PHYSICAL REVIEW VOLUME 89, NUMBER 5 MARCH 1, 1953

Quantum Field Theory in the Light of Distribution Analysis

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(Received August 29, 1952)

It is shown that part of the divergences and ambiguities of the current quantum theory of fields can be overcome by the consistent use of distribution analysis. Representing the singular functions of field theory by distributions, nongauge invariant and nonequivalent terms will be eliminated from the S-matrix in renormalizable as well as in nonrenormalizable theories without any limiting process. Instead of divergent quantities, there appear arbitrary normalization and division constants. Feynman's cutoff as well as the renormalization are automatically contained in a theory which gives a correct meaning to delta-functions. Applications to closed loop processes are discussed in detail. It is not possible to attribute definite values to the parameters of bare particles; for that, some modifications of the theory seem to be necessary.

INTRODUCTION

As is well known, the difficulties appearing in the application of covariant formalisms to meson problems are closely connected with the mathematical defects of quantum field theory. Neither formal limiting methods[1] nor realistic modifications[2] of the theory have given satisfactory results in a self-consistent way. In such a situation an examination of the mathematical foundations of field theoretical formalisms seems to us to be necessary. In our opinion the difficulties are partly due to our inability to handle the singular propagation functions correctly. In fact, it is impossible, in a strict sense, to treat delta-functions, which are not elements of classical analysis, by the methods of this analysis itself. Therefore, we are obliged to look for a generalization of classical analysis which contains Dirac functions as regular elements. In such a correct formalism, all inconsistencies due to mathematical defects of current field theory must vanish. An analysis of the required kind exists in the form of the so-called distribution analysis, introduced recently into mathematics by Schwartz.[3] Schwartz defines distributions as certain linear functionals which can be used to represent singular expressions. Such functionals are suitable for a strict foundation of the mathematical formalism of the field theory.

After a sketch of the conventional treatment of closed loop processes in Part I, we give a survey of distribution analysis in the following second part. In Secs. III and IV the new aspects of the field theory resulting from the introduction of distributions are discussed. Some formulas used extensively in the conventional formalism have to be modified. A so-called Pf symbol (pseudofunction) is introduced which enables one to consider singular functions at their singularities themselves. The Pf-symbol may be regarded as a generalization of the notion of principal value, taking account of the fact that we must define each singularity in a theory containing elements with point structure like Dirac functions. The field theory is shown to be convergent. In some cases the Pf symbol introduces an arbitrary finite "normalization constant" which is connected with a lack of invariance of certain quantities with respect to dilatation transformations of the "support" of distributions.

Feynman's cutoff and the renormalization factors are direct consequences of this constant, which appears in unobservable quantities only. The normalization constant is to be found only in processes of low order simultaneously with so-called division constants in distributions, the support of which is the origin, arising from the fact that delta-algebra contains divisors of zero. A determination of these constants is possible only for matrix elements which are subjected to general rules, for instance, to gauge invariance. By the help of the division constants, unphysical quantities will be eliminated from the S-matrix. Such elimination processes are due in a much more direct way to the normalization constants contained in one expression together with nongauge invariant or nonequivalent terms in the S-matrix. The lack of dilatation invariance named above is characteristic for particles with nonvanishing mass. According to the existence of the normalization constant there is no longer any distinction in principle between renormalizable and nonrenormalizable theories; in the latter the normalization constant yields an indirect renormalization effect.

The regulator of Pauli and Villars[1] is shown to be equivalent to the introduction of distributions with the origin as support, but such limiting processes are thoroughly superfluous. The discrepancy between the classical quadratic divergence of the photon self-energy

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[2] D. Feldman, Phys. Rev. **76**, 1369 (1949); R. Jost and J. Rayski, Helv. Phys. Acta **22**, 458 (1950); W. Heisenberg, Z. Naturforsch. **5a**, 251 (1950); A. Pais and G. E. Uhlenbeck, Phys. Rev. **79**, 145 (1950).

[3] L. Schwartz, Ann. Univ. Grenoble **21**, 57 (1945); **23**, 7 (1948); *Théorie des distributions I/II* (Hermann et Cie, Paris, 1950); Lecture notes at the Canadian Mathematical Congress, second summer seminar, University of British Columbia, August, 1949 (unpublished).

and the finite result of Wentzel[4] originates from an inadmissible integration by parts which has to be modified. Self-energies and self-charges become finite and are in agreement with the results of Schwinger and Feynman[5] if we identify the normalization constant included with the cut-off factor of these authors. We discuss further some meson processes. For instance, the postulate of gauge invariance for the two-photon decay of neutral π-mesons and the equivalence theorem for the decay of a τ-meson into π-mesons will be fulfilled correctly, giving transition elements in a unique way also for nonrenormalizable models. The discrepancies connected with the transition from the S_F-functions to $\bar{S}$ and $S^{(1)}$ as well as the problem of normal dependent terms can also be clarified. Various problems, for instance, radiative corrections, may be treated by means of the theory of distributions in a self-consistent way. It should be pointed out that some results are affected by arbitrary functions whose physical meaning is uncertain. Finally, some remarks are made on the nature of normalization constants and the structure of ni-variant propagation functions.

