (44)$$

where χ_e is still given by (10). For angles χ small compared with a radian, (44) goes over into our old

¹⁷ M. C. Wang and E. Guth, Phys. Rev. 84, 1092 (1951).

¹⁸ This formula is slightly more accurate than that given by Lewis, Eq. (19), in which $l + \frac{1}{2}$ is replaced by l .

formula (9). As in Sec. 3, we break the integral up into two parts, from 0 to k and from k to π , and, similarly to (11a) we choose k such that

$$\chi_0 \ll k \ll 1/l. \tag{44a}$$

Then, in the region from 0 to k , we may use the formula

$$1 - P_l(\chi) = \frac{1}{4}l(l+1)\chi^2 \tag{45}$$

and replace (44) by (9); then we get

$$\int_0^k Nt\sigma(\chi) \sin\chi d\chi [1 - P_l(\chi)] = \frac{1}{2}\chi_0^2 l(l+1) \int_0^k (d\chi/\chi) q(\chi), \tag{46}$$

in complete analogy with (15). The screening angle χ_a may then be introduced by (16), in analogy with Goudsmit and Saunderson.

In the interval $\chi > k$, we replace $q(\chi)$ by 1, as in (13). Then, for $l=1$, this part of the integral is elementary:

$$\frac{1}{2}\chi_0^2 \int_k^\pi \frac{\sin\chi d\chi}{1 - \cos\chi} = \frac{1}{2}\chi_0^2 \ln(1 - \cos\chi) \Big|_k^\pi = \chi_0^2 \ln(2/k). \tag{47}$$

For other values of l , Goudsmit and Saunderson have shown that

$$Nt \int_k^\pi \sigma(\chi) \sin\chi d\chi [1 - P_l(\chi)] = \frac{1}{2}\chi_0^2 l(l+1) \left[\ln(2/k) - \left(\frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{l} \right) \right]. \tag{48}$$

GS have omitted the proof of this formula as too lengthy; Lewis has given a proof which is somewhat complicated. An elementary proof is given in Appendix B.

Adding (46) and (48), and using (16) and (19), we get for the GS exponent,

$$\begin{aligned} Q_l &\equiv \int_0^\pi Nt\sigma(\chi) \sin\chi d\chi [1 - P_l(\chi)] \\ &= \frac{1}{2}\chi_0^2 l(l+1) \left[\ln 2/\chi_a - \frac{1}{2} - \left(\frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{l} \right) \right] \\ &= \frac{1}{2}\chi_0^2 l(l+1) \\ &\quad \times \left[-\ln\chi_a' + \ln 2 + C - \left(1 + \frac{1}{2} + \dots + \frac{1}{l} \right) \right]; \tag{49} \end{aligned}$$

or, using the well-known formula¹⁹

$$1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{l} = \Psi(l) + C, \tag{50}$$

with $\Psi(x) = d \ln \Gamma(x+1)/dx$, we get

$$Q_l = \frac{1}{2}\chi_0^2 l(l+1) [-\ln(\frac{1}{2}\chi_a') - \Psi(l)]. \tag{51}$$

This very simple formula is correct for all l which satisfy (44a). This limitation is the same as that of (11a) and, like the latter, introduces errors of the order of $1/\Omega_0$ at most, with Ω_0 the number of collisions (see end of Sec. 3).

We now use for Ψ the asymptotic formula,¹⁸

$$\Psi(l) = \ln(l + \frac{1}{2}) + \frac{1}{24}(l + \frac{1}{2})^{-2} + \dots \tag{52}$$

Neglecting the second term, setting $l + \frac{1}{2} = \eta = y/\chi_0$, and introducing b from (19), (51) becomes

$$Q_l = \frac{1}{4}\chi_0^2 l(l+1) [b - \ln(\frac{1}{4}y^2)]. \tag{53}$$

This differs from Molière's formula, (18), only by having the factor $l(l+1)$ instead of $(l + \frac{1}{2})^2$ outside the bracket.

The neglect of the second term in (52) is obviously justified for large l . Indeed, for large l it must be expected that the Goudsmit-Saunderson theory *should* give the same as Molière's because in this case the rapid oscillation of $P_l(\chi)$ destroys the contributions to the integral Q_l from large angles χ , while for small angles the approximations (9) and (40) are justified. However, this argument does not hold for small l , especially for $l=1$, and in this case 3 approximations are made in Molière's theory: the replacement of the Rutherford law (44) by (9), the replacement of the Legendre polynomial by the Bessel function (40), and that of the upper limit π of the angular integration by infinity. Equation (52) shows then that these approximations compensate almost exactly, the error being only 0.019 for $l=1$.

