our apparatus. Both of these effects would increase the calculated spread for a given depth thus increasing the agreement between the experimental points and the curve calculated for the exponential form of the differential cross section. Evidence in favor of this form of differential cross section has already been given by Messel and Green,[11] and Hazen *et al.*[12] has shown that the Fermi distribution tends to give lateral spreads of extensive air showers considerably greater than those reported experimentally.

CONCLUSION

We have been able to determine experimentally the variation of the intensity of charged particles with depth, the variation of the lateral spread with depth, and the variation of the number of charged particles at the maximum of the transition curve with the depth of that maximum for the nucleon induced cascade in water. In all cases there is qualitative agreement with theory. Whereas in some cases the theory of the complete cascade is not sufficiently developed for a quantitative comparison to be possible, in the case of the lateral spread our results make it seem very likely that the differential cross section in high energy nucleon-nucleon collisions cannot have a quasi-isotropic form. On the other hand the exponential form suggested by Messel and Green gives good agreement.

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[12] Hazen, Heineman, and Lennox, Phys. Rev. 86, 198 (1952).

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Relativistic Corrections to the Magnetic Moments of Nuclear Particles*

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A re-examination of the problem is reported. The results for the vector and scalar cases in the case of the deuteron are explained in terms of known correction factors for the one-body problem. For the vector equation part of the result is caused by an induction effect which is the meson theoretic generalization of Faraday's law of induction. In the scalar case the relation to the one-body result is made in a form employing an effective change in mass caused by the presence of the scalar. These interpretations are substantiated by an analysis in terms of plane waves. Simple forms are obtained for one particle in a pseudoscalar field and a tentative application to the deuteron is made and criticized.

1. INTRODUCTION AND NOTATION

RELATIVISTIC corrections to the magnetic moment of a single Dirac particle in a central potential field have been discussed by Breit[1] and by Margenau.[2] The latter has pointed out that an application of the formula applicable to the one-electron case to the deuteron problem gives effects comparable with the change of the magnetic moment expected on account of the admixture of a D-wave to the ground state of H^2. Caldirola[3] was the first to consider the relativistic correction for the case of a particle having an intrinsic magnetic moment in Pauli's sense. Caldirola's signs are either inconsistently used or applied with a misunderstanding regarding the correction factor for the proton, which differs from unity by an amount which is too small in absolute value under his postulated assumptions having been obtained as $-0.667+0.596=-0.071$ rather than $-0.667-0.596$. Since there is an almost compensating error for the neutron, the result for the deuteron is practically unaffected by this ambiguity. In view of this situation and the fact that Caldirola's work considered the particle with intrinsic Pauli moment to be in a central field, a condition which is not satisfied in the deuteron, the problem was again briefly treated by Breit.[4] In this discussion the correction for the intrinsic moment has the form

$$1-\langle T_\sigma\rangle/Mc^2,$$

where T_σ is the part of the kinetic energy owing to motion along the direction of the particle spin. The character of the field enters the result only through T_σ, and this part of the correction may therefore be used directly for the deuteron. The relativistic factor for a single charged particle in a central scalar field was also contained in this work. It was pointed out in the same note that a single particle treatment does not suffice for the calculation of effects stemming from the Dirac current of the particle's charge; the contribution to the relativistic correction arising from the charge of the

* Assisted by the joint program of the U. S. Office of Naval Research and the U. S. Atomic Energy Commission.

[1] G. Breit, Nature 122, 649 (1928).
[2] H. Margenau, Phys. Rev. 57, 383 (1940).
[3] P. Caldirola, Phys. Rev. 69, 608 (1946).
[4] G. Breit, Phys. Rev. 71, 400 (1947).

proton may in fact be affected by the two-body character of the problem.

This aspect has been treated by Sachs[5] and by Breit and Bloch,[6] the papers being concerned with hypothesized scalar and vector interactions. The calculations of Breit and Bloch contain an error in manipulation as has been ascertained for the scalar case by Adams[7] and confirmed by Breit.[8] The corrected result agrees with that obtained by Sachs[5] for the same field. The error just mentioned affected the results for the vector field as well, the same erroneously evaluated integral having entered both cases. The corrected vector case factor turned out to be unity in disagreement with the result of Sachs. The procedure used by Sachs was not convincing; the effective Hamiltonians were not covariant and the employment of momentum operators for current was made in such a way as to omit part of the effect of spin currents in the vector case. On the other hand, the main point brought out in the discussions of Breit,[4] Sachs,[5] Breit and Bloch,[6] and Primakoff[9] has been that it was not possible to make relativistic corrections with certainty and there existed considerable uncertainty in attempts at quantitative conclusions regarding the fractional content of the ^3D state in the ground state of the deuteron. From this viewpoint the result of Sachs for the vector case may be regarded as an illustration of the flexibility which could be attained if one invented a vector meson field in which coupling to the nuclear spin current would be absent. The papers mentioned have not considered the pseudoscalar interaction which appears at present to be the most probable one. In view of the absence of a consistent and error free treatment of the two-body problem it appeared desirable to reexamine the subject, establishing more obvious connections between single particle and two-body results and removing some of the uncomfortable dependence on involved calculations which has been necessary so far. A discussion of the pseudoscalar interaction is included. The answer in this case is especially simple, the correction factor being Mc^2/E for a single particle. The small magnitude of this correction suggests agreement with experiment for the deuteron, where $(\mu_N+\mu_P)-\mu_D=0.879-0.857=0.022$. Here μ_N, μ_P, μ_D are the magnetic moments of the neutron, the proton, and the deuteron, respectively. However, such an identification is difficult to justify since it involves the assumption of additivity of nucleon moments.[10–12] The possibility of additional corrections having their origin in distorting or exchange effects between nucleons is not denied, but the percentage of D state arrived at by Lévy[13] from the $PSps$ theory and from nuclear two-body data does not definitely indicate its presence.

[5] R. G. Sachs, Phys. Rev. 72, 91 (1947).
[6] G. Breit and I. Bloch, Phys. Rev. 72, 135 (1947).
[7] E. N. Adams II, Phys. Rev. 81, 1 (1951).
[8] See footnote 20 of E. N. Adams (reference 7). The authors wish to thank Dr. Adams for his correspondence relating to this matter and for making available some of his calculations.
[9] H. Primakoff, Phys. Rev. 72, 118 (1947).
[10] H. Miyazawa, Prog. Theoret. Phys. 7, 207 (1952).
[11] R. Osborn and L. Foldy, Phys. Rev. 79, 795 (1950).
[12] R. G. Sachs, Phys. Rev. 74, 433 (1948).

Notation

I, II . . . subscripts designating the two particles. For the deuteron, I designates the proton, II the neutron.

1, 2 . . . subscripts designating respectively values of a quantity for a one-body and two-body problem; these subscripts are used only when a distinction between the two cases is necessary.

E = energy including rest mass energies.

$\mathbf{p}$ = momentum (variable canonically conjugate to coordinate).

$\boldsymbol{\alpha}=(\alpha_x, \alpha_y, \alpha_z)$ = Dirac's four-row square matrix vector.

$\boldsymbol{\alpha}=\rho_1(\sigma_x, \sigma_y, \sigma_z)$, where the σ are Dirac's four-row spin matrices.

β = Dirac's $\alpha_4=\rho_3$.

$\boldsymbol{\sigma}$ = Pauli's two-row square matrix vector when it occurs in a nonrelativistic approximation.

ψ = Dirac's spinor wave function having four components per particle.

