Momentum Transfer and Angle of Divergence of Pairs Produced by Photons*

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The distribution of the momentum transferred to the pairs in pair production by photons has been derived from the Bethe-Heitler cross section. The effect of screening is evaluated using a shielded Coulomb potential for the nucleus. In the last part of the article the distribution of the angle of divergence of the pair is obtained, and the most probable angle of divergence is given by $\phi(E_+/k)\cdot 4mc^2/k$, where k and E_+ are the energies of the photon and the positron and $\phi(E_+/k)\sim 1$ for $0.2<E_+/k<0.5$.

SOME of the results of recent experiments[1] on the recoil momentum q of the nucleus in pair production by photons seem to be in disagreement with theoretical prediction.[2] Probably this disagreement is only due to experimental difficulties. Anyway, further research is needed to clarify the question.

In this paper a formula is given for the distribution of the total momentum $P=|\mathbf{p}_++\mathbf{p}_-|$ transferred to the electron pair. This formula, in comparison with the distribution-function of the recoil momentum,[2] has some advantages: (a) it is simpler; (b) it allows us to take screening into account in a simple way; (c) it gives information on the angle between the electron and the positron.

From the experimental point of view, the determination of the total momentum P has the advantage of being independent of the knowledge of the direction of the incident photon. Thus it is possible to avoid collimation errors and use all the available volume of the detector. For the same reason it is possible to study the distribution of the momentum transfer in the case of pairs produced by cosmic-ray photons, for which there is no collimation at all. The angle ω between the electron and the positron in pairs produced by cosmic-ray photons has been used[3] to guess at the energy of the pair and the energy k of the generating photon. In the last part of this paper we obtain the distribution of the angle ω and derive from it a correlation between ω and the photon energy.

1. DISTRIBUTION OF THE TRANSFERRED MOMENTUM

From the Bethe-Heitler[4] differential cross section (Born approximation), we obtain the cross section for the distribution of the vector recoil momentum $\mathbf{q}$ of the nucleus (units $\hbar=c=1$), as follows:

$$d\sigma=\sigma_0\frac{2m^2}{\pi k}\frac{g(\mathbf{q})}{q^4}d^3\mathbf{q}, \qquad (1)$$

where $\sigma_0=Z^2r_0^2/137$ is the unit cross section of Heitler, k is the energy of the photon, m the electron mass, and $g(\mathbf{q})$ the expression:

$$\begin{aligned} g(\mathbf{q})=&[1+2(m^2-q^2)x+q^2(q^2-k^2-m^2)x^2\\ &+2k^2q^2(2q^2-m^2)x^3+k^2q^4(m^2-2q^2)x^4]\cdot\Delta\\ &-[2+2(2m^2-q^2)x+(q^4-2k^2q^2-2m^2q^2-4m^4)x^2\\ &+2k^2q^2(q^2-2m^2)x^3\\ &+k^2q^2(4m^4+6m^2q^2-q^4)x^4]\cdot L, \end{aligned} \qquad (2)$$

where

$$\begin{aligned} x^{-1}&=(\mathbf{k}\cdot\mathbf{q})=kq\cos\theta_n, &\Delta&=(1-4m^2/Q^2)^{\frac{1}{2}},\\ L&=\cosh^{-1}(Q/2m), &Q&=(k^2-P^2)^{\frac{1}{2}}; \end{aligned} \qquad (3)$$

θ_n is the angle between the directions of the recoil momentum and the incident photon.

The integration of (1) over the angle θ_n gives the distribution of the recoil momentum q as obtained by Jost *et al.*[2] The result agrees with a more general distribution previously derived by us,[5] in which also the energy transfer is taken into account.[6]

To obtain the distribution of the total momentum $P=|\mathbf{p}_++\mathbf{p}_-|$ of the pair, we replace in (1) $d^3\mathbf{q}$ by $d^3\mathbf{P}=P^2dP\sin\theta d\psi$ and integrate over the angles. Assuming that θ is the angle between $\mathbf{P}$ and $\mathbf{k}$, we have $q^2=k^2+P^2-2kP\cos\theta$, $x^{-1}=k^2-kP\cos\theta$; here q varies with θ, but in (2), when we perform the integration over the angles, Δ and L remain constant. For this reason the result is expressible with only elementary functions and gives