According to (53), the factor η^2 in Molière's exponent, e.g., in Eq. (17), should be replaced by $l(l+1)$. In other words, the integrand in (20) should be multiplied by

$$\exp\left[\frac{1}{16}\chi_0^2 (b - \ln\frac{1}{4}y^2) \right]. \tag{54}$$

Now, according to the beginning of Sec. 5, the most important values of y are of order $B^{-\frac{1}{2}}$, making the parenthesis in (54) equal to B , according to (23). Therefore,

$$f_{GS} \approx f_M \exp\left(\frac{1}{16}\chi_0^2 B \right). \tag{55}$$

Finally, we consider the factor $P_l(\theta)$ in (39). Molière [reference 13, Eq. (A.1)] has derived a formula considerably more accurate than (40), *viz.*

$$P_l(\theta) = (\theta/\sin\theta)^{\frac{1}{2}} J_0(l + \frac{1}{2}\theta). \tag{56}$$

At an angle as large as 90° , this formula gives 1.067, 0.032, and -0.502 , respectively, for $l=0, 1$, and 2 , as compared with the correct values 1, 0 and -0.5 . For all except very low l , the expression remains good even up to values of θ close to 180° , and breaks down only in the immediate neighborhood of 180° . For small angles, the approximation is, of course, particularly good.

¹⁹ See reference 14, p. 19.

Combining the three corrections, we get the approximate formula

$$f_{GS} = (\theta/\sin\theta)^{\frac{1}{2}} \exp(\frac{1}{16}\chi_e^2 B) f_M + 1/24. \quad (57)$$

From its derivation, this formula should be good approximately for $\chi_e^2 B \leq 1$, i.e., until the Gaussian has a width of about one radian. At this point, the exponential in (57) gives a correction of only 6 percent so that it can in general be neglected. For larger values of $\chi_e^2 B$, i.e., larger thickness of foil, only 2 or 3 terms in the Goudsmit-Saunderson formula (39) need to be taken into account so that this formula becomes very easy to handle directly. For smaller thickness, (57) may be used, and since both the exponential and the $1/24$ are in general unimportant, the angular distribution may simply be written

$$f_M(\theta)(\theta \sin\theta)^{\frac{1}{2}} d\theta, \quad (58)$$

where f_M is the Molière distribution function as calculated in this paper.

As pointed out in the beginning of this section, t is the total length of path of the electron, rather than the foil thickness t' . The difference $t-t'$ gives effects of the same order as the difference between the Goudsmit-Saunderson and the Molière distribution.

Lewis⁵ has shown how the energy loss can be taken into account and has calculated the lateral distribution in space.

I am greatly indebted to Dr. Max Goldstein of the Los Alamos Scientific Laboratory for the calculation of Table II and the development of methods which made this calculation possible. I also wish to thank Dr. Hanson of the University of Illinois for drawing my attention to Molière's theory and for discussion of the experiments, to Stanley Cohen of Cornell University for help with Table I and the figure, and to Dr. Henry Hurwitz of the Knolls Atomic Power Laboratory for showing me the essentials of the proof of Sec. 2 in 1949.

APPENDIX A. ALTERNATIVE DERIVATION OF ASYMPTOTIC FORMULA

We consider the region of large angles in which $f^{(1)}$, Eq. (28) dominates over all other contributions. According to Table II, $f^{(0)}$ becomes unimportant for $\vartheta > 3$. We shall neglect all terms which decrease exponentially with ϑ , such as $f^{(0)}$, but keep terms which decrease as inverse powers, ϑ^{-n} . We have shown at the end of Sec. 5 that $f^{(3)}$ can be neglected to an accuracy of 1 percent or better, so that we need consider only $f^{(1)}$ and $f^{(2)}$.

We shall now show that it is possible to combine these two terms, by re-arrangement of terms in the expansion (25). For this purpose, we start again from the exact formula (20) but introduce $\lambda y = z$, rather than u , as the integration variable. We then write the exponent of the exponential as follows:

$$\frac{1}{4}y^2(-b + \ln \frac{1}{4}y^2) = (z^2/4\lambda^2)[-b - 2 \ln(2\lambda/k) + 2 \ln(z/k)], \quad (61)$$

where k is a numerical constant which will be determined later to our convenience. We further introduce the abbreviations

$$\beta \equiv \vartheta_1^{-2} \equiv [b + 2 \ln(2\lambda/k)]/\lambda^2. \quad (62)$$

Since b is in general large compared to the logarithm, the ϑ_1 , defined in (62) is close to ϑ of (24) but is not identical with it. In our asymptotic region, ϑ_1 is large and β small.