Φ = column matrix formed by first two components ("small" components) of ψ in one body problem; employment of Dirac's original representation of the α_μ is presupposed. In the two body problem Φ is operated on by spinor index matrices like the direct product of the corresponding quantities for the two particles.

Ψ, χ_{I}, χ_{II} are quantities similar to Φ with Ψ "large" in both particles; χ_{I} "small" in I and "large" in II; χ_{II} "large" in I and "small" in II.

W = energy excluding rest mass energies; W is negative for the deuteron.

M = mass of nucleon.

T = kinetic energy in nonrelativistic approximation, excluding rest mass energies.

J = negative of potential energy in nonrelativistic approximation.

$\mathbf{u}$ . . . arbitrary fixed unit vector.

u . . . index used for designation of components along $\mathbf{u}$.

$f=dJ/rdr$.

$[A, B]_+=AB+BA$.

$\mathbf{r}$ = position vector in the one-body case. In the two-body case $\mathbf{r}=\mathbf{r}_{\mathrm{I}}-\mathbf{r}_{\mathrm{II}}$.

$r=|\mathbf{r}|$.

$\mathfrak{A}$ = mesic vector potential.

$\mathcal{J}=\mathcal{J}(|\mathbf{p}-\mathbf{p}'|)$ = momentum transform of J, see Eq. (12.4).

$\mathcal{J}'=(d\mathcal{J}(x)/dx)_{x=|\mathbf{p}-\mathbf{p}'|}$.

$\mathfrak{L}(\mathbf{p})C(\mathbf{p})$ = wave function in momentum space.

$\mathfrak{L}(\mathbf{p})=(E^a+Mc^2)\beta+c(\boldsymbol{\alpha}\mathbf{p})$.

$E^a=|(M^2c^4+p^2c^2)^{\frac{1}{2}}|$.

C = four-component amplitude column matrix, describing spin orientation and sign of energy.

[13] M. Lévy, Phys. Rev. 88, 725 (1952). The authors are grateful to Dr. Lévy for making available to them a preprint of this paper.

C_- = two-component amplitude column matrix, corresponding to negative energy, consisting of the first and second components of C.
C_+ = two-component amplitude column matrix, corresponding to positive energy, consisting of the third and fourth components of C.
$\mathbf{L}=[\mathbf{r}\times\mathbf{p}]$.
$\mathbf{v}=[\mathbf{u}\times\mathbf{r}]$.
$\mathcal{C}$ = two-body wave function in momentum space.
g = pseudoscalar coupling constant.
χ = pseudoscalar nonquantized wave field.

2. LIMITATIONS OF PREVIOUS FORMAL TREATMENTS

(a) The Treatment of Sachs[5]

In this discussion the scalar and vector interactions are considered, employing explicitly a nonquantized meson field. The field produced by I is used in static approximation, and the terms corresponding to it are included in the part of the two-body Hamiltonian which belongs to II in the uncoupled problem. The Hamiltonian is then symmetrized in I and II. The assumption is made that it suffices to use a static approximation for the meson field produced by I in the process just described and that in the vector case one may substitute for $\boldsymbol{\alpha}_{\mathrm{I}}$ the approximate representation by $\mathbf{p}_{\mathrm{I}}$.

The approximation mentioned last neglects the effect of the particle spin current of I. This current carries with it a current of mesic charge. Its neglect is analogous to disregarding the magnetic field caused by the spin magnetic moment of one electron in a calculation of the magnetic moment of the second electron. According to Lenz' law one expects the spin current of the second electron to change in such a direction as to oppose the change of magnetic flux through a fixed closed curve produced by the first electron.

The Hamiltonians used by Sachs include some corrections of order v^2/c^2 but omit corrections for retardation which are formally of the same order of magnitude. Without a demonstration of the absence of effects of retardation on the magnetic moments one cannot arrive at a definite conclusion regarding the value of the magnetic moment.

The construction of Hamiltonians by the procedure described can at most establish them to the second order of the interaction constant. The treatment of the deuteron to this order is not sufficient for the reproduction of experimental results as is clear from the fact that a calculation of p-n scattering by the first Born approximation would be very inaccurate. The corrections dealt with cannot be believed in, therefore, for the whole interaction unless an additional justification can be made.

(b) The Treatment of Breit and Bloch[6]

This work makes use of the possibility of formally correcting some types of wave equations for lack of relativistic invariance of their predictions. It employs Hamiltonians thus set up employing all terms literally. While consistent from the formal viewpoint of covariance, the method lacks justification in the following respects.

No proof has been given that in addition to the terms which have been added to the Hamiltonians in order to secure covariance there are not present other terms of relative order v^2/c^2. It is well known that requirements of covariance alone do not determine a two-body Hamiltonian but only restrict the choice of possibilities. To be sure the vector equation is a rather immediate extension of the special case of electromagnetic interaction. But in this case the terms of the form

$$\tfrac{1}{2}[(\boldsymbol{\alpha}_{\mathrm{I}}\boldsymbol{\alpha}_{\mathrm{II}})J-(\boldsymbol{\alpha}_{\mathrm{I}}\mathbf{r})(\boldsymbol{\alpha}_{\mathrm{II}}\mathbf{r})(dJ/rdr)]$$

have been established[14] only to the extent of representing the energy correction through their expectation value. It is in fact possible to obtain wrong results in radiation problems by employing these terms to calculate the wave function. In the magnetic moment problem the magnetic interaction-retardation terms yield a nonvanishing contribution, and a justification of the employment of the approximate Hamiltonian therefore has to be given. A complete treatment would consist in a calculation avoiding the use of a two-body Hamiltonian, except possibly as an intermediate step, but based otherwise on a consistent field theory. Since such a theory is not available a perfect treatment is impossible. The scalar equation used by Breit and Bloch is subject to a similar criticism.

It is especially unclear and uncertain that the problem may be stated with sufficient accuracy in terms of a two-particle Hamiltonian with correction terms having a universal form in terms of J. Thus, e.g., if the correct treatment were capable of being formulated in terms of an expansion of observables in powers of J, then each term in the expansion could conceivably require different forms of relativistic corrections. The large value of the interaction constant usually denoted by f or g makes it impossible to disregard this possibility on the grounds of rapid convergence of the series in powers of J. The work of Salpeter and Bethe[15] indicates the likelihood of such effects since in hyperfine structure calculations they find additional terms of relative order $\alpha m/M$ with α, m, M standing for the fine structure constant and the masses of the particles.

3. VECTOR INTERACTION

A straightforward application of the method used by Breit and Bloch[6] gives for an S state of the deuteron system the correction factor to the proton moment owing to the proton charge as

$$(CF)_V^{(2)}=1-\langle(\sigma_{\mathrm{I}}-\sigma_{\mathrm{II}})_z\rangle[\langle T_2\rangle/(3Mc^2)], \qquad (1)$$

[14] G. Breit, Phys. Rev. 51, 248 (1937).
[15] E. E. Salpeter, Phys. Rev. 87, 328 (1952).

