$$=(-1)^k k!\lim_{\epsilon\to 0}\left\{\int_{-\infty}^{-\epsilon} dx[\varphi(x)/x^{k+1}]+(-1)^k\varphi(-\epsilon)/k\epsilon^k\right.$$
$$+(1/k)\sum_{\mu=1}^{k-1}(-1)^{k+\mu-1}\frac{\varphi^{(\mu)}(0)\epsilon^{\mu-k}}{(\mu-1)!(\mu-k)}$$
$$\left.-\frac{\varphi^{(k)}(0)}{k!}\log\epsilon\right\}+\sum_{\mu=1}^{k-1}(1/\mu)\delta^{(k)}(\varphi),$$

$$=(-1)^k k!\lim_{\epsilon\to 0}\left\{\int_{-\infty}^{-\epsilon} dx[\varphi(x)/x^{k+1}]+\frac{(-1)^k}{k\epsilon^k}\varphi(0)\right.$$
$$-\frac{\varphi^{(k)}(0)}{k!}\log\epsilon+(1/k)\sum_{\mu=1}^{k-1}\left[\frac{(-1)^k\epsilon^{\mu-k}}{\mu!}\right.$$
$$\left.\left.+\frac{(-1)^{k+\mu-1}\epsilon^{\mu-k}}{(\mu-1)!(\mu-k)}\right]\varphi^{(\mu)}(0)\right\}+\frac{(-1)^k}{k}\varphi^{(k)}(0)$$
$$+\sum_{\mu=1}^{k-1}(1/\mu)\delta^{(k)}(\varphi),$$

$$=-(-1)^{k+1}k!\,\mathrm{Pf}[(x^{-k-1})(\varphi)]+\sum_{\mu=1}^{k}(1/\mu)\delta^{(k)}(\varphi);$$

i.e., (3.10) is true for every n, q.e.d. Eq. (3.8) can be proved in the same manner.

IV. Equation (3.16) is true for $m=1$, n arbitrary, according to (3.13). Assuming (3.16) to be true for $m=k>1$ we have

$$D^{k+1}\,\mathrm{Pf}[(x^{-n})_\pm(\varphi)]=D[D^k\,\mathrm{Pf}(x^{-n})_\pm(\varphi)]$$
$$=(-1)^k\prod_{\nu=0}^{k-1}(n+\nu)D[\mathrm{Pf}(x^{-k-n})_\pm(\varphi)]$$
$$\pm[(-1)^n/(n-1)!]\sum_{\nu=0}^{k-1}(1/n+\nu)\delta^{(k+n)}(\varphi)$$
$$=(-1)^{k+1}\prod_{\nu=0}^{k}(n+\nu)\,\mathrm{Pf}[(x^{-n-k-1})_\pm(\varphi)]$$
$$\pm(-1)^n\left[(1/(k+n)!)\prod_{\nu=0}^{k-1}(n+\nu)\right.$$
$$\left.+(1/(n-1)!)\sum_{\nu=0}^{k-1}(1/n+\nu)\right]\delta^{(k+n)}(\varphi)$$
$$=(-1)^{k+1}\prod_{\nu=0}^{k}(n+\nu)\,\mathrm{Pf}[(x^{-n-k-1})_\pm(\varphi)]$$
$$\pm[(-1)^n/(n-1)!]\sum_{\nu=0}^{k}(1/n+\nu)\delta^{(k+n)}(\varphi);$$

i.e., (3.16) is true for all n, m.

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The Range Correction for Electron Pick-Up

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The extension of positive particle ranges caused by pick-up of electrons at low velocities has been studied in Ilford C2 emulsion. Ranges of Li^8 and B^8 nuclei were measured in emulsion and compared with tracks of helium and hydrogen isotopes of the same velocity. The empirical range-energy relation adduced for light nuclei is: $Z^2R/M=F(T/M)+0.12Z^3$, for $\beta>1.04Z/137$.

I. INTRODUCTION

It has recently[1] been found possible to collect and analyze physically the products formed when the high energy beam of the 184-inch cyclotron is employed to disintegrate atomic nuclei. A reliable analysis of the products of atomic number greater than two was, however, difficult because the range-energy relations in nuclear track emulsion for multiply charged fragments were uncertain. In the preliminary work in which protons bombarded carbon, no "hammer" tracks indicative of the presence of Li^8 and B^8 were found. With further searching on these plates, a few such tracks have now been found, and hammer tracks are also seen in fair abundance on plates exposed to the disintegration products of various light elements bombarded with alpha-particles or deuterons. In the present experiment about one splinter in two hundred was Li^8, and about one in 5000 was B^8.