I. FORMALISM OF CLOSED LOOP PROCESSES

We start with an outline of the formalism of interaction of bosons through virtual fermions via closed loop graphs, adopting Schwinger's[5] notation with $\hbar=c=1$. The interaction energy of a system of $n+1$ bosons interacting via fermions is given in the interaction representation by

$$H=\sum_{i=0}^{n} g_i U_i \bar{\psi}\Gamma_i \tau_i \psi,$$

where U_i represents the potential of the ith boson field obeying the Proca equation (mass μ_i) and ψ means the fermion spinor (mass m). Γ_i and τ_i mean the spin and isotopic spin operators, respectively, Γ_i corresponding to scalar coupling, pseudoscalar coupling, etc.; the g_i are coupling constants. Neglecting radiative corrections, the S-matrix element describing the decay of a boson U_0 into n bosons via closed fermion loop reads

$$\mathfrak{M}_n=\int_{-\infty}^{+\infty} d^4x_0 U_0(x_0) M_n(x_0),$$

with

$$M_n(x_0)=G\prod_{i=1}^{n}\left(\int_{-\infty}^{+\infty} d^4x_i U_i(x_i)\right) \mathrm{Sp}\sum_{k=0}^{n} \bar{S}(0,1)$$

$$\times\Gamma_1\bar{S}(1,2)\cdots\Gamma_k S^{(1)}(k,k+1)\cdots\Gamma_n\bar{S}(n,0)\Gamma_0, \quad (1.1)$$

where $c=0$ or 1 in

$$G=c\prod_{i=0}^{n} g_i$$

reveals the selection rules and $S(i,k)=S(x_i-x_k)$. If there are photon fields among the U_i, the matrix element M_n is subjected to the postulate of gauge invariance. Further, the vector coupling of scalar mesons vanishes in some cases by the divergence theorem, and finally we often have equivalence between pseudoscalar and pseudovector couplings of pseudoscalar mesons. These theorems, assured in coordinate space, are generally destroyed in the results of the actual calculation of matrix elements, so we cannot decide which terms are physically significant. In order to evaluate (1.1) the Fourier representations of $\bar{S}$ and $S^{(1)}$ are substituted, using the relation[6]

$$\sum_{r=0}^{n}(k_r^2+m^2)\prod_{i\neq r}(k_i^2+m^2)^{-1}$$

$$=\prod_{r=1}^{n}\left(\int_{-1}^{+1} da_r\right)\alpha_n\delta^{(n)}\left(m^2+\sum_{i=0}^{n}\beta_i k_i^2\right), \quad (1.2)$$

with

$$2^{i+1}\beta_i=(1+a_{i+1})\prod_{k=0}^{i}(1-a_k), \quad \alpha_n=(-2)^n\prod_{i=0}^{n}\beta_i(1-a_i^2)^{-1},$$

$$a_0=0, \quad a_{n+1}=1, \quad \sum_{i=0}^{n}\beta_i=1, \quad \delta^{(n)}(x)=d^n\delta(x)/dx^n.$$

Then the transformation,

$$k_i\to k+\lambda_i(p_j), \quad \lambda_i(p_j)=\sum_{t\neq i} p_t\sum_{s=t}^{i-1}\beta_s,$$

is carried out, neglecting boundary values. Using the spherical symmetry in the momentum variable k we set $k_\mu k_\nu\to\frac{1}{4}\delta_{\mu\nu}k^2=D_1k^2, \cdots$. Finally we derive from

$$\int_{-\infty}^{+\infty} d^4k\,\exp(izk^2)=i\pi^2\epsilon(z)z^{-2},$$

with $\epsilon(z)=z/|z|$, the formula

$$\int_{-\infty}^{+\infty} d^4k(k^2)^n\exp(izk^2)=i^{1+n}\pi^2(n+1)!\epsilon(z)z^{-(n+2)}, \quad (1.3)$$

and find

$$M_n=\frac{G}{4(2\pi)^{4n+2}}\prod_{k=1}^{n}\left[\int_{-\infty}^{+\infty} d^4p_k\bar{U}_k(p_k)\int_{-1}^{+1} da_k\right]$$

$$\times\left(\sum_{j=0}^{n_0}A_j\right)\alpha_n i^{n+j+1}(j+1)!\int_{-\infty}^{+\infty} dz\epsilon(z)z^{n-j-2}$$

$$\times\exp\left[iz\left(m^2+\sum_{l=0}^{n}\beta_l\lambda_l^2\right)\right]\exp(-ip_0x), \quad (1.4)$$

[4] G. Wentzel, Phys. Rev. 74, 1070 (1948).

[5] J. Schwinger, Phys. Rev. 74, 1439 (1948); 75, 651, 1912 (1949); R. P. Feynman, Phys. Rev. 76, 749, 769 (1949); F. J. Dyson, Phys. Rev. 75, 486, 1736 (1949).