$$d\sigma=4\sigma_0\frac{F}{(k^2-P^2)^2}PdP, \qquad (4)$$

where $2m=1$ and

$$F=\left[F_1+\frac{Q^2}{2k^2}(F_1-\Delta)\right]\operatorname{sech}^{-1}\frac{Q}{k}-\left(F_2+\frac{Q^2}{6k^2}F_3\right)\frac{P}{k}, \qquad (5)$$

with F_1, F_2, F_3 functions of Q only:

$$\begin{aligned} F_1&=(2+2/Q^2-1/Q^4)L-(1+1/Q^2)\Delta,\\ F_2&=\tfrac{1}{6}(16+21/Q^2-17/Q^4)L-\tfrac{1}{12}(28+17/Q^2)\Delta,\\ F_3&=\tfrac{1}{2}(4-1/Q^2)L-\tfrac{1}{2}(2+1/Q^4)\Delta. \end{aligned} \qquad (6)$$

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[1] G. E. Modesitt and H. W. Koch, Phys. Rev. 77, 175 (1950).

[2] Jost, Luttinger, and Slotnick, Phys. Rev. 80, 189 (1950).

[3] Bradt, Kaplon, and Peters, Helv. Phys. Acta 23, 24 (1950).

[4] H. Bethe and W. Heitler, Proc. Roy. Soc. (London) A146, 83 (1934).

[5] A. Borsellino, Nuovo cimento 4, 112 (1947), Eq. (42).

[6] We take the opportunity to correct a misprint in the paper of Jost *et al.*, reference 2; the numerator of the last term in the first line of Eq. (43) ought to be $4Q^4-1$.

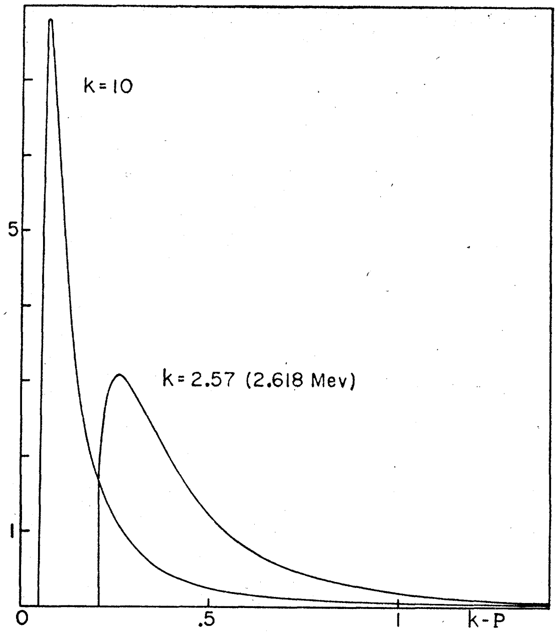

FIG. 1. Distribution of the transferred momentum P for photons of energy $k=2.57$ and $k=10$ (units $2mc^2$). In the abscissa: differences $k-P$ in units $2mc$; the distributions are both normalized to 1, dividing the cross section $d\sigma/dP$ by the total cross section.

As it must be, F is equal to zero at the extreme values, 0 and $(k^2-1)^{\frac{1}{2}}$, that P can reach. For small P, we have $F\sim P/k$, and the cross section is very small and increases as P^2/k^5. The most probable value of P is reached near the maximum value P_M of P, where the cross section for high energy ($k\gg1$) varies as

$$k^{\frac{3}{2}}\cdot\ln(2k)\cdot(P_M-P)^{\frac{1}{2}}.$$

The distribution of transferred momentum P is represented in Fig. 1 for two photon energies: $k=2.57$ (2.618 Mev) and $k=10$. On the abscissa is plotted the difference $k-P$ (in units of $2mc$) and on the ordinate the value of $d\sigma/dP$ given by (4) and (5), divided by the total cross section, to make the area under each curve equal to one. We see that, with increasing energy, the distribution becomes strongly peaked near $P\sim k$, i.e., when the pair receives nearly all the available momentum.

Instead of the P-distribution, it is easier to study the equivalent distribution of Q, which is given by:

$$d\sigma=4\sigma_0(F/Q^3)dQ, \tag{7}$$

where Q varies now between 1 and k.

This quantity Q has a simple invariant meaning: it is the energy of the pair in the center-of-mass system of the electron and the positron.