The unmistakable appearance of the tracks which they produce make Li^8 and B^8 extremely useful isotopes to employ in studying the range-energy relations for multiply charged ions. At low velocities a positive particle tends to be neutralized by electrons, thus reducing its rate of energy loss. It is the purpose of this experiment to utilize the tracks of Li^8 and B^8 to evaluate the range correction arising from this effect.

II. IDENTIFICATION AND MEASUREMENT OF TRACKS

Tracks are identified by plotting the range, R, *versus* the radius of curvature, ρ, for each track. The tracks of

[1] Walter H. Barkas and J. Kent Bowker, Phys. Rev. **87**, 207 (1952).

TABLE I. Calculation of normalization factors. T is the kinetic energy in Mev calculated from the momentum of the particle. M is the atomic weight of the ion, R is the measured range in microns, and $(4R/M)_W$ is taken from Wilkins' tables for the value of T/M listed.

	Isotope	T/M	$(4R/M)$	$(4R/M)_W$	Ratio
Plate I	He^4	1.264	21.8 ± 0.3	21.27	1.025
	He^4	1.334	23.6 ± 0.3	22.94	1.031
	He^3	2.221	49.6 ± 0.3	48.30	1.027
	Adopted factor 1.027 ± 0.002				
Plate II	He^4	2.236	50.0 ± 0.3	48.80	1.025
	He^3	3.936	121.9 ± 0.6	118.50	1.029
	Adopted factor 1.027 ± 0.002				
Plate III	He^4	4.700	160.5 ± 0.6	158.00	1.016
	He^3	8.187	401.7 ± 1.2	396.00	1.015
	Adopted factor 1.0155 ± 0.0005				

a particular nuclear species will then fall on a characteristic locus. For the present measurements a beryllium ribbon was bombarded by the internal alpha-particle beam of the cyclotron. The disintegration products were observed in three intervals of radius of curvature simultaneously by placing Ilford C2 200 micron plates in accurately known positions. Particles leaving the target reached the plates after being bent 180° in the magnetic field. The particles entered the emulsion making a small angle of dip with the surface. The position of the track and its entrance angles were sufficient data from which to calculate the momentum. The radial distribution of the magnetic field in the cyclotron is accurately known. Definite identification of hammer tracks as B^8 as well as Li^8 was made by a study of the ρ *vs* R curves. Loci assignable to B^8 bent in the magnetic field while carrying 5, 4, and 3 units of charge were found as well as loci of Li^8 carrying 3, 2, and 1 units of charge. Additional short unresolved hammer tracks were seen which were attributed to B^8 carrying one or two units of charge. We designate by Z' the actual number of charges carried by an ion of atomic number Z while it is deflected in the magnetic field. Its momentum then is $Z'(e/c)H\rho$.

The range is taken to be the length of the path along the trajectory in the emulsion between the extremities of the first and last grain. Badly scattered tracks were not measured. The grain density was so high, particularly for tracks of He, Li, and B, that no appreciable error could have been introduced by failure to see all of the track. No correction for such effects was made.

In the plates the tracks of many nuclear species coming from the target are found, including isotopes of hydrogen and helium. These tracks are important to the experiment as their ranges provide an over-all calibration of the geometry, magnetic field intensity, and emulsion stopping power.

III. THE RANGE EXTENSION

As a basis for the analysis of range data, I have assumed that a particle of charge Ze loses energy at the rate[2]

$$dT/dR=Z^2f(\beta), \tag{1}$$

where T is the kinetic energy in Mev of the ion, R is its residual range in microns, and βc its velocity. Since T/M is a function solely of the velocity, M being the ionic atomic weight, Eq. (1) can be written:

$$g(\beta)d\beta=(Z^2/M)dR, \tag{2}$$

with $g(\beta)$ a function of the velocity alone. When the nucleus is moving with a sufficiently high velocity, one assumes that it is completely stripped so that the charge is not a function of the velocity. In a range interval where this condition holds, one may integrate (2) and obtain:

$$\int_{\beta_0}^{\beta} g(\beta)d\beta=G(\beta)-G(\beta_0)=Z^2R/M-Z^2R_0/M. \tag{3}$$

