ordering of the spins resulting from magnetic spin interaction alone to occur at about 10^{-7} degree K. However measurements by Weinstock, Abraham, and Osborne[10] give a positive slope of the melting pressure curve down to 0.5°K.

The structure or density of solid He^3 is not yet known. It is quite possible that solid He^3 is different from other solids regarding the alignment of the nuclear spin. Solid He^3 might be so very loosely packed that, for similar reasons as in the liquid state, a spin alignment occurs in the solid state at a much higher temperature than assumed by Pomeranchuk, and that, at a finite temperature, $S_{\text{solid}} < R \ln 2$.

Another possibility would be that, as liquid He^3 is compressed towards the solidification line, it may gradually or discontinuously be squeezed into a structure almost as localized as in the solid state, and lose the characteristics of a smoothed potential liquid. In this case the spins would become disorganized under compression *before* solidification. This would mean that the thermal expansion coefficient of the liquid under sufficient pressure would become negative, since $(\partial V/\partial T)_p = -(\partial S/\partial p)_T$.

Weinstock, Abraham, and Osborne[11] proposed to extrapolate the vapor pressure data in such a way as to give $S_{\text{liq}} > R \ln 2$, $T \geqslant 0.5$°K. In view of our interpretation ($S_{\text{liq}} \to 0$ when $T \to 0$), this possibility is in no way demanded by the experimental data and appears to us less likely than either of the two alternatives mentioned above.

An experiment to decide directly the degree of nuclear spin alignment in liquid as well as in solid He^3 by measuring the intensity of the nuclear resonance adsorption is being conducted at this University.

[10] Weinstock, Abraham, and Osborne, Phys. Rev. **85**, 158 (1952).

[11] Weinstock, Abraham, and Osborne, Phys. Rev. **89**, 787 (1953).

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Bohm's Interpretation of the Quantum Theory in Terms of "Hidden" Variables

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An analysis of Bohm's theory, emphasizing the role of probability in it, is presented.

IN two recent articles[1] David Bohm has suggested a "deterministic" interpretation of quantum mechanics based on the introduction of "hidden" variables. It is the purpose of this note to analyze his interpretation and especially the role of probability in it, by examining the complete mathematical formulation of the theory. We are led to the conclusion that if Eq. (10) below must be postulated, then the suggested interpetation is not an ordinary statistical mechanics of a deterministic theory, which is the kind of interpretation many physicists have hoped for and which some have thought this to be. On the other hand, if Eq. (10) can be deduced from the other postulates of the theory, as Bohm attempts to prove in his latest[1] paper, then the theory is essentially an ordinary statistical mechanics of a deterministic theory.

I. DETERMINISTIC MECHANICS OF A SINGLE PARTICLE

In order to present our analysis, we shall first give a precise formulation of Bohm's theory for a single particle in an external potential $V(x)$. Bohm postulates that the particle of mass m has a position $x(t)$ at time t, and thus also a velocity $\dot{x}(t)$. These quantities $x(t)$ and $\dot{x}(t)$ are the so-called "hidden" variables. In addition to $x(t)$ and $\dot{x}(t)$, there is a "quantum mechanical field" $P(x, t)$. The quantities $P(x, t)$ and $x(t)$ satisfy the following equations:

$$\dot{P} + \nabla \cdot (m^{-1} P \nabla S) = 0, \tag{1}$$

$$\dot{S} + \frac{1}{2m}(\nabla S)^2 + V(x) - \frac{\hbar^2}{2m}\frac{\nabla^2 P^{\frac{1}{2}}}{P^{\frac{1}{2}}} = 0, \tag{2}$$

$$\dot{x} = \nabla S/m. \tag{3}$$

In these equations $S(x, t)$ is a modified Hamilton-Jacobi function which is not assigned a physical interpretation,[2] and $\hbar$ is Planck's constant.

The above equations must be supplemented by the following initial conditions in order to determine a unique solution:[3]

$$P(x, 0) = P_0(x), \tag{4}$$

$$S(x, 0) = S_0(x), \tag{5}$$

$$x(0) = x_0. \tag{6}$$

[1] David Bohm, Phys. Rev. **85**, 166, 180 (1952); **89**, 458 (1953).

[2] An alternative formulation, not employing S, is possible but does not alter the main discussion to be given below.

[3] It is interesting to notice that the function $S_0(x)$ must be given for all x in order to determine a single trajectory. In contradistinction to this, in classical mechanics a trajectory is uniquely

From Eqs. (3) and (5) we notice that $\dot{x}(0)$ is not arbitrary but is given by

$$\dot{x}(0)=\nabla S_0(x_0)/m. \tag{7}$$

Therefore if $\dot{x}(0)$ is given, $S_0(x)$ must be so chosen as to satisfy Eq. (7).

Equations (1)–(6) constitute the formulation of the deterministic theory and determine the trajectory $x(t)$, the velocity $\dot{x}(t)$ and the field $P(x, t)$ uniquely.

