in zinc-orthosilicate[32] contributing to the moment, are lower than the experimentally determined values. These discrepancies can be accounted for by assuming (1) incomplete spin-pairing of pairs and clusters of manganese ions, with resultant contributions to the magnetic moment, or (2) that the distribution of manganese activator ions in zinc-orthosilicate is not of a random nature.

Interaction and Luminescence

If the decrease in luminescence emission intensity with increasing manganese proportion were due only to a decrease in the effective number of emitting centers, then neither the temperature break-point nor the lifetime of the excited state should be affected. However, the fact that both break-point and lifetime are functions of the manganese proportion can be explained by activator interactions which cause an increase in the probability of radiationless transitions. Figure 6 shows the relationship between emission intensity and the Weiss constant, under electron excitation, with high and low current density, for manganese proportions from 1 to 10 percent. It is seen that the decrease in emission intensity occurs for the same range of Mn proportion as the increase in the Weiss constant.

The effectiveness of the magnetic method for determining the ionization state and degree of interaction of small amounts of paramagnetic impurities should prove of value in studies of other systems, as well as in trapping, and the effect of luminescence poisons.

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[32] By *isolated* is meant having no manganese ion in next-adjacent available site; see H. W. Leverenz, reference 23, pp. 477–480. P. D. Johnson and F. E. Williams, J. Chem. Phys. **18**, 323 (1950), assumed for ZnF_2:Mn phosphors that only those manganese ions which do not have other manganese ions at nearest cation sites are capable of luminescing. See also H. W. Leverenz and D. O. North, Phys. Rev. **85**, 930 (1952).

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The Evaluation of the Energy Matrix of the Tensor Forces

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It is shown how the use of tensor operators enables a simple calculation of the tensor forces in nuclear configurations by the usual spectroscopic methods. The number of independent parameters necessary to define the energy in the nuclear configuration l^n (or j^n) is found to be $2l$ (or $2j$), whereas it is shown to be only $[4l/3]$ (or $[2j+2/3]$) in the l^n (or j^n) configuration of equivalent nucleons. The energy matrix is given by means of a closed formula for the case of d^n and $(d_{5/2})^n$ configurations (of equivalent nucleons). It is found that in the latter configuration the order of levels for a short-range potential (of the tensor forces) is the same as for short-range central forces.

I. INTRODUCTION

IN a recent paper[1] it has been shown that the tensor forces possess the pairing property, i.e., this interaction is diagonal with respect to the seniority v and the term values of the configuration l^n (or j^n) of equivalent nucleons differ from the corresponding states of the l^v (or j^v) configuration only by the term $\frac{1}{2}(n-v)E_0$. E_0 is the energy of $l^2\,{}^1S$, which vanishes for tensor forces (or $j^2, J=0$). The proof of this fact is based on an expansion of the tensor force interaction between two nucleons into a sum of products of double tensors $\mathbf{s}_1\mathbf{C}_1^{(k)}$ and $\mathbf{s}_2\mathbf{C}_2^{(k)}$ of the two nucleons, where $\mathbf{s}_i$ is the spin vector of the ith nucleon and $\mathbf{C}_{iq}^{(k)}$ differ only by a constant factor from the spherical harmonics of order k which depend on the coordinates of the ith nucleon. This expansion will be used throughout this paper in order to obtain further results on the tensor forces.

The matrix elements of the tensor force interaction were calculated in the case of d^2 and p^2 by Marvin,[2] who took for the potential the special case of $1/r^3$ (which appears in the electromagnetic spin-spin interaction). The results of his long and complicated calculations are very simple, they contain only two independent parameters in the case of d^2 (and only one for p^2). Also, these parameters can be easily expressed by the ordinary Slater coefficients of the potential $1/r^3$. The decomposition described above enables a simpler calculation of the matrix elements of the tensor forces. We shall use a general potential and see to what cause the simplification which occurs in Marvin's results is due. The method used offers a natural definition of the radial parameters for the tensor forces, with the help of which

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[1] G. Racah and I. Talmi, Physica (to be published).
[2] H. H. Marvin, Phys. Rev. **71**, 102 (1947).

the number of the independent parameters in the l^n (or j^n) configuration will be found.

By means of the parameters defined, we give the procedure for the calculation of the energy matrix in a nuclear configuration. This will enable us to write down the results of Trees,[3] who used Marvin's results to calculate the terms of the d^n configuration by means of a closed formula.

Using the method of harmonic oscillator wave functions,[4] we will give a method to calculate the radial parameters in terms of simple integrals. By considering these integrals, a discussion of the effect of tensor forces on the levels of the $(d_{\frac{5}{2}})^n$ configuration is given. It is found that for short-range potentials the tensor forces behave in a similar manner to that of short-range central interactions.

II. DECOMPOSITION OF THE INTERACTION

It is well known that the tensor force interaction between two nucleons is the scalar product of two tensors of the second degree, one of which is a function of the spins of the two nucleons and the other is a function of their spatial coordinates. In order to express this interaction by means of operators which operate on the (spin and space) coordinates of a single nucleon, we shall write it down as a sum of tensor products[5] of the double tensors $\mathbf{s}_1\mathbf{C}_1^{(k)}$ and $\mathbf{s}_2\mathbf{C}_2^{(k')}$. The k and k' of tensors whose matrix elements diagonal with respect to l_1 and l_2 do not vanish must be even and satisfy $0\leqslant k\leqslant 2l_1$, $0\leqslant k'\leqslant 2l_2$.