In this expression $\beta_0 c$ is some lower velocity still sufficiently high that the nucleus remains stripped, and R_0 is the corresponding range. Thus for $\beta>\beta_0$ one can write:

$$Z^2R/M=G(\beta)+B_Z. \tag{4}$$

In so far as the above assumptions are valid, the range of an arbitrary particle is derivable from the universal function $G(\beta)$ and the constant B_Z, which is characteristic of the element, and is a measure of the range extension caused by electron pick-up. If Z^2R/M is plotted against β, a separate locus may be anticipated for each element, but at high velocities the curves will be expected merely to be displaced by a constant amount from each other, corresponding to the differences in the values of the B_Z. In particular, if the range of an alpha-particle or other helium isotope is measured at the same velocity as that of an ion of atomic number Z, then

$$Z^2R/M-4(R/M)_{He}=B_Z-B_2 \tag{5}$$

is obtained by subtraction, and the effect relative to helium can be obtained.

IV. NORMALIZATION OF RANGES

The range increments being evaluated are generally small and are obtained from the differences of two large numbers. Errors are easily introduced under these conditions and a method of analysis should be used which minimizes insofar as possible the systematic errors. In this article a calibration procedure for each plate was employed which eliminates many of the potential sources of difficulty. A careful study of the range-energy relations for Ilford C2 emulsions was made by Wilkins,[3] who based his results primarily on

[2] M. S. Livingston and H. A. Bethe, Revs. Modern Phys. **9**, 245 (1937).

[3] J. J. Wilkins, Atomic Energy Research Establishment (Harwell) Report G/R 664 (1951) (unpublished).

alpha-particle points. In the present experiment, the plates were subjected to normal ($\sim$50 percent) humidity before being placed in the cyclotron vacuum where they remained about fifteen minutes before bombardment. It is well known that a period of many hours is required for the water content of emulsions to stabilize so that it is hardly to be expected that the stopping power of the plates employed in this experiment was identical to that used by Wilkins in his calculations. Such a difference, if it exists, would cause all ranges to deviate systematically by roughly the same percentage from those of Wilkins. Since the range is approximately proportional to a power of the momentum, a small percentage error in the momentum also can be corrected by a percentage adjustment in the range. I have, therefore, normalized Wilkins' data to the existing conditions by adjusting his ranges to match the measured ranges of helium tracks in the plates. To good approximation, this procedure eliminates from the data small errors in geometry, magnetic field intensity, and stopping power. I rely on Wilkins' results chiefly in assuming that the shape of his curve is correct over a limited interval.

In Table I, measured values of Z^2R/M are tabulated for groups of helium tracks from each of the plates used in this study. The values predicted by Wilkins are listed in a parallel column. The measured range has been divided by Wilkins' range, and the ratio listed in the last column. Wilkins' ranges have been multiplied by the ratio appropriate to the plate being studied in obtaining the normalized quantity $(4R/M)_N$ of Table II.

The normalization factors for plates I and II check and are sufficiently near unity that they are fully explainable in view of the possible difference in stopping powers mentioned above, or possibly as an effect traceable to the uncertainty in the absolute value of the magnetic field intensity. On the other hand, there appears to be a significant change in the ratio for the third plate. Since the relative momenta are known with high accuracy, this effect can most simply be interpreted as a one percent inconsistency in Wilkins' ranges for alpha-particles of $\approx$20 Mev when compared with those in the region of 5–10 Mev.

V. RESULTS

In Table II are listed the ranges of groups of particles which were identified and measured in this study. The helium isotopes were employed in finding the normalization factors shown in Table I and are not listed again in Table II. The energies calculated from the momenta are entered in Table II as well. The hydrogen isotopes listed were measured in order to make the data complete, but to evaluate B_1-B_2 accurately lower energy particles should be used. Actually the individual determinations of B_1-B_2 appear more consistent than one might have expected. The standard deviations given are statistical and do not include systematic errors arising from possible incorrect normalization factors. For long ranges the determination of B_1-B_2 depends very sensitively on the normalization. The present measurement of B_1-B_2 is somewhat lower in absolute value than the figure of -1.38 microns adopted by Wilkins.[3] The quantities B_Z are much larger for lithium and boron than for hydrogen, and, therefore, are not so sensitive to normalization errors. Whereas for the other isotopes points were determined using 25 to 100 individual tracks, the B^8 tracks were so rare and the pick-up effect so large that each point was determined using five tracks, all of which, however, lay within two percent of the average momentum for the group.