the two spins being supposedly oriented along the z axis, with T_2 representing the energy of relative motion. Subscripts I, II refer to the proton and neutron, respectively. For the parallel spin orientation in the ^3S state the whole factor reduces to

$$(CF)_V{}^{(2)}=1. \tag{1.1}$$

This result may be understood as follows. For the single particle problem the factor[2]

$$(CF)_V{}^{(1)}=1-2\langle T_1\rangle/(3Mc^2) \tag{2}$$

is applicable in this case. Here the nonrelativistic approximation to T_1 suffices. In this approximation

$$\langle T_2\rangle=2\langle T_1\rangle, \tag{2.1}$$

questions of frames of reference being irrelevant within the accuracy of the calculation. Thus the correction factor is expected to be

$$1-\langle T_2\rangle/(3Mc^2). \tag{2.2}$$

The mesic field induction effect is omitted, however, in this estimate. This effect arises from the fact that particle II (neutron) produces a mesic vector potential

$$\mathfrak{A}_{\mathrm{I}}$$

at the location of particle I. The vector potential modifies the single particle Hamiltonian for I by the correction

$$\delta H_{\mathrm{I}}=-(\boldsymbol{\alpha}_{\mathrm{I}}\mathfrak{A}_{\mathrm{I}}) \tag{2.2$'$}$$

in the interaction energy,

$$H_{\mathrm{I}}'=-J+\delta H_{\mathrm{I}}, \tag{2.2$''$}$$

which represents the interaction of particle I with the meson field. As in the electromagnetic case,

$$\mathfrak{A}_{\mathrm{I}}=-\boldsymbol{\alpha}_{\mathrm{II}}J$$

and

$$H_{\mathrm{I}}'=-J[1-(\boldsymbol{\alpha}_{\mathrm{I}}\boldsymbol{\alpha}_{\mathrm{II}})]. \tag{2.3}$$

Taking into account additional terms arising from the inclusion of δH_{I} in the calculation of the one-body problem, one obtains an addition to the expression representing small components in terms of large ones, *viz.*,

$$\delta\Phi_{\mathrm{I}}=(J/2Mc^2)(\boldsymbol{\sigma}_{\mathrm{I}}\boldsymbol{\alpha}_{\mathrm{II}})\Psi_{\mathrm{I}}, \tag{2.4}$$

the subscripts I on Φ and Ψ indicating a one-body treatment of the proton (i.e. I). This change in Φ_{I} produces a change

$$\delta(\psi_{\mathrm{I}}{}^*[\mathbf{r}_{\mathrm{I}}\times\boldsymbol{\alpha}_{\mathrm{I}}]_u\psi_{\mathrm{I}})=(J/Mc^2)(\Psi_{\mathrm{I}}{}^*[\mathbf{r}_{\mathrm{I}}\times\boldsymbol{\alpha}_{\mathrm{II}}]_u\Psi_{\mathrm{I}}) \tag{2.5}$$

which is found to be expressible approximately as the mean value of

$$(\hbar/2M^2c^3)[\boldsymbol{\sigma}_{\mathrm{II}}(\mathbf{r}_{\mathrm{I}}\mathbf{r})-\mathbf{r}(\mathbf{r}_{\mathrm{I}}\boldsymbol{\sigma}_{\mathrm{II}})]f, \tag{2.6}$$

with

$$f=dJ/rdr \tag{2.7}$$

and

$$\mathbf{r}=\mathbf{r}_{\mathrm{I}}-\mathbf{r}_{\mathrm{II}},\quad r=|\mathbf{r}|. \tag{2.7$'$}$$

For s terms the expression yields Eq. (1.1). In the above account of the work the presentation is such as though the two-body wave function were a product of one-body functions for particles I and II. The calculations have been carried, however, also for linear combinations of products, so as to apply also in configuration space. The limitations existing at this stage will become more apparent in the analysis of the problem in terms of plane waves.

The results obtained by the separate consideration of the two particles agree with calculations made by employing the covariant Hamiltonians as in the work of Breit and Bloch.[6] In the notation of this reference and in units making $\hbar=c=1$,

$$(\psi,[\mathbf{r}_{\mathrm{I}}\times\boldsymbol{\alpha}_{\mathrm{I}}]\psi)_{Vec}=\left(\Psi^{(1)},\left\{\left[-\frac{1}{M}+\frac{W+J}{2M^2}+\frac{p_{\mathrm{I}}{}^2-p_{\mathrm{II}}{}^2}{4M^3}\right](\mathbf{L}_{\mathrm{I}}+\boldsymbol{\sigma}_{\mathrm{I}})-\mathbf{D}+\frac{\mathbf{C}_V}{4M^2}\right\}\Psi^{(1)}\right), \tag{3}$$

where

$$\mathbf{D}=\{i[\mathbf{r}_{\mathrm{I}}\times\boldsymbol{\nabla}_{\mathrm{I}}J]+(\boldsymbol{\nabla}_{\mathrm{I}}J)(\mathbf{r}_{\mathrm{I}}\boldsymbol{\sigma}_{\mathrm{I}})-\boldsymbol{\sigma}_{\mathrm{I}}(\mathbf{r}_{\mathrm{I}}\boldsymbol{\nabla}_{\mathrm{I}}J)\}/(4M^2), \tag{3.1}$$

$$[A,B]_+=AB+BA, \tag{3.2}$$

$$\mathbf{C}_V=[(\boldsymbol{\sigma}_{\mathrm{II}}\mathbf{p}_{\mathrm{II}}),[X,[\mathbf{r}_{\mathrm{I}}\times\boldsymbol{\sigma}_{\mathrm{I}}]]_+]_+ \tag{3.2$'$}$$

$$X=-\tfrac{1}{2}(\boldsymbol{\sigma}_{\mathrm{I}}\boldsymbol{\sigma}_{\mathrm{II}})J+\tfrac{1}{2}(\boldsymbol{\sigma}_{\mathrm{I}}\mathbf{r})(\boldsymbol{\sigma}_{\mathrm{II}}\mathbf{r})f, \tag{3.3}$$

$$\Psi^{(1)}=(\Psi,\Psi)^{-\frac{1}{2}}\Psi=[1+(p_{\mathrm{I}}{}^2+p_{\mathrm{II}}{}^2)/8M^2+\cdots]\Psi. \tag{3.3$'$}$$

Calculation gives for s terms

$$\langle\mathbf{D}\rangle=-\langle r^2f\boldsymbol{\sigma}_{\mathrm{I}}\rangle/(12M^2)=\langle T_2\boldsymbol{\sigma}_{\mathrm{I}}\rangle/(6M^2), \tag{3.4}$$

$$\left(\Psi^{(1)},\left[\left(-\frac{1}{M}+\frac{W+J}{2M^2}+\frac{p_{\mathrm{I}}{}^2-p_{\mathrm{II}}{}^2}{4M^3}\right)(\mathbf{L}_{\mathrm{I}}+\boldsymbol{\sigma}_{\mathrm{I}})-\mathbf{D}\right]\Psi^{(1)}\right)=-\frac{1}{M}\left\langle\left(1-\frac{T_2}{3M}\right)\boldsymbol{\sigma}_{\mathrm{I}}\right\rangle, \tag{3.5}$$

and in general

$$\langle\mathbf{C}_V\rangle=2\langle[\boldsymbol{\sigma}_{\mathrm{II}}(\mathbf{r}\mathbf{r}_{\mathrm{I}})-\mathbf{r}(\boldsymbol{\sigma}_{\mathrm{II}}\mathbf{r}_{\mathrm{I}})]f\rangle \tag{3.6}$$

which becomes

$$\langle\mathbf{C}_V\rangle=-(4/3)\langle T_2\boldsymbol{\sigma}_{\mathrm{II}}\rangle,\quad (s\text{ terms}). \tag{3.7}$$

These calculations differ from corresponding steps in the work of Sachs[5] through the consistent use of the operators $\boldsymbol{\alpha}$ and from the work of Breit and Bloch[6] through the correct evaluation of $\mathbf{C}_V$. Inserting the values listed in Eqs. (3.4), (3.5), (3.7) into Eq. (3)

there results

$$\langle[\mathbf{r}_{\mathrm{I}}\times\boldsymbol{\alpha}_{\mathrm{I}}]\rangle_{Vec}=-\frac{1}{M}\left\langle\left(1-\frac{T_2}{3M}\right)\boldsymbol{\sigma}_{\mathrm{I}}\right\rangle-\frac{\langle T_2\boldsymbol{\sigma}_{\mathrm{II}}\rangle}{3M^2},\quad(4)$$

which is seen to agree with Eq. (1).