The distribution function (20) becomes now

$$f(\theta)\theta d\theta = (d\lambda/\lambda) \int_0^\infty z dz J_0(z) \times \exp(-\frac{1}{4}\beta z^2) \exp[(z^2/2\lambda^2) \ln(z/k)]. \quad (63)$$

Expanding the last exponential, the first term will, upon integration, give a function like $f^{(0)}$ which decreases exponentially with angle λ , and can therefore be neglected. The second term will give a function like $f^{(1)}$ which will be our main contribution. The third term gives, except for a constant factor $(8\lambda^4)^{-1}$:

$$F^{(2)} = \int_0^\infty z dz J_0(z) \exp(-\frac{1}{4}\beta z^2) z^4 (\ln z/k)^2. \quad (64)$$

Our simplification will now be achieved by making $F^{(2)}$ equal to zero by appropriate choice of the free numerical constant k . Then the distribution function is reduced to $F^{(1)}$ alone.

In the limit of small β , the integral (64) can be evaluated analytically and gives

$$\lim_{\beta \rightarrow 0} F^{(2)} = -128(\frac{3}{2} + \ln 2 - C - \ln k). \quad (65)$$

To make this zero, we have to choose

$$k = 5.0325 \quad (65a)$$

or nearly 5. The result (65) is closely related to Molière's asymptotic formula for $f^{(2)}$, Eq. (34).

For somewhat larger β , it is possible to estimate k by considering the integral (63) in the complex plane. We first substitute for $J_0(z)$ the real part of the Hankel function $H_0^{(1)}(z)$. Then we replace the integral along the positive real axis of z by one along the following contour: We follow the imaginary axis up to $z = 2i/\beta$ and then go parallel to the positive real axis. To evaluate the integral along the imaginary axis, we write $z = ix$, with x real, and get for the integral in (63):

$$-\text{Re} \int_0^{2/\beta} x dx H_0^{(1)}(ix) e^{i\beta x^2} \times \exp[-(x^2/2\lambda^2)\{\ln(x/k) + \pi i/2\}]. \quad (66)$$

Since most of the contribution comes from large x , see Eq. (70), we may replace the Hankel function by its asymptotic expression,

$$H_0^{(1)}(ix) = -i(2/\pi x)^{\frac{1}{2}} e^{-x}. \quad (67)$$

As in Eq. (25), it is now simplest to expand the last exponential in (66). The first term in this expansion, 1 is purely real. The integral in (66) is then purely imaginary, which means that the integral along the imaginary axis gives no contribution at all. This leaves only the second part of the contour parallel to the real axis which obviously gives a contribution proportional to the value of the integrand at $x=2/\beta$ which is about $\exp(-x+\frac{1}{4}\beta x^2) = \exp(-1/\beta) = \exp(-\vartheta_1^2)$. This explains the exponential decrease of $f^{(0)}$, Eq. (27). (For β moderate or large, this term is of course large.)

The second term in the expansion of the exponential is

$$-(x^2/2\lambda^2)[\ln(x/k) + \pi i/2]. \tag{68}$$

Only the imaginary part matters, so that the integral becomes

$$(\pi/2)^{\frac{1}{2}}(2\lambda^2)^{-1} \int_0^{2/\beta} x^3 \exp(-x + \frac{1}{4}\beta x^2) x^{-\frac{1}{2}} dx. \tag{69}$$

We now consider the integrand of (69), which we define as

$$x^3 \exp(-x + \frac{1}{4}\beta x^2). \tag{69a}$$

The factor $x^{-\frac{1}{2}}$ is better included with dx as will be shown below, Eq. (72). This integrand has a maximum at

$$x_1 = \beta^{-1}[1 - (1 - 6\beta)^{\frac{1}{2}}]. \tag{70}$$

In the limit of small β which interests us particularly, x_1 has the value 3. This is large enough to use the asymptotic formula (67) but is very small compared with $2/\beta$. The main contribution to the integral (69) comes then from the neighborhood of x_1 and the integral can be evaluated by a saddle point method. The resulting asymptotic formula is similar to (35). For larger β , x_1 increases; for $\beta = \frac{1}{6}$, it reaches the value $1/\beta = 6$. For still larger β , there is no longer any solution x_1 : The integrand (69a) then has no maximum for real x but increases monotonically to $x = 2/\beta$. Then the main contribution comes from the neighborhood of that point: We get into the region where the Gaussian dominates. Thus we should expect that our asymptotic theory will hold reasonably well for $\beta \ll \frac{1}{6}$ or $\vartheta_1^2 \gg 6$, $\vartheta_1 > 2.45$. This is in agreement with Table III which shows that at about $\vartheta = 2.45$, formula (37) breaks down and simultaneously the Gaussian $f^{(0)}$ becomes more important than $f^{(1)}$.