VI. EVALUATION OF B_Z

If Z^* is the effective charge carried by a slowly moving nucleus, then:

$$B_Z=\int_0^{\beta 0}[\langle Z^2/Z^{*2}\rangle_{\text{Av}}-1]g(\beta)d\beta. \tag{6}$$

TABLE II. Observed ranges and derived range corrections.

Ion	T_{Mev}	R_{micron}	Z^2R/M	$(4R/M)_N$	$Z^2R/M-(4R/M)_N$
H^3	2.890	42.6±0.4	14.1 ±0.13	15.01	−0.91±0.13
H^2	2.508	41.1±0.2	20.4 ±0.1	21.39	−0.99±0.10
H^3	6.217	129.7±0.4	42.99±0.13	43.81	−0.82±0.13
H^2	9.265	312.6±1.5	154.1 ±0.75	154.90	−0.8 ±0.75
H^1	5.006	177.8±0.7	176.4 ±0.7	177.60	−1.20±0.7
		Adopted (B_1-B_2) · · · · · · · ·			−0.9
Li^8	4.325	9.1 ±0.12	10.21±0.15	7.50	2.71±0.15
Li^8	5.793	11.6 ±0.1	13.01±0.13	10.50	2.51±0.13
Li^8	9.138	19.46±0.2	21.64±0.25	18.72	2.92±0.25
Li^8	10.69	22.97±0.2	25.76±0.25	23.51	2.25±0.25
Li^8	22.41	64.9 ±0.22	72.80±0.26	69.72	3.08±0.26
		Adopted (B_3-B_2) · · · · · · · ·			2.7
B^8	18.47	20.99±0.4	65.4±1.3	52.40	13.0±1.3
B^8	26.83	34.78±0.4	108.4±1.3	93.80	14.6±1.3
B^8	60.09	115.6 ±1.0	360.2±3.0	350.00	10.2±3.0
		Adopted (B_5-B_2) · · · · · · · ·			13.5

The cross section for electron capture by a stripped nucleus first becomes comparable to the cross section for electron loss when β falls to $\approx Z/137$, and the capture cross section rises very rapidly for lower velocities. Knipp and Teller[4] collected experimental data obtained by workers with slow alpha-particles which showed that the root-mean-square charge carried by the alpha-particle varies with β to $\beta\approx 2/137$, and approaches the value 2 asymptotically above $\beta=2/137$. For a nucleus of atomic number Z this behavior will be approximated by putting $\langle 1/Z^{*2}\rangle_{\text{Av}}=\lambda^2/137^2\beta^2$ for $\beta<\lambda Z/137$, and $Z^*=Z$ for $\beta>\lambda Z/137$, with λ a number near unity.

$g(\beta)$, which for fast particles is an odd function of β, contains as a divisor the fraction of the emulsion electrons which is effective in stopping. When the velocity

[4] J. Knipp and E. Teller, Phys. Rev. **59**, 659 (1941).

of the moving particle is small, the more tightly bound electrons do not contribute to the energy loss. Those whose velocity exceeds that of the moving particle, in first approximation, interact with it in an adiabatic manner. The number of effective electrons is estimated by Bohr[5] to vary with β over a considerable range when $\beta < Z_2^{\frac{2}{3}}/137$, where Z_2 is the atomic number of the stopping material. $g(\beta)$ should then be an even function of β in this region. Empirically the range data[3] for protons between 0.1 and 1.0 Mev are found to fit the formula $g(\beta) = 4.08 \times 10^5 \beta^2$ very well. In principle, $g(\beta)$ for lower velocities can be obtained directly from the range-energy relation for negative particles such as mesons. Since meson ranges for the velocities of interest here would be the order of 0.1 micron, it is not, therefore, practical to obtain $g(\beta)$ in this way. From other information it nevertheless appears probable that the number of emulsion electrons with velocities less than βc may continue to fall off roughly with β to a point somewhat below $\beta = 1/137$. For example, the data of Wilcox[6] imply that $d\beta/dR$ is quite a good constant for alpha-particles penetrating gold, when β is in the range 6×10^{-3} to 1.5×10^{-2}. These considerations carry one down to residual ranges of less than a micron, so that no large range error can be introduced by assuming $g(\beta) = 4.08 \times 10^5 \beta^2$ to arbitrarily low velocities.