Comparing Eq. (3.5) with Eqs. (2), (2.2) one sees that all terms in the two body treatment with the exclusion of $\mathbf{C}_V$ account for the one-body effects caused by I (proton) being exposed to the action of J and without inclusion of effects of $\mathfrak{A}_{\mathrm{I}}$ which are treated in Eqs. (2.2′), (2.2″), (2.3), (2.4), and (2.5). The correction for the generalization of the magnetic induction effect treated in these equations corresponds in the two-body calculation to the terms arising from $\mathbf{C}_V$. These terms arise from X of Eq. (3.3), and this term arises from the generalization of the magnetic interaction between two charges with inclusion of the correction for retardation in the action of the electrostatic potential. The values in Eqs. (2.6), (3.6) agree. The two ways of obtaining $(CF)_V^{(2)}$ are thus closely related contributions arising from the same physical effects contributing equal amounts in both considerations. The effect of the correction for retardation is seen to be immaterial for the present problem.

4. SCALAR INTERACTION

A calculation along the lines of Breit and Bloch[6] gives in units for which $\hbar=c=1$,

$$(\psi,[\mathbf{r}_{\mathrm{I}}\times\boldsymbol{\alpha}_{\mathrm{I}}]\psi)_{Sc}=\left(\Psi^{(1)},\left\{\left[-\frac{1}{M}+\frac{W-J}{2M^2}+\frac{p_{\mathrm{I}}^2-p_{\mathrm{II}}^2}{4M^3}\right](\mathbf{L}_{\mathrm{I}}+\boldsymbol{\sigma}_{\mathrm{I}})+\mathbf{D}+\frac{\mathbf{C}_{Sc}}{4M^2}\right\}\Psi^{(1)}\right),\quad(5)$$

where $\mathbf{D}$ is as in Eqs. (3.1), (3.4), while

$$\mathbf{C}_{Sc}=[(\boldsymbol{\sigma}_{\mathrm{II}}\mathbf{p}_{\mathrm{II}}),[Y,[\mathbf{r}_{\mathrm{I}}\times\boldsymbol{\sigma}_{\mathrm{I}}]]_+]_+\quad(5.1)$$

with

$$Y=\tfrac{1}{2}J(\boldsymbol{\sigma}_{\mathrm{I}}\boldsymbol{\sigma}_{\mathrm{II}})+\tfrac{1}{2}(\boldsymbol{\sigma}_{\mathrm{I}}\mathbf{r})(\boldsymbol{\sigma}_{\mathrm{II}}\mathbf{r})f.\quad(5.2)$$

A straightforward rearrangement gives

$$\langle(\mathbf{u}\cdot\mathbf{C}_{Sc})\rangle=\langle(\boldsymbol{\sigma}_{\mathrm{II}}\cdot\mathbf{curl}_{\mathrm{II}}\{J\mathbf{u}_{\mathrm{I}}+f\mathbf{r}(\mathbf{u}_{\mathrm{I}}\mathbf{r})\})\rangle,\quad(5.3)$$

where

$$\mathbf{u}_{\mathrm{I}}=[\mathbf{u}\times\mathbf{r}_{\mathrm{I}}],\quad(5.4)$$

with $\mathbf{u}$ standing for an arbitrary fixed unit vector. The two parts in curly braces on the right side of Eq. (5.3) contribute equal and opposite amounts so that

$$\langle\mathbf{C}_{Sc}\rangle=0.\quad(5.5)$$

Collecting terms one finds with the aid of Eqs. (3.4), (3.5), (5.5) by substitution into Eq. (5) that

$$\langle[\mathbf{r}_{\mathrm{I}}\times\boldsymbol{\alpha}_{\mathrm{I}}]\rangle_{Sc}=-\frac{1}{M}\left\langle\left(1-\frac{W}{M}+\frac{T_2}{3M}\right)\boldsymbol{\sigma}_{\mathrm{I}}\right\rangle,\quad(6)$$

in agreement with Eq. (14) of Sachs.[5] The difference from Breit and Bloch[6] is caused by an error in the calculation of $\langle\mathbf{C}_{Sc}\rangle$. On the other hand, the calculation of Sachs has left out of consideration the effect of C_{Sc} which arises from corrections for lack of covariance of the two-body equation if one employs only the term in $\beta_{\mathrm{I}}\beta_{\mathrm{II}}J$. If one omitted the corrections for covariance in the vector case the effect of $\mathbf{C}_V$ would have been absent. It is thus seen that the inclusion of corrections for covariance is essential in the general case. The agreement of Eq. (6) with the result of Sachs is seen to arise from Eq. (5.5) which has not been discussed in the literature before and explains the apparently accidental correctness of the answer in the work of Sachs.

On the other hand Eq. (5.5) may be seen to mean directly that the addition to the Hamiltonian of a term proportional to

$$H''=(\boldsymbol{\alpha}_{\mathrm{I}}\boldsymbol{\alpha}_{\mathrm{II}})J+(\boldsymbol{\alpha}_{\mathrm{I}}\mathbf{r})(\boldsymbol{\alpha}_{\mathrm{II}}\mathbf{r})f\quad(7)$$

has no effect on the magnetic moment to within terms of relative order v^2/c^2. This term contains the same combinations as the term which has to be added to the interaction $(1-\boldsymbol{\alpha}_{\mathrm{I}}\boldsymbol{\alpha}_{\mathrm{II}})J$ to obtain the Hamiltonian used by Breit and Bloch and in the present paper for the discussion of the vector case. It was found for the latter that the same result can be obtained by the procedure used in Eqs. (2)$\cdots$(2.6) as by that used in Eqs. (3)$\cdots$(4). The latter includes the generalization of electrodynamic retardation effects as in Eq. (7) which are again seen to have made no difference for the magnetic moment problem. The fact that terms of this type have no effect could have been deduced from the fact that

$$H''=-(\boldsymbol{\alpha}_{\mathrm{I}}\boldsymbol{\nabla}_{\mathrm{I}})(\boldsymbol{\alpha}_{\mathrm{II}}\boldsymbol{\nabla}_{\mathrm{II}})K,\quad(7.1)$$

where

$$K=\int^r rJdr.$$

The introduction of a term of form

$$(\boldsymbol{\alpha}_{\mathrm{I}}\boldsymbol{\nabla}_{\mathrm{I}})G\quad(7.2)$$

can be counterbalanced by making

$$\psi=\exp(-iG/\hbar c)\psi'.\quad(7.3)$$

Since G is in this case free of the α_{I} and β_{I}, the calculation of $[\mathbf{r}_{\mathrm{I}}\times\boldsymbol{\alpha}_{\mathrm{I}}]$ is unaffected by the change from ψ to ψ' and the retardation terms cannot affect the magnetic moment.