The most probable values of Q are now close to 1 and, since F increases as slowly as $\ln Q$, the distribution, after the maximum, drops down as $1/Q^3$. Therefore, if the energy is large enough, we are interested essentially in values for which $Q\ll k$. In this case, formula (5) gets simplified as follows:

$$F=F_1\ln(2k)-F_2^*, \tag{8}$$

with

$$F_2^*=F_2+F_1\ln Q. \tag{9}$$

From (8) one sees that the distribution of Q is not very sensitive to the energy of the photon. In Fig. 2 the distribution of Q is given for the same energies as in Fig. 1 and for $k=25$.

2. EFFECT OF SCREENING

Until now, we have not taken into account the screening of the field of the nucleus due to the atomic electrons. To evaluate this effect, we use a shielded Coulomb potential $V(r)=Zer^{-1}\exp(-a_0r)$ to represent the potential of the atom, and we choose[7] the parameter $a_0=mcZ^{\frac{1}{3}}(108\hbar)^{-1}$, in such a way that the total cross section for very high energies agrees with the values deducible from the Fermi-Thomas atom.

With this potential we must only replace in (1) the denominator q^4 by $(q^2+q_0^2)^2$, where $q_0=\hbar a_0$. The calculations are elementary as before. The screening is effective for $k\gg1$, therefore we need consider only the case $Q\ll k$. The distribution of the Q values is again given by (7), where F is now

$$F=F_1\ln\frac{2k}{(1+4k^2q_0^2/Q^4)^{\frac{1}{2}}}-F_2^*, \tag{10}$$

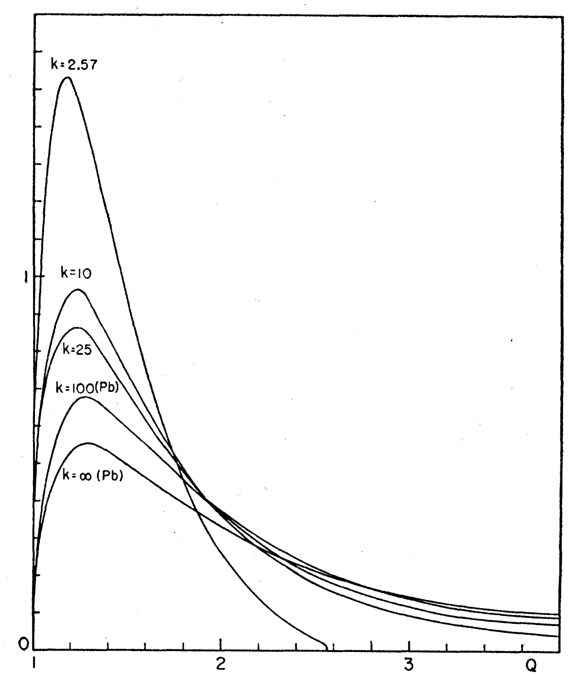

FIG. 2. Q distributions for the same photon energies k as in Fig. 1 and for $k=25$ (no screening); the curves (Pb) are calculated for lead and photon energies $k=100$ (partial screening) and $k=\infty$ (complete screening). All the distributions are normalized to 1.

[7] H. A. Bethe, Proc. Cambridge Phil. Soc. 30, 524 (1934).

which, for $2kq_0 \ll 1$ (no screening), coincides with (8). In the opposite case of complete screening, we obtain a distribution independent of energy:

$$F = F_1 \ln(216Q^2Z^{-\frac{1}{3}}) - F_2^*. \tag{11}$$

The differential cross section (7), with F given by (10), also yields, when integrated, correct values of the total cross section at intermediate energies (partial screening); therefore it is probable that (10) gives correct Q distributions at these energies. In Fig. 2, the distribution of Q for lead is given, at $k=100$ and for complete screening ($k=\infty$).