The interaction of the tensor forces between nucleons 1 and 2,

$$\begin{aligned}S_{12}V(r_{12})&=\left[\frac{(\mathbf{s}_1\cdot\mathbf{r}_{12})(\mathbf{s}_2\cdot\mathbf{r}_{12})}{r_{12}^2}-\tfrac{1}{3}(\mathbf{s}_1\cdot\mathbf{s}_2)\right]V(r_{12})\\ &=\Bigg\{r_1^2\left[\left(\mathbf{s}_1\cdot\frac{\mathbf{r}_1}{r_1}\right)\left(\mathbf{s}_2\cdot\frac{\mathbf{r}_1}{r_1}\right)-\tfrac{1}{3}(\mathbf{s}_1\cdot\mathbf{s}_2)\right]\\ &\quad+r_2^2\left[\left(\mathbf{s}_1\cdot\frac{\mathbf{r}_2}{r_2}\right)\left(\mathbf{s}_2\cdot\frac{\mathbf{r}_2}{r_2}\right)-\tfrac{1}{3}(\mathbf{s}_1\cdot\mathbf{s}_2)\right]\\ &\quad-r_1r_2\left[\left(\mathbf{s}_1\cdot\frac{\mathbf{r}_1}{r_1}\right)\left(\mathbf{s}_2\cdot\frac{\mathbf{r}_2}{r_2}\right)+\left(\mathbf{s}_1\cdot\frac{\mathbf{r}_2}{r_2}\right)\right.\\ &\qquad\left.\times\left(\mathbf{s}_2\cdot\frac{\mathbf{r}_1}{r_1}\right)-\tfrac{2}{3}(\mathbf{s}_1\cdot\mathbf{s}_2)\right]\Bigg\}\frac{V(r_{12})}{r_{12}^2},\end{aligned}\quad(1)$$

can be written as follows, after introducing the tensors[6] $\mathbf{C}_i^{(k)}=(4\pi/2k+1)^{\frac12}\mathbf{Y}_i^{(k)}$ (where $Y_{im}^{(k)}$ are the spherical harmonics of order k which are functions of the coordinates of the ith nucleon):

$$\begin{aligned}&\Big[\sum_{\rho\rho'}(-1)^{-\rho+\rho'}s_{1-\rho}s_{2\rho'}(2/3)^{\frac12}(1-\rho1\rho'|112,-\rho+\rho')\\ &\quad\times(r_1^2C_{1\rho-\rho'}^{(2)}+r_2^2C_{2\rho-\rho'}^{(2)})-\sum_{\rho\rho'qq'}(-1)^{q-q'}s_{1-\rho}s_{2\rho'}\\ &\quad\times 5r_1r_2V(112;q,-q',-q+q')V(112;-\rho,\rho',q-q')\\ &\quad\times(C_{1q}^{(1)}C_{2-q'}^{(1)}+C_{2q}^{(1)}C_{1-q'}^{(1)})\Big]\frac{V(r_{12})}{r_{12}^2}.\end{aligned}\quad(2)$$

Expansion of $V(r_{12})/r_{12}^2$ in the usual way, namely,

$$\begin{aligned}\frac{V(r_{12})}{r_{12}^2}&=\sum_{k=0}^{\infty}f_k(r_1,r_2)P_k(\cos\omega_{12})\\ &=\sum_{k=0}^{\infty}f_k(r_1,r_2)\sum_{m=-k}^{+k}(-1)^mC_{1-m}^{(k)}C_{2m}^{(k)},\end{aligned}$$

gives after expressing products of spherical harmonics as linear combinations of these functions:

$$\begin{aligned}&\sum_{krm\rho\rho'}(-1)^{-\rho+\rho'+m}f_k(r_1,r_2)s_{1-\rho}s_{2\rho'}\frac{(2/3)^{\frac12}(2r+1)}{[5(2k+1)]^{\frac12}}\\ &\quad\times c^r(2,-\rho+\rho',k,m)(1-\rho1\rho'|112,-\rho+\rho')\\ &\quad\times(r_1^2C_{1\rho-\rho'-m}^{(r)}C_{2m}^{(k)}+r_2^2C_{2\rho-\rho'-m}^{(r)}C_{1m}^{(k)})\\ &\quad-\sum_{krsm\rho\rho'qq'}10r_1r_2f_k(r_1,r_2)(-1)^{q-q'}s_{1-\rho}s_{2\rho'}\\ &\quad\times C_{1q-m}^{(r)}C_{2m-q'}^{(s)}\frac{(2r+1)(2s+1)}{3(2k+1)}c^r(1qkm)\\ &\quad\times c^s(km1q')V(112;q,-q',-q+q')\\ &\qquad\times V(112;-\rho,\rho',q-q').\end{aligned}\quad(3)$$

With the help of II (52) this can be written as

$$\begin{aligned}&\sum_{krm\rho\rho'}(-1)^{1+(r+k)/2}f_k(r_1,r_2)(2/15)^{\frac12}(2r+1)\\ &\quad\times(\tfrac12 C_{2rk})^{\frac12}(-1)^{\rho-\rho'}s_{1-\rho}s_{2\rho'}(r_1^2C_{1\rho-\rho'-m}^{(r)}C_{2m}^{(k)}\\ &\quad+r_2^2C_{2\rho-\rho'-m}^{(r)}C_{1m}^{(k)})(1-\rho1\rho'|112,-\rho+\rho')\\ &\quad\times(r,\rho-\rho'-m,k,m|rk2,\rho-\rho')\\ &\quad-\sum_{krsm\rho\rho'qq'}(-1)^{k+1+(r+s)/2}10r_1r_2f_k(r_1,r_2)\\ &\quad\times(\tfrac14 C_{1rk}C_{1sk})^{\frac12}(2r+1)(2s+1)(-1)^{m-q'}s_{1-\rho}s_{2\rho'}\\ &\quad\times C_{1q-m}^{(r)}C_{2m-q'}^{(s)}V(112;-\rho,\rho',q-q')\\ &\quad\times V(112;q,-q',-q+q')V(1rk;-q,q-m,m)\\ &\qquad\times V(1sk;q',m-q',-m).\end{aligned}\quad(4)$$

The first sum has already the desired form of a tensor product of $\mathbf{s}_1\mathbf{C}_1^{(r)}$ and $\mathbf{s}_2\mathbf{C}_2^{(k)}$; in order to bring the second sum also to this form we transform the product $V(1rk;-q,q-m,m)V(1sk;q',m-q',-m)$ by means

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[3] R. E. Trees, Phys. Rev. **82**, 683 (1951).

[4] I. Talmi, Helv. Phys. Acta **XXV**, 185 (1952).

[5] G. Racah, *Group Theory and Spectroscopy*, Lecture notes, Princeton, 1951 (unpublished).