The form of the two-body result for the scalar case can be explained in terms of the one-body result.[4] The latter is

$$(CF)_{Sc}^{(1)}=1-\langle T_1/3Mc^2\rangle-W_1/Mc^2,\quad(8)$$

while Eq. (6) gives

$$(CF)_{Sc}^{(2)}=1+\langle T_2/3Mc^2\rangle-W_2/Mc^2.\quad(8.1)$$

The connection becomes more obvious by eliminating

W by means of

$$W_i=\langle T_i-J\rangle,\quad (i=1,2). \tag{8.2}$$

One then has expressions equivalent to Eqs. (8), (8.1)

$$(CF)_{Sc}{}^{(1)}=1-\langle 4T_1/3Mc^2\rangle+\langle J_1/Mc^2\rangle, \tag{8.3}$$

$$(CF)_{Sc}{}^{(2)}=1-\langle 2T_2/3Mc^2\rangle+\langle J_2/Mc^2\rangle. \tag{8.4}$$

Noting that T_2 includes the kinetic energy of the neutron and comparing the one- and two-body answers for the same mean kinetic energies of the proton one has Eq. (2.1). With this identification it follows from Eqs. (8.3), (8.4) that

$$(CF)_{Sc}{}^{(2)}/(CF)_{Sc}{}^{(1)}=\langle Mc^2-J_1\rangle/\langle Mc^2-J_2\rangle, \tag{8.5}$$

corresponding to expectation, since for the one-body scalar equation J occurs in the Hamiltonian only in the term

$$-\beta(Mc^2-J), \tag{8.6}$$

and since the nonrelativistic magnetic moment contains the factor $1/M$. The part of the effect of J which is not explicitly taken into account in the kinetic energy is the change of the mass M to the effective mass $M-\langle J/c^2\rangle$.

Comparing this interpretation with the corresponding conditions for the vector case which have been discussed in relation to Eqs. (2), (2.1), (2.2) one sees that in both cases the one-body answers when supplemented by the argument concerning absence of retardation effects either in the form of Eqs. (7) to (7.3) or of Eqs. (5.3) to (5.5) give a complete account of the situation for vector and scalar interactions. It is essential here that in the two-body problem J may be considered as playing the same role for the motion of I as though the motion of II were not part of the problem.

5. PLANE WAVE REPRESENTATIONS

Comparison of one-body results as in Eqs. (2) and (8.3) shows the occurrence of an extra factor 2 in the correction term containing T in the answer for the scalar interaction. The work described in the present section has been done in order to obtain a clearer understanding of the reason for this difference. It turned out that a classification of contributions from terms diagonal and nondiagonal in the sign of energy gives a simple account of the facts. The wave function ψ can be represented by

$$\psi_\mu(\mathbf{r})=h^{-\frac{3}{2}}\int(\mathfrak{L}(\mathbf{p})C(\mathbf{p}))_\mu e^{i\mathbf{kr}}d\mathbf{p}, \tag{9}$$

where

$$\mathbf{k}=\mathbf{p}/\hbar,\quad c=1, \tag{9.1}$$

$$\mathfrak{L}(\mathbf{p})=(E^a+M)\beta+\boldsymbol{\alpha}\mathbf{p},\quad E^a=|(M^2+p^2)^{\frac{1}{2}}|, \tag{9.2}$$

$$(\mathfrak{L}(\mathbf{p})C(\mathbf{p}))_\mu=\mathfrak{L}_{\mu\nu}(\mathbf{p})C_\nu(\mathbf{p}). \tag{9.3}$$

The convention of omitting summation signs over common indices is used and units are adjusted so as to have $c=1$. The 4-row column matrix C describes the probability of a state with specified spin orientation and sign of energy, provided the reference system for $\boldsymbol{\alpha}$ and β is such as to make β diagonal. The representation used below is such that

$$\beta_{\mu\mu}=1,\quad (\mu=1,2);\quad \beta_{\mu\mu}=-1,\quad (\mu=3,4). \tag{9.4}$$

Accordingly C_1, C_2 are relative probability amplitudes for $E<0$ while C_3, C_4 are similar amplitudes for $E>0$. The notation

$$(C_-)=\begin{pmatrix}C_1\\C_2\end{pmatrix},\quad (C_+)=\begin{pmatrix}C_3\\C_4\end{pmatrix} \tag{9.5}$$

shows this relationship to the sign of E. The probability of finding the particle in the momentum range $d\mathbf{p}$ with specified spin and energy sign designated by σ is

$$N^2|C_\sigma(\mathbf{p})|^2d\mathbf{p}, \tag{9.6}$$

where N^2 arises from

$$(\mathfrak{L}^2)_{\mu\nu}=N^2\delta_{\mu\nu} \tag{9.7}$$

and is given by

$$N^2=2E^a(E^a+M). \tag{9.8}$$

Charge conjugation is not needed and is therefore not used. The wave function in momentum space in the usual sense is seen to be

$$\mathfrak{L}(\mathbf{p})C(\mathbf{p})$$

and multiplication of $\psi_\mu(\mathbf{r})$ by a Cartesian coordinate y is equivalent to the application of $-\hbar\partial/i\partial p_y$ to the momentum space wave function.

It is convenient to employ the quantity

$$\mathbf{v}=[\mathbf{u}\times\mathbf{r}] \tag{10}$$

where $\mathbf{u}$ is an arbitrary fixed unit vector. In terms of $\mathbf{v}$ the component

$$[\mathbf{r}\times\boldsymbol{\alpha}]_u=(\mathbf{v}\boldsymbol{\alpha}). \tag{10.1}$$

The evaluation of the expectation value of this quantity yields the magnetic moment arising from the Dirac current. It is seen from Eq. (9) that this evaluation when put in terms of the C_σ involves the quantity $\mathfrak{L}(\mathbf{v}\boldsymbol{\alpha})\mathfrak{L}$, a general form for which can be obtained by direct calculation. One finds

$$\begin{aligned}\mathfrak{L}(\mathbf{v}\boldsymbol{\alpha})\mathfrak{L}=&\,2\rho_3(E^a+M)(L_u+\hbar\sigma_u)\\&+\rho_3(\hbar/E^a)[(\boldsymbol{\sigma}\mathbf{p})p_u-p^2\sigma_u]+\rho_1\{[(\boldsymbol{\sigma}\mathbf{p}),L_u]_+\\&+2\hbar p_u-[E^a(E^a+M),[\mathbf{r}\times\boldsymbol{\sigma}]_u]_+\},\end{aligned} \tag{10.2}$$

where the component of orbital angular momentum enters as

$$L_u=(\mathbf{Lu}),\quad \mathbf{L}=[\mathbf{r}\times\mathbf{p}] \tag{10.3}$$

and components of other quantities along $\mathbf{u}$ are similarly designated by the subscript u. In Eq. (10.2) the first two terms contain $\rho_3=\beta$ and are, therefore, diagonal in the sign of energy. The remaining part of $\mathfrak{L}(\mathbf{v}\boldsymbol{\alpha})\mathfrak{L}$ contains the 4-row square matrices only as ρ_1 and $\boldsymbol{\sigma}$ and is nondiagonal in the sign of energy.