3. DISTRIBUTION OF THE ANGLE OF DIVERGENCE ω

The quantity Q depends on the angle ω between the electron and the positron, and on the ratio in which the

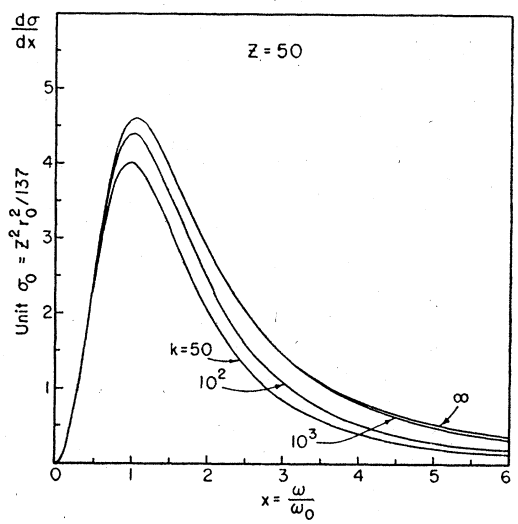

FIG. 3. Distribution of the angle of divergence ω between the two members of a pair for photon energies $k=50$ (51 Mev); 10^2; 10^3; ∞ (complete screening) and atomic number $Z=50$. On the abscissa: ratio ω/ω_0, where $\omega_0 = kmc^2/(E_+E_-)$; on the ordinate: cross section $d\sigma/dx$ in unit $\sigma_0 = Z^2r_0^2/137$. The curves are given for equipartition ($a=E_+/k=\frac{1}{2}$); otherwise the ordinates must be multiplied by $4a(1-a)$.

energy of the photon is divided between the two electrons.

If the energy is large enough ($E_+, E_- \gg mc^2$), we may write for Q:

$$Q^2 = E_+E_-(\omega_0^2+\omega^2) = [4a(1-a)]^{-1}(1+\omega^2/\omega_0^2), \tag{12}$$

where a is the ratio E_+/k, and ω_0 is the characteristic angle:

$$\omega_0 = kmc^2/(E_+E_-). \tag{13}$$

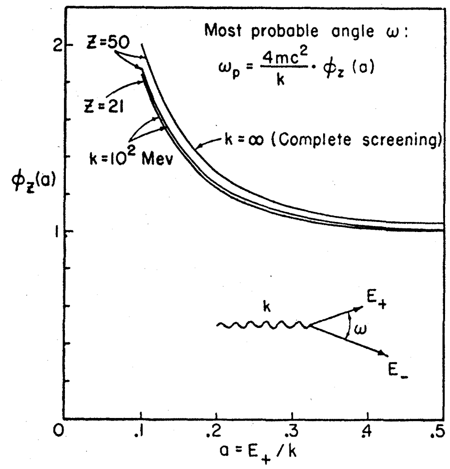

FIG. 4. $\phi_Z(a)$ *versus* $a=E_+/k$.

If one puts $x=\omega/\omega_0$, one obtains for the cross section (7)

$$d\sigma = 16\sigma_0 a(1-a)Fxdx/(1+x^2)^2, \tag{14}$$

which for a fixed value of a gives the distribution of the angle ω. In the region where the factor $x/(1+x^2)^2$ has a maximum ($x=1/\sqrt{3}$), the function F is still rapidly increasing and this shifts the value at which the maximum of the distribution lies towards greater values of x.

Figure 3 represents the distribution of the angle ω, according to (14), for different photon energies and atomic number $Z=50$, in the case of equipartition ($a=\frac{1}{2}$).

The angle ω_p at which the angular distribution has its maximum can be written in the form:

$$\omega_p = (4mc^2/k)\cdot\phi_Z(a), \tag{15}$$

where $\phi_Z(a)$ is a function of a and also depends on the energy k and the atomic number Z. In Fig. 4 the values of $\phi_Z(a)$ for $k=100$ Mev and $k=\infty$ (complete screening) for two different values of Z are given. It is evident from this that the dependence on k and Z is very small. For complete screening the curve for $Z=21$ practically coincides with that for $Z=50$. For equipartition, we have $\phi_Z \sim 1$.

Formula (15) may be of use in guessing at the energy of the pair from the observed value of the angle ω. Bradt *et al.*[3] equated ω to twice the rms angle between the electron and the incident photon, as calculated by Stearns.[8] As a result of the contributions of the angles beyond the maximum of the distribution, this rms angle is greater by a factor of $\ln(k/mc^2)$ than the most probable value given by (15).

Thus it seems more reasonable to the author to equate ω to ω_p. Following the other method, the deduced photon energy turns out to be smaller by a factor of $\ln(k/mc^2)$ than the value obtained using the method proposed here.

[8] M. Stearns, Phys. Rev. **76**, 836 (1949).