[6] G. Racah, Phys. Rev. **62**, 438 (1942), which will be referred to as II; the notation defined in it will be used throughout this paper.

of the W function according to the formula

$$\sum_{\varphi}(-1)^{f+\varphi}V(acf;\,\alpha-\gamma-\varphi)V(bdf;\,\beta-\delta\varphi)$$
$$=\sum_{e\epsilon}(-1)^{a+d-f+\epsilon}(2e+1)W(abcd;\,ef)$$
$$\times V(abe;\,\alpha\beta-\epsilon)V(cde;\,-\gamma-\delta\epsilon). \quad (5)$$

We then sum the product of $V(11e;\,q,\,-q',\,-q+q')$, thus obtained, and $V(112;\,q,\,-q',\,-q+q')$ over q, q', and m for fixed $q-m$ and $m-q'$, which yields $\delta(2,e)/(2e+1)$, so that we finally obtain

$$S_{12}V(r_{12})=\sum_{kr}(-1)^{1+(r+k)/2}f_k(r_1,r_2)(2/15)^{\frac{1}{2}}$$
$$\times(2r+1)(\tfrac{1}{2}C_{2rk})^{\frac{1}{2}}\{r_1^2([\mathbf{s}_1\times\mathbf{s}_2]^{(2)}\cdot[\mathbf{C}_1^{(r)}\times\mathbf{C}_2^{(k)}]^{(2)})$$
$$+r_2^2([\mathbf{s}_1\times\mathbf{s}_2]^{(2)}\cdot[\mathbf{C}_2^{(r)}\times\mathbf{C}_1^{(k)}]^{(2)})\}$$
$$-\sum_{krs}(-1)^{k+(r-s)/2}r_1r_2f_k(r_1,r_2)(2r+1)(2s+1)$$
$$\times(\tfrac{1}{4}C_{1rk}C_{1sk})^{\frac{1}{2}}W(11rs;\,2k)([\mathbf{s}_1\times\mathbf{s}_2]^{(2)}$$
$$\cdot\{[\mathbf{C}_1^{(r)}\times\mathbf{C}_2^{(s)}]^{(2)}+[\mathbf{C}_2^{(r)}\times\mathbf{C}_1^{(s)}]^{(2)}\}). \quad (6)$$

In this expression of the interaction, the angular and spin-dependent parts appear in a form which shows their tensorial properties. For a definite configuration, after the integration over r_1 and r_2, the elements of the energy matrix are sums of matrix elements of definite products of tensors which can be found in every case by the formulas of the tensor algebra and the usual methods of spectroscopy.

In particular, the matrix elements for the case of two nucleons with orbital angular momenta l_1 and l_2 in LS coupling (the case of jj coupling will be treated below) are given by the general formula[3]

$$(\alpha SLJM|S_{12}V(r_{12})|\alpha'S'L'JM)$$
$$=(-1)^{S+L'-J}(\alpha SL\|S_{12}V(r_{12})\|\alpha'S'L')W(SLS'L';\,J2), \quad (7)$$

and by

$$(SL\|[\mathbf{s}_1\mathbf{C}_1^{(r)}\times\mathbf{s}_2\mathbf{C}_2^{(s)}]^{(22)}\|S'L')$$
$$=5\sum_{S''L''}(SL\|s_1C_1^{(r)}\|S''L'')(S''L''\|s_2C_2^{(s)}\|S'L')$$
$$\times W(S1S'1;\,S''2)W(LrL's;\,L''2), \quad (8)$$

in addition to II (44) and Sec. 5 of II.

III. THE l^n CONFIGURATION

We shall now calculate the interaction matrix of the tensor forces in the l^n configuration. We introduce the unit tensor operators by means of II (51):

$$(l\|C_i^{(k)}\|l)=(-1)^{k/2}(2l+1)(\tfrac{1}{2}C_{kll})^{\frac{1}{2}}(l\|u_i^{(k)}\|l), \quad (9)$$

and the double tensor $\mathbf{V}^{(1k)}=\sum_{i=1}^{n}\mathbf{s}_i\mathbf{u}_i^{(k)}$ which operate on the coordinates of the whole group of the l nucleons. The values of the radial integrals over r_1^2 are equal in this case to those over r_2^2; as a result we obtain, by summing over all the nucleon pairs ij,

$$(l^n\alpha SL\|\sum_{i<j}S_{ij}V(r_{ij})\|l^n\alpha'S'L')$$
$$=-\sum_{kr}\int\int R_l^2(r_1)R_l^2(r_2)\tfrac{1}{2}(r_1^2+r_2^2)f_k(r_1,r_2)dr_1dr_2$$
$$\times(2/15)^{\frac{1}{2}}(2r+1)(2l+1)^2(\tfrac{1}{2}C_{2rk})^{\frac{1}{2}}(\tfrac{1}{4}C_{kll}C_{rll})^{\frac{1}{2}}$$
$$\times(l^n\alpha SL\|[\mathbf{V}^{(1k)}\times\mathbf{V}^{(1r)}]^{(22)}\|l^n\alpha'S'L')$$
$$+\sum_{krs}\int\int R_l^2(r_1)R_l^2(r_2)r_1r_2f_k(r_1,r_2)dr_1dr_2$$
$$\times(2r+1)(2s+1)(2l+1)^2(\tfrac{1}{4}C_{1rk}C_{1sk})^{\frac{1}{2}}$$
$$\times(\tfrac{1}{4}C_{rll}C_{sll})^{\frac{1}{2}}W(11rs;\,2k)$$
$$\times(l^n\alpha SL\|[\mathbf{V}^{(1r)}\times\mathbf{V}^{(1s)}]^{(22)}\|l^n\alpha'S'L'). \quad (10)$$

In the transition from (6) to (10), a term of the form

$$\sum_{i=1}^{n}(l^n\alpha SL\|[\mathbf{s}_i\mathbf{C}_i^{(r)}\times\mathbf{s}_i\mathbf{C}_i^{(s)}]^{(22)}\|l^n\alpha'S'L')$$

was dropped out since the tensor $[\mathbf{s}_1\times\mathbf{s}_2]^{(2)}$ vanishes because of the triangular conditions (such a term would be linear in n).