For $(C_-)=0$, taking into account the normalization convention of Eqs. (9.6), (9.8), it follows from Eq. (10.2) that

$$-\langle(\mathbf{v}\boldsymbol{\alpha})\rangle=\left\langle\frac{L_u+\hbar\sigma_u}{E^a}+\hbar\frac{(\boldsymbol{\sigma}\mathbf{p})p_u-p^2\sigma_u}{2(E^a)^2(E^a+M)}\right\rangle_{NC},\quad (C_-=0),\quad (11)$$

where the calculation of the expectation value of the right side is supposed to be done by inserting the operator occurring on the right side between $N(C_+^*)^T$ and $N(C_+)$, multiplying by $h^{-\frac{3}{2}}d\mathbf{p}$ and integrating; the superscript T stands for transposition. It may be shown that for s terms, i.e. states for which Dirac's $k=-1$, where $k=\pm(j+\frac{1}{2})$, the quantity $C_+(\mathbf{p})$ is spherically symmetric in $\mathbf{p}$ space. In this specialization the contribution to $-\langle(\mathbf{v}\boldsymbol{\alpha})\rangle$ arising from positive energy plane waves is

$$\begin{aligned}-\langle(\mathbf{v}\boldsymbol{\alpha})\rangle_{++}&=\left\langle\frac{\hbar\sigma_u}{E^a}-\frac{\hbar p^2\sigma_u}{3(E^a)^2(E^a+M)}\right\rangle_{NC}\\&\cong(\hbar/M)\langle[1-(2p^2/3M)]\sigma_u\rangle_{NC}\\&=(\hbar/M)\langle[1-(4T/3M)]\sigma_u\rangle_{NC},\quad (s\text{ terms}).\quad (11.1)\end{aligned}$$

The inherent relativistic correction factor for the positive energy part of the wave function is seen to be $1-4\langle T\rangle/(3Mc^2)$.

The results for the one-body scalar interaction problem are now readily interpretable. The correction factor apart from the effect of $\langle J\rangle$ in Eq. (8.3) is accounted for by the effect of states with $E>0$. The term in $\langle J\rangle$ must be caused therefore by cross product terms between C_+ and C_-. One finds in fact that the wave equation takes the form

$$\begin{aligned}(E_0+E^a)(C_-)&=\int(\mathcal{J}/N^2)\{A(\mathbf{p},\mathbf{p}')(C_-')+B(\mathbf{p},\mathbf{p}')(C_+')\}d\mathbf{p}',\\(E_0-E^a)(C_+)&=\int(\mathcal{J}/N^2)\{B(\mathbf{p},\mathbf{p}')(C_-')-A(\mathbf{p},\mathbf{p}')(C_+')\}d\mathbf{p}',\end{aligned}\quad (12)$$

where

$$(C_+')=[C_+(\mathbf{p}')]\quad (12.1)$$

and similarly for the other (C). The other quantities in the Schrödinger equation needing explanation are

$$A(\mathbf{p},\mathbf{p}')=(E^a+M)(E'^a+M)-(\boldsymbol{\sigma}\mathbf{p})(\boldsymbol{\sigma}\mathbf{p}'),\quad (12.2)$$

$$B(\mathbf{p},\mathbf{p}')=(E^a+M)(\boldsymbol{\sigma}\mathbf{p}')+(E'^a+M)(\boldsymbol{\sigma}\mathbf{p}),\quad (12.3)$$

$$\mathcal{J}=h^{-3}\int J(\mathbf{r})\exp\{i(\mathbf{k}-\mathbf{k}')\mathbf{r}\}d\mathbf{r},\quad (12.4)$$

$$E_0=\text{energy of stationary state.}\quad (12.5)$$

Solving for (C_-) from the first line of Eq. (12) one has

$$(C_-)\cong(1/4M^2)\int(\boldsymbol{\sigma}(\mathbf{p}+\mathbf{p}'))\mathcal{J}(C_+')d\mathbf{p}',\quad (12.6)$$

and employing this form in the ρ_1 containing part of Eq. (10.2) one obtains three terms arising from the three parts inside the curly braces. The last term gives the largest contribution on account of the factor $E^a(E^a+M)$. A simplification takes place in the evaluation of the effect of this term because $[\mathbf{r}\times\boldsymbol{\sigma}]$ operating on $\mathcal{J}$ gives an odd function in $\mathbf{p}-\mathbf{p}'$ while the remainder of the integrand is even. For this reason the only contributions to be considered come from $[\mathbf{r}\times\boldsymbol{\sigma}]$ applied to $\boldsymbol{\sigma}\mathbf{p}$. One has thus by means of Eq. (12.6)

$$\begin{aligned}-\langle\rho_1[E^a(E^a+M),[\mathbf{r}\times\boldsymbol{\sigma}]_u]_+\rangle&=-(4/4M^2)\int C_+^{*T}(\mathbf{p})(-h/i)[-\boldsymbol{\sigma}\times\boldsymbol{\sigma}]_u\\&\quad\times\mathcal{J}E^a(E^a+M)C_+(\mathbf{p}')d\mathbf{p}d\mathbf{p}'\\&=-(1/M^2)\int C_+^{*T}(\mathbf{p})N\hbar\sigma_u\mathcal{J}NC_+(\mathbf{p}')d\mathbf{p}d\mathbf{p}'\\&=-(\hbar/M^2)\langle\sigma_uJ\rangle.\quad (12.7)\end{aligned}$$

Here the first factor 4 takes account of repeated doubling coming from double order once for $C_-^{*T}C_+$ and $C_+^{*T}C_-$ and once for $[\quad]_+$. Adding the right side of Eq. (12.7) to the value of $\langle(\mathbf{v}\boldsymbol{\alpha})\rangle_{++}$ as in Eq. (11.1) the correction factor of Eq. (8.3) is reproduced. The term in $\langle J\rangle$ has thus been verified to be caused by matrix elements nondiagonal in the sign of energy.

For the vector case the Schrödinger equation in the C_+, C_- form takes the form

$$\begin{aligned}(E_0+E^a)(C_-)&=-\int(\mathcal{J}/N^2)\{A_V(\mathbf{p},\mathbf{p}')(C_-')+B_V(\mathbf{p},\mathbf{p}')(C_+')\}d\mathbf{p}',\\(E_0-E^a)(C_+)&=-\int(\mathcal{J}/N^2)\{-B_V(\mathbf{p},\mathbf{p}')(C_-')+A_V(\mathbf{p},\mathbf{p}')(C_+')\}d\mathbf{p}',\end{aligned}\quad (13)$$

where

$$A_V(\mathbf{p},\mathbf{p}')=(E^a+M)(E'^a+M)+(\boldsymbol{\sigma}\mathbf{p})(\boldsymbol{\sigma}\mathbf{p}'),\quad (13.1)$$

$$B_V(\mathbf{p},\mathbf{p}')=(E^a+M)(\boldsymbol{\sigma}\mathbf{p}')-(E'^a+M)(\boldsymbol{\sigma}\mathbf{p}),\quad (13.2)$$

and the notation is otherwise as for Eq. (12). From the first line of Eq. (13) there follows the approximation

$$(C_-)\cong(1/4M^2)\int(\boldsymbol{\sigma}(\mathbf{p}-\mathbf{p}'))\mathcal{J}(C_+')d\mathbf{p}'.\quad (13.3)$$

A procedure similar but slightly more laborious than that in Eq. (12.7) gives

$$\langle(\mathbf{v}\boldsymbol{\sigma})\rangle_{+-}\cong(\hbar/2M^2)\int(C_+^{*T})\{2\sigma_u\mathcal{J}+[\sigma_u(\mathbf{p}-\mathbf{p}')^2+(p_u'-p_u)(\boldsymbol{\sigma}(\mathbf{p}-\mathbf{p}'))]\mathcal{J}'/|\mathbf{p}-\mathbf{p}'|\}(C_+')d\mathbf{p}d\mathbf{p}',\quad (13.4)$$

with the understanding that both orders $C_+^{*T}C_-$ and $C_-^{*T}C_+$ are included and with

$$\mathcal{J}'=(d\mathcal{J}(x)/dx)_{x=|\mathbf{p}-\mathbf{p}'|}. \quad (13.5)$$

Inserting the value for (C_-) available in Eq. (13.3) into the second line of Eq. (13) one obtains the C space transform of the nonrelativistic Schrödinger equation. From this equation there follows the relation

$$[(p^2/M)+(E^a-E_0)(\mathbf{p}\nabla_p)](C_+)$$
$$=\int[p^2-(\mathbf{p}\mathbf{p}')][\mathcal{J}'/|\mathbf{p}-\mathbf{p}'|](C_+')dp', \quad (13.6)$$

which when used with Eq. (13.4) gives a contribution canceling $\frac{1}{2}$ of the term $-4\langle T/3Mc^2\rangle$ in the correction factor.