II. STATISTICAL MECHANICS OF A SINGLE PARTICLE[4]

Probability is supposed to enter the theory through incomplete knowledge of initial data, just as in classical statistical mechanics. Bohm implies, but does not state explicitly, that $P_0(x)$ and $S_0(x)$ are known exactly, but that x_0 is not. Instead an initial probability distribution $\phi_0(x)$ of x is known, and it is desired to determine the subsequent probability distribution $\phi(x, t)$ of x at time t.[5] By applying the law of conservation of probability we conclude that $\phi(x, t)$ satisfies

$$\dot{\phi}+\nabla\cdot[\phi(x, t)\nabla S/m]=0, \tag{8}$$

$$\phi(x, 0)=\phi_0(x). \tag{9}$$

In Eq. (8), $\nabla S(x, t)/m$ represents the velocity field obtained by solving Eqs. (1) and (2) subject to the initial conditions, Eqs. (4) and (5).

Equations (8) and (9) determine the probability distribution $\phi(x, t)$ corresponding to given $\phi_0(x)$, $P_0(x)$ and $S_0(x)$. The function ϕ characterizes the statistical mechanics of a single particle, the deterministic mechanics of which is given by Eqs. (1)–(6). Although none of the quantities $\phi(x, t)$, $\phi_0(x)$, $P_0(x)$ or $S_0(x)$ was mentioned by Bohm, it appears that they are essential for an understanding of the theory.

III. RELATION OF $P(x, t)$ AND $\phi(x, t)$

By comparing Eqs. (1) and (8) we notice that P and ϕ satisfy the same equation, but nevertheless they are not equal, since they satisfy the independent initial conditions, Eqs. (4) and (9), respectively. The initial function $P_0(x)$ is supposed to represent the initial values of the real quantum-mechanical field present in a particular experiment. The function $\phi_0(x)$ is supposed to represent the initial probability distribution of x, characteristic of some ensemble of experiments. In the statistical treatment of a deterministic theory the initial probability $\phi_0(x)$ is arbitrary, and therefore, it will not generally equal $P_0(x)$. Therefore, in general, $P(x, t)$ will differ from $\phi(x, t)$.

The only instance in which $P(x, t)=\phi(x, t)$ is that in which the initial data are accidentally equal, that is if

$$P_0(x)=\phi_0(x). \tag{10}$$

Only in this case can $P(x, t)$ be given the dual interpretation of a probability distribution characterizing an ensemble of experiments, and a real field present in a single experiment. Thus it is only in this case that the probabilities deduced from the statistical mechanics of Bohm's deterministic theory will coincide with those given by quantum mechanics [with $\psi(x, 0)=P_0^{\frac{1}{2}}(x)\times e^{iS_0(x)/\hbar}$].

In order to assure agreement between these two theories in all cases, it is necessary to add to Bohm's theory the postulate that Eq. (10) always holds. This is a physical postulate which is inherently probabilistic. Therefore, if it is accepted, the resulting theory is no longer an ordinary statistical mechanics of a deterministic theory. Probability enters the theory not only through incomplete knowledge of initial data but also in a deeper way—it affects the initial P field. In other words, if an ensemble of experiments is prepared, the initial P field in each is determined by the nature of the ensemble, and all experiments in the ensemble will have the same initial P field. This surprising hypothesis somewhat strains the notion of a "real" P field. However, if Eq. (10) can be deduced from the other postulates of the theory, as Bohm contends in his latest paper, then probability enters the theory just as in any statistical mechanics.

There is another remarkable difference between Bohm's theory and an ordinary statistical mechanics of a deterministic theory. In an ordinary statistical mechanics the underlying deterministic theory is also a special case of the statistical theory. It is obtained from the latter by assuming that the initial probability distribution is a delta-function. In Bohm's theory, however, only that particular case of the deterministic theory in which $P_0(x)$ is a delta-function can be obtained by specializing the statistical theory. This follows from Eq. (10), since to obtain the deterministic theory $\phi_0(x)$ must be a delta-function, and therefore $P_0(x)$ is also.

The conclusion is that Bohm's interesting interpretation of quantum mechanics, based on the introduction of "hidden" variables, is not an ordinary statistical mechanics of a deterministic theory, unless Eq. (10) can be shown to follow from the other postulates. If not, the theory involves probability in an additional and deeper manner, expressed by Eq. (10).

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determined by giving merely $\nabla S_0(x)$ at $x=x_0$, the initial point on the trajectory. This difference is related to the structure of Eqs. (1)–(3), and implies that $S(x, t)$ should also be given a physical interpretation, since its initial values must be determined in order to formulate a problem correctly.

[4] Statistical mechanics, as used here, does not refer to equilibrium theory, but merely to the introduction of probability into a deterministic theory through incomplete knowledge of initial data.

[5] It is significant that Bohm does not seek the joint probability distribution of x and $\dot{x}$. This is not sought because $\dot{x}$ is uniquely related to x and t by $\dot{x}=\nabla S(x, t)/m$ [see explanation after Eq. (9)]. This is related to the fact that quantum mechanics does not yield joint probability distributions of conjugate variables.