Considering the equation analogous to (8),

$$(\alpha SL\|[\mathbf{V}^{(1r)}\times\mathbf{V}^{(1s)}]^{(22)}\|\alpha'S'L')$$
$$=5\sum_{\alpha''S''L''}(\alpha SL\|V^{(1r)}\|\alpha''S''L'')(\alpha''S''L''\|V^{(1s)}\|\alpha'S'L')$$
$$\times W(1S1S';\,S''2)W(rLsL';\,L''2), \quad (11)$$

we see that the energy matrix is given if the matrix elements $(l^n\alpha SL\|V^{(1k)}\|l^n\alpha'S'L')$ are calculated. To evaluate them is a problem which is encountered in the calculation of the energy of central interactions. In the special case $n=2$ they are simply

$$(l^2SL\|V^{(1k)}\|l^2S'L')$$
$$=-\tfrac{1}{2}[(-1)^{S+L}+(-1)^{S'+L'}](6)^{\frac{1}{2}}$$
$$\times[(2S+1)(2S'+1)(2L+1)(2L'+1)]^{\frac{1}{2}}$$
$$\times W(\tfrac{1}{2}S\tfrac{1}{2}S';\,\tfrac{1}{2}1)W(lLlL';\,lk). \quad (12)$$

For higher n one should use the methods developed by Racah[5–8] and others.[9,10] The calculation may be facilitated by the fact that the tensors $V^{(1k)}$ which should be calculated have all $1+k$ odd. It follows from the triangular conditions and from the fact that the symmetry of a wave function of two nucleons is $(-1)^{S+L}$ that, for odd $1+k$,[7]

$$(l^2SL\|V^{(1k)}\|l^2\,{}^1S)=(l^2\,{}^1S\|V^{(1k)}\|l^2SL)=0 \quad (13)$$

for every S and L. The group theoretical meaning of this fact is that such tensors are representations of the infinitesimal elements of the group which leaves in-

[7] G. Racah, Phys. Rev. 63, 367 (1943).
[8] G. Racah, Phys. Rev. 76, 1352 (1949).
[9] H. A. Jahn, Proc. Roy. Soc. (London) **201**, 516 (1950).
[10] B. H. Flowers, Proc. Roy. Soc. (London) **212**, 248 (1952).

variant the antisymmetric form—the wave function of $l^2\,{}^1S$—i.e., the symplectic group in $2(2l+1)$ dimensions $\mathrm{Sp}(4l+2)$. Therefore, the tensors $\mathbf{V}^{(1k)}$ with odd $1+k$ are diagonal with respect to the quantum numbers which characterize these representations. In the special case of equivalent nucleons (the l^n configuration of protons or neutrons only) the double tensors $\mathbf{V}^{(1k)}$ with $1+k$ odd are diagonal with respect to the seniority and independent of n[7]. For this case, it follows from (10) that in LS coupling also the energy is diagonal with respect to the seniority and independent of n, as was found empirically by Trees[3] in the case of d^n.

It is natural to define the sum of radial integrals which multiplies a definite matrix

$$(l^n\alpha SL\|[\mathbf{V}^{(1r)}\times\mathbf{V}^{(1s)}]^{(22)}\|l^n\alpha' S'L'), \qquad (14)$$

which depends only on the angular and spin-dependent parts of the wave functions, as the parameters by means of which the energy can be expressed. These parameters are the analogs for the case of tensor forces of the Slater coefficients in the case of central forces. As r and s must be even and satisfy together with 2 the triangular conditions, it follows that in the l^n configuration there are $2l$ possible matrices (14). In the general case the $2l$ matrices (14) are independent and the number of independent parameters necessary to define the energy is $2l$. In the case of equivalent nucleons, however, the $2l$ matrices (14) are not independent. The proof of this fact and the number of independent parameters in this case can be found by the following group-theoretical reasoning. The operator $\mathbf{V}^{(1k)}$ with even k has the tensorial properties of the wave function of l^2 with $L=k$. Such a state belongs to the representation (20) or (00) of the rotation group in $2l+1$ dimensions $R(2l+1)$. The operator $[\mathbf{V}^{(1r)}\times\mathbf{V}^{(1s)}]^{(22)}$ corresponds in the above manner to a D state (of l^4) which belongs to a representation of $R(2l+1)$ contained in the products $(20)\times(00)$ and $(20)\times(20)$. The first product [which is in fact (20)], contains only one such state. The other product can be decomposed into $(00)+(20)+(22)+(40)$ of which to (00) belongs only an S state; there is only one state $(20)D$ (which is the state already mentioned) and the other D states are $(22)D$ and $(40)D$. But we have to take into consideration only states which have nonvanishing matrix elements between the states of odd L, which are the only triplet states of the configuration l^2 of equivalent nucleons (as the tensor forces vanish in this case either between two singlet states or between a singlet and a triplet state). The states of odd L (for l^2) belong to the representation (11); therefore, the relevant D states belong to a representation of $R(2l+1)$ which is contained in the product $(11)\times(11)$; this is either (20) or (22) but not (40). Therefore, the number of independent matrices (14) is equal to the number of states $(20)D$ and $(22)D$. This number is exactly the number of D states (of l^4) which belong to the representation [22] of the unitary group in $2l+1$ dimensions $U(2l+1)$, as this representation breaks upon restriction to $R(2l+1)$ as follows: $[22]\rightarrow(22)+(20)+(00)$ (and to (00) belong S states only). This latter number is given in a recent paper of Gamba and Verde.[11] Putting $j=l$, $J=2$ in their formula for $n=4$ we obtain $[4l/3]$ (the largest integer smaller than $4l/3$). This is the number of independent parameters necessary to define the energy in the l^n configuration of equivalent nucleons.