The representation of the answer in terms of plane waves has been used so far only for the one-body problem. It will now be made use of for the two-body case. It is fortunately not necessary to make a new calculation because it is possible to justify the adoption of Eq. (11.1) for the two-body case, with T standing foı the kinetic energy of the proton. Similarly one can justify the employment of Eq. (12.7) with J standing for $J(|\mathbf{r}_{\mathrm{I}}-\mathbf{r}_{\mathrm{II}}|$ essentially on the grounds that Eq. (12.6) can be carried over to the two-particle case with a slightly changed meaning. The somewhat intuitive approach used in connection with Eq. (8.5) can be thus explained in terms of J causing transitions to (C_-) and the form of the correction factor remaining the same for the scalar case provided the variables used are the kinetic energy of the proton and the mean J. Similarly in the vector case Eq. (13.3) shows how J is responsible for the existence of (C_-) and cancellation of half of $4\langle T\rangle/(3Mc^2)$ occurs because the contribution in terms of $\mathcal{J}$ happens to be expressible in terms of T with the aid of Eqs. (13.4), (13.5), (13.6).

In the argument just presented there are some gaps which were left in order to present the essential features concisely and which will now be filled in. The two-body 16-component ψ is analyzed as

$$\psi=h^{-3}\int\mathfrak{C}(\mathbf{p}_{\mathrm{I}},\mu_{\mathrm{I}};\mathbf{p}_{\mathrm{II}},\mu_{\mathrm{II}})\times\exp\{i(\mathbf{k}_{\mathrm{I}}\mathbf{r}_{\mathrm{I}}+\mathbf{k}_{\mathrm{II}}\mathbf{r}_{\mathrm{II}})\}d\mathbf{p}_{\mathrm{I}}d\mathbf{p}_{\mathrm{II}}, \quad (14)$$

the spin index μ being indicated as an argument of the wave function in momentum space. The function $\mathfrak{C}$ can be represented as

$$\mathfrak{C}(\mathbf{p}_{\mathrm{I}},\mu_{\mathrm{I}};\mathbf{p}_{\mathrm{II}},\mu_{\mathrm{II}})=[\mathfrak{L}(\mathbf{p}_{\mathrm{I}})C^{\mathrm{I}}(\mathbf{p}_{\mathrm{I}})]_{\mu\mathrm{I}}[\mathfrak{L}(\mathbf{p}_{\mathrm{II}})C^{\mathrm{II}}(\mathbf{p}_{\mathrm{II}})_{\mu\mathrm{II}}, \quad (14.1)$$

because only states restricted by

$$\mathbf{p}_{\mathrm{I}}+\mathbf{p}_{\mathrm{II}}=0 \quad (14.2)$$

need to be considered. For each value of the relative momentum there enter in the representation four possibilities regarding sign of energy, two for each particle, corresponding to C_+^{I}, C_-^{II}, etc. Since one is concerned with the calculation of the expectation value of a single particle operator, *viz.* $[\mathbf{r}_{\mathrm{I}}\times\boldsymbol{\alpha}_{\mathrm{I}}]$, the only combinations that matter are those diagonal in the sign of C^{II}. For this reason J and $\mathcal{J}$ affect the cross product terms in a manner analogous to that in the one-particle case. It will be noted that $\mathcal{J}$ in the present case is precisely of the same form as in the one-body problem since it comes in through the introduction of the relative momentum. The quantity C^{II} takes no part in the operations and the one-body result for cross terms can be transferred to the two-body case. Another point needing mention is the slight change regarding the operator L_u^{I} for $+$, $+$ combinations. This operator has no effect in the one body problem on account of spherical symmetry of C_+ for s states. In the two-body case it also has no effect to the order that matters, provided the presence of the D state is disregarded. The authors have not succeeded in reducing this part of the argument to a simple form. It is based on the possibility of separating spin and angular variables for the $^3S_1$ state in the general manner used by Critchfield.[16]

6. PSEUDOSCALAR INTERACTION

The single particle problem has been calculated by means of the Hamiltonian

$$H=-c(\boldsymbol{\alpha}\mathbf{p})-Mc^2\beta-ig\beta\gamma^5\chi, \quad (15)$$

where χ is the pseudoscalar nonquantized field and the conventions

$$\gamma^4=-\beta, \qquad \gamma^k=-i\beta\alpha_k=\rho_2\sigma_k,$$
$$\gamma^5=\gamma^1\gamma^2\gamma^3\gamma^4=\rho_1, \qquad -i\beta\gamma^5=\rho_2 \quad (15')$$

are used. Expressing the "small components" Φ in terms of the "large components" Ψ one finds by straightforward calculation

$$\int(\psi^{*T}[\mathbf{r}\times\boldsymbol{\alpha}]\psi)d\mathbf{r}\bigg/\int(\psi^{*T}\psi)d\mathbf{r}$$
$$=-(c/E)\int(\Psi^{*T}(\mathbf{L}+\hbar\boldsymbol{\sigma})\Psi)d\mathbf{r}\bigg/\int(\Psi^{*T}\Psi)d\mathbf{r}$$
$$=-(c/E)\langle\mathbf{L}+\hbar\boldsymbol{\sigma}\rangle_\Psi, \quad (16)$$

where it is supposed that the energy of the state ψ is E and the subscript Ψ indicates averaging in terms of the two-component function Ψ. The result is independent of the pseudoscalar field except insofar as Ψ is affected by it. Although Eq. (16) suggests that the correction factor to the magnetic moment is Mc^2/E, the special character of the pseudoscalar interaction makes it necessary to exercise caution in applications.

It appears probable that Eq. (16) will be more useful in applications to heavy nuclei because the approxi-

[16] C. L. Critchfield, Phys. Rev. 71, 258 (1947).

mation of a fixed field is presumably more justifiable in such cases than for the deuteron. If one were to form a model of a heavy nucleus by analogy with atomic models the simplest hypothesis would consist in making χ spherically symmetric. If one did, stationary states containing mixtures of wave functions of even and odd parity would result showing the inconsistency of such an assumption with the pseudoscalar character of χ. The simple *a priori* reasons for considering L to be a good quantum number can thus hardly be transferred from atomic theory to the present case. It is nevertheless customary to assume that L is not far from being a good quantum number and the L meant in this connection is usually defined in terms of Ψ rather than ψ, nuclear theory being customarily stated nonrelativistically. Within the limitations of these assumptions there is a definite meaning to a statement that a nuclear particle is in an s-state when the nucleus as a whole is in a stationary state. Under such conditions Eq. (16) gives the simple correction factor

$$Mc^2/E \tag{16.1}$$

to the nonrelativistic moment. This factor, as well as the general result in the form of Eq. (16), means that the relation between the magnetic moment and the energy which applies to free particles is not disturbed by the introduction of the pseudoscalar interaction.