We now turn to the third term in the expansion of the exponential in (66) which is

$$-(x^2/2\lambda^2)^2[\ln(x/k) + \pi i/2]^2. \tag{71}$$

The imaginary part of this which is the only part giving a contribution, yields the integral

$$F^{(2)} \sim \int x^5 \exp(-x + \frac{1}{4}\beta x^2) \ln(x/k) x^{-\frac{1}{2}} dx. \tag{72}$$

If β is neglected, the integrand is $x^5 e^{-x}$ and has a maxi-

mum at $x=5$. Therefore the integral, evaluated by the method of steepest descent, will be zero if k is set equal to 5. It was shown in (65a) that this is very nearly correct, the exact value of k being 5.0325. This is the reason why we included $x^{-\frac{1}{2}}$ with dx ; had we included it with the integrand, the maximum would occur at $x_2=4.5$ which would give less accurate results.

For finite values of β , the exponential has a maximum at

$$x_2 = \beta^{-1}[1 - (1 - 10\beta)^{\frac{1}{2}}]. \tag{73}$$

The method of steepest descent will therefore fail for $\beta > \frac{1}{10}$ or $\vartheta_1 < 10^{\frac{1}{2}} = 3.16$; at this point, $x_2=10$. For smaller β , the method can be applied and shows that $F^{(2)}=0$, if we set

$$k = x_2. \tag{73a}$$

Therefore, if we define ϑ_1 by (62) and insert for k the value (73), the distribution function is given by $f^{(1)}(\vartheta_1)$ alone, without any contribution $f^{(2)}$.

According to (73), k depends on ϑ_1 so that ϑ_1 is defined by an implicit equation which is not convenient. However, in the region in which the method is useful at all, (73) may be expanded, giving²⁰

$$k = x_2 = 5/(1 - 5\beta/2 + \dots). \tag{74}$$

The useful range is somewhat better covered by putting

$$k = 5/(1 - 3\beta). \tag{74a}$$

This is accurate within better than 1 percent for $\vartheta_1 > 4$, $\beta < \frac{1}{16}$. Inserting into (62), we get then

$$\lambda^2 \vartheta_1^{-2} = b + 2 \ln(2\lambda/5) + 2 \ln(1 - 3\vartheta_1^{-2}), \tag{75}$$

which is still implicit but now very simple.

The last term in (75) is small and can therefore be treated approximately. In particular, it will be shown presently that ϑ_1 is very nearly equal to ϑ in the region $\vartheta_1 \sim 3.5$, i.e., where the asymptotic treatment just begins to be valid and where therefore $\ln(1 - 3\vartheta_1^{-2})$ is as large as it can get. Therefore we replace ϑ_1 by ϑ in the last term of (75), and we can then get a relation between ϑ and ϑ_1 , using (24) and (23):

$$\begin{aligned} \vartheta_1 &= B^{\frac{1}{2}} \vartheta [B + 2 \ln(2\vartheta/5) + 2 \ln(1 - 3\vartheta^{-2})]^{-\frac{1}{2}} \\ &= B^{\frac{1}{2}} \vartheta [B + 2 \ln(0.4\vartheta - 1.2\vartheta^{-1})]^{-\frac{1}{2}}. \end{aligned} \tag{76}$$

From this it follows that $\vartheta \approx \vartheta_1$ in the important region. Furthermore, this result provides a check on the theory of this Appendix: Molière's $f^{(2)}$ vanishes for $\vartheta = 3.80$ (Table II) so that his distribution function is given by $B^{-1} f^{(1)}(\vartheta)$ alone for this particular value of ϑ . On the other hand, our ϑ_1 is so defined that the distribution function is $B^{-1} f^{(1)}(\vartheta_1)$ for all (sufficiently large) ϑ_1 . Therefore we should have $\vartheta_1 = \vartheta$ at the ϑ for which Molière's $f^{(2)}$ vanishes. The agreement is reasonably satisfactory.

²⁰ This expansion in the denominator is somewhat more accurate than a direct expansion, and will actually be more convenient Eq. (76).