It is therefore possible in this case to express the $2l$ matrices (14) in terms of $[4l/3]$ of them; these might be chosen for the sake of convenience to have the lowest possible r and s. A given matrix (14) which is equal in the case of l^2 to a certain linear combination of the $[4l/3]$ matrices is equal to the same combination also in the case of l^n. In fact, in the case of l^2 the following relation exists:

$$[\mathbf{V}^{(1r)}\times\mathbf{V}^{(1s)}]^{(22)}=[\mathbf{s}_1\mathbf{u}_1^{(r)}\times\mathbf{s}_2\mathbf{u}_2^{(s)}]^{(22)} +[\mathbf{s}_2\mathbf{u}_2^{(r)}\times\mathbf{s}_1\mathbf{u}_1^{(s)}]^{(22)} \qquad (15)$$

(as $[\mathbf{s}_i\mathbf{u}_i^{(r)}\times\mathbf{s}_i\mathbf{u}_i^{(s)}]^{(22)}$ vanishes). In the case of l^n the analogous expression is

$$[\mathbf{V}^{(1r)}\times\mathbf{V}^{(1s)}]^{(22)}=\sum_{i\neq j}[\mathbf{s}_i\mathbf{u}_i^{(r)}\times\mathbf{s}_j\mathbf{u}_j^{(s)}]^{(22)}. \qquad (16)$$

Therefore, if the expression of

$$[\mathbf{s}_i\mathbf{u}_i^{(r)}\times\mathbf{s}_j\mathbf{u}_j^{(s)}]^{(22)}+[\mathbf{s}_j\mathbf{u}_j^{(r)}\times\mathbf{s}_i\mathbf{u}_i^{(s)}]^{(22)}$$

as a definite linear combination is valid for any pair ij, it is valid also for their sum (16). These facts facilitate essentially the calculation as it is rather easy to construct (14) for l^2 by means of (11) and (12), and then express all the matrices (14) in terms of $[4l/3]$ of them. As a result, one has to calculate in the case of l^n only $[4l/3]$ matrices (14) (with lowest possible r and s).

IV. THE ENERGY MATRIX OF THE CONFIGURATION d^n

In order to see an example how the procedure described above is carried out, we give in detail the case of d^n of equivalent nucleons. The matrices (14) which enter the calculation are $[\mathbf{V}^{(12)}\times\mathbf{V}^{(10)}]^{(22)}$, $[\mathbf{V}^{(12)}\times\mathbf{V}^{(12)}]^{(22)}$, $[\mathbf{V}^{(14)}\times\mathbf{V}^{(12)}]^{(22)}$, and $[\mathbf{V}^{(14)}\times\mathbf{V}^{(14)}]^{(22)}$. All of them can be expressed in terms of $[8/3]=2$ of them, say, $[\mathbf{V}^{(12)}\times\mathbf{V}^{(10)}]^{(22)}$ and $[\mathbf{V}^{(12)}\times\mathbf{V}^{(12)}]^{(22)}$. After the calculation of these four matrices for d^2, we find the relations

$$\begin{aligned}[\mathbf{V}^{(14)}\times\mathbf{V}^{(12)}]^{(22)}&=-\frac{1}{9}[\mathbf{V}^{(12)}\times\mathbf{V}^{(10)}]^{(22)}-\frac{5}{9}[\mathbf{V}^{(12)}\times\mathbf{V}^{(12)}]^{(22)},\\ [\mathbf{V}^{(14)}\times\mathbf{V}^{(14)}]^{(22)}&=-\frac{5\sqrt{22}}{54}[\mathbf{V}^{(12)}\times\mathbf{V}^{(10)}]^{(22)}+\frac{5\sqrt{22}}{54}[\mathbf{V}^{(12)}\times{}^{(12)}]^{(22)}.\end{aligned} \qquad (17)$$

[11] A. Gamba and M. Verde, Nuovo cimento **IX**, 544 (1952).

Introducing these relations into (10) and inserting there the proper values of the k, r, and s, we can express the energy matrix of the d^n configuration (of equivalent nucleons) by means of the closed formula

$$\begin{aligned}(d^n vSL\|\sum_{i<j} S_{ij}V(r_{ij})\|d^n vS'L') &= A(d^n vSL\|[\mathbf{V}^{(12)}\times\mathbf{V}^{(10)}]^{(22)}\|d^n vS'L') \\ &\quad + B(d^n vSL\|[\mathbf{V}^{(12)}\times\mathbf{V}^{(12)}]^{(22)}\|d^n vS'L') \\ &= A(S\|S\|S)W(1S1S';S2)(d^n vSL\|V^{(12)}\|d^n vS'L') \\ &\quad + B5\sum_{S''L''}(d^n vSL\|V^{(12)}\|d^n vS''L'') \\ &\quad \times(d^n vS''L''\|V^{(12)}\|d^n vS'L') \\ &\quad \times W(1S1S';S''2)W(2L2L';L''2). \qquad (18)\end{aligned}$$

In this formula A and B are the radial integrals

$$\begin{aligned}A=\frac{5}{\sqrt{21}}\int\int\Big[&-(r_1^2+r_2^2)f_0(r_1,r_2) \\ &-\frac{1}{7}(r_1^2+r_2^2)f_2(r_1,r_2)+\frac{4}{21}(r_1^2+r_2^2)f_4(r_1,r_2) \\ &+\frac{2}{3}2r_1r_2f_1(r_1,r_2)-\frac{91}{441}2r_1r_2f_3(r_1,r_2) \\ &-\frac{5}{99}2r_1r_2f_5(r_1,r_2)\Big]R_2^2(r_1)R_2^2(r_2)dr_1dr_2, \qquad (19a)\end{aligned}$$

$$\begin{aligned}B=\frac{5}{\sqrt{21}}\int\int\Big[&\tfrac{1}{3}2r_1r_2f_1(r_1,r_2)-\frac{2}{9}2r_1r_2f_3(r_1,r_2) \\ &+\frac{5}{99}2r_1r_2f_5(r_1,r_2)\Big]R_2^2(r_1)R_2^2(r_2)dr_1dr_2. \qquad (19b)\end{aligned}$$