It may be shown that a relation similar to Eq. (16) holds in terms of the four component wave function ψ. The wave equation corresponding to the Hamiltonian of Eq. (15) is

$$-(E/c)\psi=[(\boldsymbol{\alpha}\mathbf{p})+Mc\beta-(g/c)\rho_2\chi]\psi. \tag{17}$$

Multiplying by $(\mathbf{v}\boldsymbol{\alpha})$ it becomes

$$-(E/c)(\mathbf{v}\boldsymbol{\alpha})\psi =\{(\mathbf{v}\boldsymbol{\alpha})(\boldsymbol{\alpha}\mathbf{p})-[Mc\beta-(g/c)\rho_2\chi](\mathbf{v}\boldsymbol{\alpha})\}\psi. \tag{17.1}$$

Multiplying by ψ^{*T}, integrating over $\mathbf{r}$, and employing the complex conjugate of Eq. (17) to eliminate the combination $Mc\beta-(g/c)\rho_2\chi$ one obtains

$$-2(E/c)(\psi,(\mathbf{v}\boldsymbol{\alpha})\psi) =(\psi,[(\mathbf{v}\boldsymbol{\alpha})(\boldsymbol{\alpha}\mathbf{p})+(\boldsymbol{\alpha}\mathbf{p})(\mathbf{v}\boldsymbol{\alpha})]\psi). \tag{17.2}$$

Linearizing the operator on the right side of the last equation with respect to the Dirac σ's and employing commutation relations for coordinates and momenta, one finds after a short calculation

$$(\psi,[\mathbf{r}\times\boldsymbol{\alpha}]\psi)=-(c/E)(\psi,[\mathbf{L}+\hbar\boldsymbol{\sigma}]\psi). \tag{18}$$

The form of this result is similar to that of Eq. (16). For two states with different energies E, E' one obtains by employing ψ'^{*T} in place of ψ^{*T}

$$(\psi',[\mathbf{r}\times\boldsymbol{\alpha}]\psi)=-[2c/(E+E')](\psi',[\mathbf{L}+\hbar\boldsymbol{\sigma}]\psi), \tag{18'}$$

which furnishes nondiagonal matrix elements. The last result shows the presence of vanishing denominators between states with the same absolute value but opposite signs of E and is nugatory in such cases. The existence of Eq. (18) shows that the relationships to conditions in heavy nuclei which have been discussed in connection with Eq. (16) are more general than the usual nonrelativistic approximation.

The usefulness of Eq. (16) in applications to the deuteron is questionable. If it were justifiable to consider the ground state of this nucleus as a linear combination of wave functions consisting of products of proton and neutron wave functions and if these states contained only proton functions with energies between approximately $W+Mc^2$ and Mc^2, one would be justified in employing a mean E in place of E in Eq. (16.1). The discussion of other cases in the section concerned with plane waves has shown however that states with $E<0$ matter, so that this simple procedure is not correct in the general case.

By analogy with the cases of the scalar and vector interactions one can attempt to estimate the correction factor for the deuteron by attributing the difference between the factor $1-4\langle T_1\rangle/(3Mc^2)$ and

$$Mc^2/E_1=1-W_1/(Mc^2) \tag{19}$$

as arising from cross product terms between states with opposite signs of energy. This difference corresponds to the inclusion of a factor

$$1+[4\langle T_1\rangle/3-W_1]/(Mc^2) =1+[\langle T_1\rangle/3-(W_1-\langle T_1\rangle)]/(Mc^2) \tag{19.1}$$

in the one-body case, as an allowance for the effect of cross terms. If one replaces $W_1-\langle T_1\rangle$ by $W_2-\langle T_2\rangle$ on the grounds that the mean potential energy must be responsible for the proportion of states with $E<0$ and if one supposes that $\langle T_1/3\rangle$ is present in Eq. (19.1) as a direct property of the proton, then the correction factor for the deuteron should be

$$[1-2\langle T_2\rangle/(3Mc^2)]\{1+\langle T_2\rangle/6-(W_2-\langle T_2\rangle)\} =1+[\langle T_2\rangle/2-W_2]/(Mc^2). \tag{19.2}$$

This formula is subject to considerable uncertainties and doubts. The interaction between the particles is inherently not of the central field type. It is therefore rather questionable that there is a simple connection between the properties of a one-particle s state and the ^3S state of the two-particle system. There is no proof that the latter does not contain linear combinations of products of wave functions corresponding to vector coupling of states with $L>0$. For a nonquantized pseudoscalar field the sources of the $e^{-\kappa r}/r$ terms are in fact proportional to

$$\partial(\psi_\alpha{}^{*T}\sigma_{\alpha\beta}{}^j\Psi_\beta)/\partial x^j$$

and are decidedly directional in character. The fact[13] that for the two-body problem the net result of the quantized pseudoscalar theory is to give a relatively small proportion of the ^3D state does not remove this difficulty. Furthermore there is no proof that the contributions of L_u arising from C_+ are sufficiently small in the present case to be neglected. There is besides no assurance that the formation of virtual nucleon pairs which is known to be important in applications to nuclear forces does not appreciably affect the conclusions. This question is so closely connected with the perturbing influence between nucleons that it becomes hard to separate it from the general question of additivity of nuclear moments.[10–12] It is necessary therefore to regard Eq. (19.1) as a speculation.

7. THE DIAMAGNETIC EFFECT AND CONCLUDING REMARKS

In addition to the relativistic effects, the comparison of calculated with measured values should include the consideration of the diamagnetic effect caused by the shielding of a nuclear moment by the current system of the deuteron. This effect is related to the diamagnetic effect calculated for atoms by Lamb.[17] It has been apparently omitted in previous discussions. Formally such an omission amounts to disregarding magnetic interaction terms between the particles. While these terms produce smaller effects on the mutual energy between the particles they produce, nevertheless, effects which are formally of the same order of magnitude as the relativistic corrections. It is estimated to be smaller than the direct effects of $\langle T \rangle$ or $\langle J \rangle$ but not necessarily negligible in comparison with W/Mc^2. In the general case it is necessary to consider the difference in the rates of precession of $\mathbf{L}$ and $\boldsymbol{\sigma}$. For s terms in the single body case only the precession of the spin matters and the effect is then very similar to that calculated by Lamb. Estimates made on this basis indicate that this effect is small compared to the others considered here. On the other hand there is an obvious inconsistency in applying a one-body treatment to this problem. Some of the smallness of the effect results from the spherical symmetry of the s term and this condition is not satisfied on account of the tensor force.

The treatment of relativistic corrections has been discussed in the present paper in terms of somewhat arbitrary assumptions concerning the form of the two-body equations, the primary object being to explain the differences between the different results in terms of effects which can be described in simple language. The existence of additional effects is, of course, not excluded by the fact that the formal calculations have been interpreted in another way. The fact that two of the Hamiltonians used are covariant to the relative order v^2/c^2 is not a sufficient condition for their correctness and the results may not be regarded as final. In fact the more detailed discussion has brought out reasons for believing that additional terms including the interaction constant with the meson field will occur in a more complete treatment. In this connection it may be especially desirable to draw attention to the fact that the possibility of clearly distinguishing between relativistic corrections and nonadditivity of nuclear moments has not been established. The effect of the tensor force has been omitted on the grounds that the deuteron is probably predominantly in the ^3S state. The errors committed at this point have not been ascertained and are probably impossible to separate from the questions raised in connection with the pseudoscalar interaction.

[17] W. E. Lamb, Jr., Phys. Rev. 60, 817 (1941).