The ratio to Rutherford scattering, (32), becomes now, using (63), (31), (62), (26):

$$R = \int z dz J_0(z) \exp(-\frac{1}{4}z^2/\vartheta_1^2) \frac{1}{4}z^2 \ln(z/k) \\ = \frac{1}{2}\vartheta_1^4 f^{(1)}(\vartheta_1). \quad (77)$$

Since the value of k matters only for the part proportional to $\exp(-\vartheta_1^2)$, it is often convenient to use (77) directly with (76) and Table II. Alternatively, we may use the asymptotic formula (33), getting

$$R = (1 - 5\vartheta_1^{-2})^{-4/5}. \quad (78)$$

Inserting (76), expanding the reciprocal of R and dropping terms of relative order B^{-2} or ϑ^{-6} leads directly to our old formula (36).

Finally, we may write down our complex integral (66) without expanding the exponential; it is

$$(2/\pi)^{\frac{1}{2}} \int x^{\frac{1}{2}} dx \exp(-x + \frac{1}{4}\beta x^2) \\ \cdot \exp[-\frac{1}{2}x^2\lambda^{-2} \ln(x/k)] \sin(\pi x^2/4\lambda^2). \quad (79)$$

The approximation used earlier in this Appendix, of considering only $f^{(1)}$, is equivalent to replacing the sine by its argument. Since the important x are around 3, and λ is very large (greater than 10) in the single-scattering region, this is a good approximation.

APPENDIX B. DERIVATION OF THE GOUDSMIT-SAUNDERSON FORMULA

We shall here prove Eq. (48). Since k is chosen to satisfy (44a), we may neglect screening. Setting $\cos\chi = x$, we thus have to calculate [see (44)]

$$K_l = 2 \int_{-1}^{1-\epsilon} dx (1-x)^{-2} [1 - P_l(x)], \quad (81)$$

where $\epsilon \approx \frac{1}{2}k^2$ and the factor 2 is inserted for convenience in the following. Because of (44a), P_l at $\chi = k$ satisfies (45), or, since we now use x as the argument:

$$1 - P_l(1-\epsilon) = \frac{1}{2}l(l+1)\epsilon. \quad (82)$$

We first integrate (81) by parts:

$$K_l = 2 \frac{1 - P_l(x)}{1-x} \Big|_{-1}^{1-\epsilon} + \int_{-1}^{1-\epsilon} \frac{2dx}{1-x} P_l'(x). \quad (83)$$

The integrated part gives, according to (82)

$$l(l+1) - 1 + P_l(-1). \quad (83a)$$

In the integral, we write

$$\frac{2}{1-x} = \frac{1+x}{1-x} + 1 = \frac{1-x^2}{(1-x)^2} + 1. \quad (83b)$$

Then the term with 1 can be integrated, giving

$$1 - P_l(-1), \quad (83c)$$

which cancels the last two terms of (83a) (a term of order ϵ has been neglected). In the other term, we perform another integration by parts, giving

$$K_l = l(l+1) + \frac{1}{1-x} (1-x^2) P_l' \Big|_{-1}^{1-\epsilon} \\ - \int_{-1}^{1-\epsilon} \frac{dx}{1-x} \frac{d}{dx} [(1-x^2) P_l']. \quad (84)$$

Using (82) one finds that the second term cancels the first. In the last term, we use the differential equation for the Legendre polynomials and get

$$K_l = l(l+1) A_l, \quad (85)$$

$$A_l = \int_{-1}^{1-\epsilon} dx P_l/(1-x), \quad (85a)$$

which is a considerable simplification.

In order to be able to extend the integral to 1 rather than $1-\epsilon$, we take the difference

$$A_{l-1} - A_l = \int_{-1}^1 dx (P_{l-1} - P_l)/(1-x). \quad (86)$$

Here we use the relation between spherical harmonics,²¹

$$l(P_{l-1} - xP_l) = (1-x^2)P_l'. \quad (87)$$

Then (86) becomes

$$A_{l-1} - A_l = l^{-1} \int_{-1}^1 dx (1+x) P_l' - \int_{-1}^1 P_l dx. \quad (88)$$

Integrating the first integral once more by parts, and using the fact that

$$\int P_l dx = 0 \quad (89)$$

for any $l \neq 0$ (orthogonality), we get

$$A_{l-1} - A_l = l^{-1} (1+x) P_l \Big|_{-1}^1 = 2/l. \quad (90)$$

Inserting back into (85), we find then

$$K_l = l(l+1) \left[A_1 - 2 \left(\frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{l} \right) \right]; \quad (91)$$

and inserting $A_1 = \frac{1}{2}K_1$ from (47), we have proved Eq. (48).

²¹ See reference 14, p. 115.