These are the general expressions in which no reference to a special form of the potential or of the wave functions was done. In general, there exist between the $\varphi_k(r_1,r_2)$ defined by $V(r_{12})=\sum_{k=0}^{\infty}\varphi_k(r_1,r_2)P_k(\cos\omega_{12})$ and the $f_k(r_1,r_2)$, defined above, the relations

$$\begin{aligned}\varphi_k(r_1,r_2)&=(r_1^2+r_2^2)f_k(r_1,r_2) \\ &-\frac{k}{2k-1}2r_1r_2f_{k-1}(r_1,r_2)-\frac{k+1}{2k+3}2r_1r_2f_{k+1}(r_1,r_2). \qquad (20)\end{aligned}$$

There are special cases, however, in which it is possible to express $(r_1^2+r_2^2)f_k(r_1,r_2)$ and $2r_1r_2f_k(r_1,r_2)$ in terms of the $\varphi_k(r_1,r_2)$. For the potential $V(r_{12})=1/r_{12}^3$ it is easily found that

$$\begin{aligned}(r_1^2+r_2^2)f_k(r_1,r_2) &= \tfrac{1}{3}(2k+3)\varphi_k(r_1,r_2)+\tfrac{2}{3}(2k+1) \\ &\quad\times[\varphi_{k+2}(r_1,r_2)+\varphi_{k+4}(r_1,r_2)+\cdots], \qquad (21a)\end{aligned}$$

$$\begin{aligned}2r_1r_2f_k(r_1,r_2)&=\tfrac{2}{3}(2k+1) \\ &\quad\times[\varphi_{k+1}(r_1,r_2)+\varphi_{k+3}(r_1,r_2)+\cdots]. \qquad (21b)\end{aligned}$$

Therefore, the integrations in A and B are over

$$-\frac{5}{\sqrt{21}}[(\varphi_0(r_1,r_2)-\tfrac{1}{5}\varphi_2(r_1,r_2))-\tfrac{2}{3}(\tfrac{1}{5}\varphi_2(r_1,r_2)-\tfrac{1}{9}\varphi_4(r_1,r_2))]$$

and

$$\frac{50}{3\sqrt{21}}(\tfrac{1}{5}\varphi_2(r_1,r_2)-\tfrac{1}{9}\varphi_4(r_1,r_2)),$$

respectively. The M^0 and M^2 of Marvin's paper[2] are given in our notation by

$$\begin{aligned}M^0&=-\frac{1}{24}\int\int[\varphi_0(r_1,r_2)-\tfrac{1}{5}\varphi_2(r_1,r_2)]\times R_2^2(r_1)R_2^2(r_2)dr_1dr_2, \\ M^2&=-\frac{1}{24}\int\int[\tfrac{1}{5}\varphi_2(r_1,r_2)-\tfrac{1}{9}\varphi_4(r_1,r_2)]\times R_2^2(r_1)R_2^2(r_2)dr_1dr_2.\end{aligned} \qquad (22)$$

Therefore, we obtain in this case,

$$\begin{aligned}A&=\frac{120}{\sqrt{21}}(M^0-\tfrac{2}{3}M^2)=40\sqrt{21}\left(M_0-\frac{14}{3}M_2\right), \\ B&=-\frac{400}{\sqrt{21}}M^2=-\frac{19600}{\sqrt{21}}M_2.\end{aligned} \qquad (23)$$

Putting these values in (18), one obtains the results of Trees.[3]

If the wave functions of the harmonic oscillator are used, it is always possible to express the radial parameters defined above (in this case the A and B) as linear combinations of the integrals[4] I_l of the function $V(r_{12})$. The easiest way to do it is to express the $2r_1r_2f_k(r_1,r_2)$ with the help of (20) as (linear) functions of $\varphi_k(r_1,r_2)$ and $(r_1^2+r_2^2)f_k(r_1,r_2)$. These latter functions are the coefficients of $P_k(\cos\omega_{12})$ in the expansion

$$\begin{aligned}(r_1^2+r_2^2)\frac{V(r)}{r^2}&=\frac{1}{2}\frac{4R^2+r^2}{r^2}V(r)=\frac{1}{2}V(r)+\frac{2R^2}{r^2}V(r) \\ &=\sum_{k=0}^{\infty}(r_1^2+r_2^2)f_k(r_1,r_2)P_k(\cos\omega_{12}). \qquad (24)\end{aligned}$$

The radial integrals over these coefficients can be evaluated by the method of Sec. 5 of reference 4; the only difference is the appearance of the function $\frac{1}{2}(4R^2+r^2)V(r)/r^2$ instead of $V(r)$. The integral I_0 of $V(r)/r^2$ as well as I_0 of $V(r)$, which appear in the course of the calculation must disappear in the results. For the case of d^n the results are

$$\begin{aligned}A&=-(5/12\sqrt{21})(7I_1-20I_2+21I_3), \\ B&=\tfrac{1}{6}\sqrt{(7/3)}(7I_1-5I_2).\end{aligned} \qquad (25)$$

V. TRANSITION TO jj COUPLING

In the jj coupling scheme it is not convenient to work with the double tensors $\mathbf{s}_i\mathbf{u}_i^{(k)}$ as these are no more irreducible. We shall decompose such double tensors into a sum of tensors $\mathbf{t}_i^{(1k)K}$ of degree K, irreducible with respect to $\mathbf{j}_i$. This decomposition is given by

$$t_{iQ}^{(1k)K}=[\mathbf{s}_i\times\mathbf{u}_i^{(k)}]_Q^{(K)}=\sum_{pq}s_{ip}u_{iq}^{(k)}(1pkq|1kKQ). \tag{26}$$

It can be shown that from this definition it follows that $\mathbf{t}_i^{(1k)K}$ has nonvanishing matrix elements between states with the same j_i only if $(-1)^K=(-1)^{1+k}$; this restricts the values of K, appearing in the calculation of a given configuration, to $k+1$ and $k-1$. Using the relation

$$\begin{aligned}([\mathbf{T}_1^{(\kappa_1)}\times\mathbf{T}_2^{(\kappa_2)}]^{(\kappa)}\cdot[\mathbf{U}_1^{(k_1)}\times\mathbf{U}_2^{(k_2)}]^{(\kappa)})\\ =\sum_k(-1)^{\kappa_1+k_2}(2\kappa+1)W(\kappa_1k_1\kappa_2k_2;k\kappa)\\ \times([\mathbf{T}_1^{(\kappa_1)}\times\mathbf{U}_1^{(k_1)}]^{(k)}\cdot[\mathbf{T}_2^{(\kappa_2)}\times\mathbf{U}_2^{(k_2)}]^{(k)}),\end{aligned} \tag{27}$$

we find for the interaction (6) between two nucleons the expression

$$V_{12}=\sum_{kk'K}A_{kk'}(\mathbf{t}_1^{(1k)K}\cdot\mathbf{t}_2^{(1k')K}). \tag{28}$$

Here the coefficients $A_{kk'}$ are sums of definite radial integrals, easily determined from (6) by means of (27).