One may now evaluate the integral of Eq. 6,

$$B_Z \approx 4.08 \times 10^5 \int_0^{\lambda Z/137} \left[\left(\frac{\lambda Z}{137\beta} \right)^2 - 1 \right] \beta^2 d\beta = 0.106 \lambda^3 Z^3.$$

By using the present experimental data, one can obtain the coefficient, a, assuming $B_Z = aZ^3$. From $B_1 - B_2$, $a = 0.129$; from $B_3 - B_2$, $a = 0.142$; and from $B_5 - B_2$, $a = 0.115$. The last value presumably contains the smallest error so it should be weighted most. An adopted value of 0.12 appears most reasonable, and when equated to $0.106\lambda^3$, implies $\lambda = 1.04$.

The purpose of this analytical study of B_Z is primarily to seek a *form* in which to express the empirical data. For this the Z^3 law has the great advantages of simplicity and uniform applicability to all light nuclei. As a test of its validity for a higher value of Z, one may compare the value of $B_6 - B_2$ calculated from $B_Z = 0.12Z^3$ with $B_6 - B_2$ determined by Miller.[7] The calculated value is 25.0 microns and his measurement is 23.9 ± 0.75 microns.

Because of the simplifications introduced in its derivation, some further remarks on the evaluation of B_Z are, however, merited: For β above $Z/137$ the effective charge[8] approaches Z only asymptotically, and in this respect the assumed behavior is an idealization. The effect of discrete electron shells is also smoothed out in this approximation. The introduction of the parameter λ serves to make a suitable adjustment for these effects, and in the integrated result the error should be minimized. For calculating ranges with β less than $Z/137$, however, the present assumptions are less valid.

Regarding the meaning of $\langle Z^2/Z^{*2} \rangle_{\text{Av}}$: (a) $\langle Z^{*2} \rangle_{\text{Av}}$ is the mean square charge effective for energy loss. Although $\langle 1/Z^{*2} \rangle_{\text{Av}}$ should behave generally like the reciprocal of the mean square charge carried by the ion, these quantities probably should not be identified with each other for mathematical reasons, and because the ionization produced by a partially shielded nucleus may not be reduced in virtue of the attached electrons to the full extent of the reduction of the square of the net charge. (b) Even for bare nuclei the strict Z^2 dependence of the rate of energy loss implies by Eq. (1) may fail at low velocities because the time of interaction for a given energy transfer to an electron increases with Z. Such an effect, however, is not evident in the empirical[3] energy loss data for protons and alpha-particles in emulsion.

VII. NEGLECTED EFFECTS

Equation (1) should include a small term corresponding to loss of energy to nuclei. This function contains the particle mass explicitly. The rate of energy loss by nuclear collisions becomes relatively more important as the velocity decreases, and it increases with the mass of the moving particle. However, on examining the range data for the hydrogen and helium isotopes in Tables I and II it is difficult to conclude that an observable isotope effect is present. Theoretically one would not expect to detect the effect until the stopping by electrons virtually ceases ($\beta < 1/137$). In any case the electron pick-up effect predominates in the region of importance here. It should be noted that the additional energy loss to nuclei tends to cause a reduction rather than an extension of the range.

In the present analysis certain small experimental effects have also been neglected. The range is somewhat poorly defined since it depends slightly on the sensitivity of the emulsion grains, and also to some extent on their size and closeness of packing.

Finally, the theoretical ideas employed in finding the form of B_Z are idealizations. In particular, it has not been proved that λ, although never deviating much from unity, is independent of Z, and the behavior of $g(\beta)$ for the lowest velocities remains questionable.

Miss Esther Jacobson made many of the range measurements and carried out the numerical reduction of a large part of the data for this study. Dr. Helge Tyren also measured many of the ranges and participated actively in the exposure of the plates. I am indebted to Dr. S. Tamor for a useful discussion of the complex problems of energy loss at low velocities.

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[5] Niels Bohr, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **18**, No. 8 (1948), p. 102.

[6] Howard A. Wilcox, Phys. Rev. **74**, 1743 (1948).

[7] James F. Miller, University of California Radiation Laboratory Report 1902, unpublished.

[8] See, e.g., Theodore Hall, Phys. Rev. **79**, 504 (1950).