From this expression the energy of a state with a definite J (the energy matrix is diagonal with respect to the total angular momentum J) can be found by use of the formulas of tensor algebra and the usual methods of spectroscopy.

The equation which should be used is II (33):

$$\begin{aligned}(\alpha JM|(\mathbf{T}_a^K\cdot\mathbf{T}_b^K)|\alpha'JM)=\frac{1}{2J+1}\sum_{\alpha''J''}(-1)^{J-J''}\\ \times(\alpha J\|T_a^K\|\alpha''J'')(\alpha''J''\|T_b^K\|\alpha'J).\end{aligned} \tag{29}$$

In order to calculate the energy levels in the case of two nucleons with l_1j_1 and l_2j_2, respectively, we have to use II (44) and[12]

$$\begin{aligned}(\tfrac{1}{2}lj\|t_i^{(1k)K}\|\tfrac{1}{2}lj)=(3/2)^{\frac{1}{2}}(2j+1)(2K+1)^{\frac{1}{2}}\\ \times[(2l+2)W(l\tfrac{1}{2}j1;l+\tfrac{1}{2},\tfrac{1}{2})W(\tfrac{1}{2}ljk;l+\tfrac{1}{2},l)\\ \times W(1jkj;l+\tfrac{1}{2},K)+2lW(l\tfrac{1}{2}j1;l-\tfrac{1}{2},\tfrac{1}{2})\\ \times W(\tfrac{1}{2}ljk;l-\tfrac{1}{2},l)W(1jkj;l-\tfrac{1}{2},K)].\end{aligned} \tag{30}$$

In the case of the j^n configuration we introduce the tensors $\mathbf{T}^{(1k)K}=\sum_{i=1}^n\mathbf{t}_i^{(1k)K}$ which operate on the coordinates of the whole group of the j nucleons. With the help of these tensors, the interaction (28), summed over all pairs ij of interacting nucleons, can be brought into the form

$$\begin{aligned}\sum_{i<j}V_{ij}&=\sum_{kk'K}A_{kk'}\sum_{i<j}(\mathbf{t}_i^{(1k)K}\cdot\mathbf{t}_j^{(1k')K})\\ &=\tfrac{1}{2}\sum_{kk'K}A_{kk'}[(\mathbf{T}^{(1k)K}\cdot\mathbf{T}^{(1k')K})-\sum_{i=1}^n(\mathbf{t}_i^{(1k)K}\cdot\mathbf{t}_i^{(1k')K})]\\ &=\tfrac{1}{2}\sum_{kk'K}A_{kk'}(\mathbf{T}^{(1k)K}\cdot\mathbf{T}^{(1k')K})\\ &\quad-\frac{n}{2}\sum_{kk'K}A_{kk'}(\mathbf{t}_1^{(1k)K}\cdot\mathbf{t}_1^{(1k')K}).\end{aligned} \tag{31}$$

On the calculation of $(j^n\alpha J\|T^{(1k)K}\|j^n\alpha'J')$ it should be said what was said before on $(l^n\alpha SL\|V^{(1k)}\|l^n\alpha'S'L')$. Only if $n=2$ we have a simple expression

$$\begin{aligned}(j^2J\|T^{(1k)K}\|j^2J')=(-1)^K[(-1)^J+(-1)^{J'}]\\ \times[(2J+1)(2J'+1)]^{\frac{1}{2}}W(jJjJ';jK)(j\|t_1^{(1k)K}\|j),\end{aligned} \tag{32}$$

in which $(j\|t_1^{(1k)K}\|j)$ is given by Eq. (30).

In the j^n configuration, tensors $\mathbf{T}^K$ with K odd [as are the tensors which appear in (31)] are representations of the infinitesimal operators of the group of transformations which leave invariant the wave function of j^2, $J=0$; this group is the symplectic group in $2j+1$ dimensions $-\mathrm{Sp}(2j+1)$. The tensors considered are therefore diagonal with respect to the quantum numbers which characterize these representations. In particular, in the case of the j^n configuration of equivalent nucleons tensors $\mathbf{T}^K$ with K odd are diagonal with respect to the seniority[1,10] and independent of n. The energy in this case, however, has also a term linear in n, which is equal to $\frac{1}{2}(n-v)E_0$, where E_0 is the energy of the state j^2, $J=0$.

The $A_{kk'}$ introduced in (28) can be defined as the radial parameters in the jj coupling scheme. In the nuclear configuration j^n the number of independent parameters is equal to the number of matrices $(\mathbf{T}^{(1k)K}\cdot\mathbf{T}^{(1k')K})$ with different K, which is $2j$. In the j^n configuration of equivalent nucleons, these matrices are not independent and the actual number of parameters in this case is the number of independent matrices $(\mathbf{T}^{(1k)K}\cdot\mathbf{T}^{(1k')K})$ (matrices $\mathbf{T}^{(1k)K}$ with the same K but different k are proportional). This number is obtained from an analogous argument to that adapted in the l^n configuration. The tensors $\mathbf{T}^K$ with odd K are now to be considered, instead of $\mathbf{V}^{(1k)}$ with even k, but the representation of $\mathrm{Sp}(2j+1)$ to which these tensors (regarded as states of j^2) belong is (20). Similarly, the antisymmetric wave functions of j^2 have even J (instead of the odd L values we had to consider before), but these belong now to the representation (11) of $\mathrm{Sp}(2j+1)$. We have to look for states with $J=0$ (instead of the D states) which belong to a representation, contained in $(20)\times(20)$ as well as in $(11)\times(11)$; this can be either (00) or (22) (to (20) does not belong a $J=0$ state). The representation [22] of $U(2j+1)$ (for j^4) breaks upon restriction to $\mathrm{Sp}(2j+1)$ into $(00)+(11)+(22)$ of which

[12] This formula in a more general form as well as all the important formulas of the tensor algebra is presented in a paper which will be published by U. Fano and G. Racah.

TABLE I. Energies of the $(d_{5/2})^2$ and $(d_{5/2})^3$ configurations.

State	$(d_{5/2})^2$ Energy	State	$(d_{5/2})^3$ Energy
$J=4$	$\frac{1}{5}I_1 - \frac{1}{5}I_3$	$J=9/2$	$\frac{3}{5}I_1 - \frac{18}{35}I_2 + \frac{3}{25}I_3$
$J=2$	$\frac{1}{5}I_1 - \frac{4}{5}I_2 + \frac{23}{25}I_3$	$J=5/2$	$\frac{7}{5}I_1 - 2I_2 + \frac{7}{5}I_3$
$J=0$	$\frac{7}{5}I_1 - 2I_2 + \frac{7}{5}I_3$	$J=3/2$	$\frac{3}{5}I_1 - \frac{12}{7}I_2 + \frac{9}{5}I_3$

to (11) does not belong any term with $J=0$. As a result we obtain the required number by use of the formula given in reference 11 in the case $n=4$, $J=0$; this number is found to be $[(2j+2)/3]$.

It is therefore possible to express the $2j$ matrices $(\mathbf{T}^K\cdot\mathbf{T}^K)$ in terms of $[(2j+2)/3]$ of them. Also in this case we can find these expressions in the case of j^2 and these hold also in the case of j^n, although the reason given for this fact in the case of l^n is no more valid. Instead of it we have the situation that if

$$(j^2|(\mathbf{T}^K\cdot\mathbf{T}^K)|j^2)=\sum_{K'} a_{K'}(j^2|(\mathbf{T}^{K'}\cdot\mathbf{T}^{K'})|j^2), \quad (33)$$

in the case of j^n there exists the relation

$$(j^n|(\mathbf{T}^K\cdot\mathbf{T}^K)|j^n)=\sum_{K'} a_{K'}(j^n|(\mathbf{T}^{K'}\cdot\mathbf{T}^{K'})|j^n) +n(n-2)[\sum_{K'} a_{K'}(j^n|(\mathbf{t}_1^{K'}\cdot\mathbf{t}_1^{K'})|j^n) -(j^n|(\mathbf{t}_1^K\cdot\mathbf{t}_1^K)|j^n)]. \quad (34)$$

The second term on the right-hand side must vanish as it is the same for all the states and vanishes for $v=0$ (for which the products $(\mathbf{T}^K\cdot\mathbf{T}^K)$ with odd K vanish), which proves the statement. It may be noted that because of (15) we can apply the transformation (27) directly on (10) in which the matrices (14) are expressed by means of $[4l/3]$ of them, a fact which facilitates the calculation.

VI. THE LEVELS OF THE $(d_{5/2})^n$ CONFIGURATION

We can now write immediately down the energy matrix of the $(d_{5/2})^n$ configuration. In this case $[4\cdot 2/3]=2$, and also $[(2\cdot 5/2+2)/3]=2$, so that there are 2 independent parameters and starting from (18) no further reduction is necessary. Transformation of (18) with the help of (27) yields the final formula:

$$((d_{\frac{5}{2}})^nJM|\sum_{i<j} V_{ij}|(d_{\frac{5}{2}})^nJM) = -\frac{5}{2J+1}\Big[\frac{A}{\sqrt{15}}((d_{\frac{5}{2}})^nJ\|(\mathbf{T}^{(10)1}\cdot\mathbf{T}^{(12)1})\|(d_{\frac{5}{2}})^nJ) +B\frac{7}{10}\sqrt{\frac{7}{3}}((d_{\frac{5}{2}})^nJ\|(\mathbf{T}^{(12)1}\cdot\mathbf{T}^{(12)1})\|(d_{\frac{5}{2}})^nJ) +\frac{B}{35}\sqrt{\frac{7}{3}}((d_{\frac{5}{2}})^nJ\|(\mathbf{T}^{(12)3}\cdot\mathbf{T}^{(12)3})\|(d_{\frac{5}{2}})^nJ)\Big] +n\Big[-A\frac{6}{25}\sqrt{\frac{7}{3}}+B\frac{6}{35}\sqrt{\frac{7}{3}}\Big]. \quad (35)$$

As there are at most 6 equivalent nucleons in a $d_{5/2}$ state, the relevant configurations are $(d_{5/2})^2$ [which is equivalent to $(d_{5/2})^4$] and $(d_{5/2})^3$. We obtain the energies of these configurations by inserting in (35) the values of A and B from (25), and we write them down multiplied by 12 [in accordance with the usual definition $S_{12}=3(\boldsymbol{\sigma}_1\cdot\mathbf{r}_{12})(\boldsymbol{\sigma}_2\cdot\mathbf{r}_{12})/r_{12}^2-(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)$]; the results are given in Table I.

In the short-range approximation,[4] I_2 and I_3 can be neglected in comparison to I_1 (it has no sense to use here the limit of a δ-function as then also I_1 vanishes). On the other hand, in the long-range limit, $I_1=I_2=I_3$. We can now see what are the energy values of the tensor forces in the $(d_{5/2})^2$ and $(d_{5/2})^3$ configurations. If the potential $V(r_{12})$ is negative (or positive and multiplied by P_x, which gives the same results in this case) the order of levels resulting from the tensor forces is $J=0$ (ground level), $J=2$, $J=4$ in $(d_{5/2})^2$ in the short-range approximation and up to the long-range limit (for reasonable form of the potential); in $(d_{5/2})^3$ the order will be $J=5/2$ (ground level), $J=3/2$, $J=9/2$. We see that the results for tensor forces are essentially those for short-range (attractive) ordinary forces, in spite of their very different appearance.

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