[6] H. Fukuda *et al.*, Prog. Theoret. Phys. 5, 283, 352 (1950).

where A_i is defined by

$$\mathrm{Sp}\left(\prod_{j=0}^{n}(i\gamma(k+\lambda_j)-m)\Gamma_{j+1}\right)=\sum_{i=0}^{n_0}A_i\cdot(k^2)^i;$$

$n_0=n/2$, $(n+1)/2$ if n even, odd. $\bar{U}_k(p_k)$ is the Fourier transform of $U_k(x_k)$. M_n is seen to be convergent if $n>3$. By evaluating the spur in A_i for respective sets of mesons and expanding the integrand in powers of μ_0/m, we obtain the transition probabilities of various processes. However, as is well known, the results are quite ambiguous.

II. PRINCIPLES OF DISTRIBUTION ANALYSIS

In order to generalize classical analysis in the sense of Schwartz[3] let us consider the space D of all continuous complex functions $\varphi(x)$, defined on a linear vector space which is represented by the real points x with coordinates $x_1, \cdots, x_n$. For simplicity we take one coordinate only, $x=x_1$. The functions $\varphi(x)$ shall be differentiable to any order and vanish with their derivatives $d^n\varphi/dx^n=\varphi^{(n)}(x)$ identically beyond a compact, for instance, finite, manifold in the space of the x: $\varphi^{(n)}(\pm\infty)\equiv 0$. We define the support of $\varphi(x)$ to be the closed manifold of those points x for which $\varphi(x)\neq 0$. On the space D of the functions φ we consider functionals $T(\varphi)$: To each φ of D a complex number $T(\varphi)$ is attached, and, if φ runs over D, the attached system of complex numbers represents the functional $T(\varphi)$. These functionals shall be linear [i.e., $T(c_1\varphi_1+c_2\varphi_2)=c_1T(\varphi_1)+c_2T(\varphi_2)$] and continuous [i.e., if $\varphi_i\to\varphi_0$ then $T(\varphi_i)\to T(\varphi_0)$]. Such functionals are called distributions. Now we attribute to each function $f(x)$, which is summable in the sense of Lebesgue, a special distribution $f(\varphi)$ defined by

$$f(x)[\varphi]\equiv f(\varphi)=\int_{-\infty}^{+\infty}dx f(x)\varphi(x). \qquad (2.1)$$

This Lebesgue integral does not change its value if $f(x)$ is changed, for instance, in a finite number of points x. Then we identify $f(x)$ with $f(\varphi)$: instead of calculating with the function $f(x)$ we operate with the associated functional $f(\varphi)$. Hence ordinary functions appear as special distributions. The support of the distribution $T(\varphi)$ is defined as the smallest manifold of those points x beyond which T vanishes. Further, we *define* a special distribution $\delta(\varphi)$ by

$$\delta(\varphi)=\varphi(0), \quad \delta_x(\varphi)=\varphi(x). \qquad (2.2)$$

This is not equal to $\int dx\delta(x)\varphi(x)$ with the usual delta function definition, since the Lebesgue integral allows a modification of the integrand $\delta(0)=0$, which gives the integral the value zero; $\delta(\varphi)$ is a distribution which is not attached to a function. In the space of distributions we define the "distribution-derivative" $D[T(\varphi)]$ by

$$D[T(\varphi)]=-T(\varphi'). \qquad (2.3)$$

It is clear that every distribution is differentiable to any order:

$$D^n[T(\varphi)]=(-1)^nT(\varphi^{(n)}). \qquad (2.4)$$

Instead of D^nT we write often $T^{(n)}$. $T(\varphi^{(n)})$ is a linear and continuous form in $\varphi^{(n)}$, and therefore a distribution in φ. Equation (2.3) will be established in Appendix I. In particular, we have from (2.4)

$$\delta^{(n)}(\varphi)=(-1)^n\delta(\varphi^{(n)})=(-1)^n\varphi^{(n)}(0). \qquad (2.5)$$

Let us consider a distribution $f(\varphi)$ associated with a summable function $f(x)$ whose ordinary function-derivative $df(x)/dx\equiv[f'(x)]$ is summable. The distribution-derivative $Df(\varphi)$ of $f(\varphi)$ reads according to (2.3) and (2.1):

$$f^{(1)}(\varphi)\equiv Df(\varphi)=-f(\varphi')=-\int_{-\infty}^{+\infty}f(x)\varphi'(x)dx.$$

By partial integration we obtain, on account of $\varphi(\pm\infty)\equiv 0$,

$$Df(\varphi)=\int_{-\infty}^{+\infty}[f'(x)]\varphi(x)dx=[f'(x)](\varphi), \qquad (2.6)$$

i.e.,

$$f^{(1)}(\varphi)\equiv Df(\varphi)=[f'(x)](\varphi).$$

This implies that, if $f(x)$ and $[f'(x)]$ are summable, the distribution-derivative $Df(\varphi)\equiv f^{(1)}(\varphi)$ of a distribution $f(\varphi)$ associated with $f(x)$ coincides with the distribution which is attributed to the function-derivative $[f'(x)]$ of $f(x)$, i.e., with $[f'(x)](\varphi)$. If $f(x)$ is summable, but $g(x)=[f^{(n)}(x)](n\geqslant 1)$ is not, the distribution-derivatives $D^nf(\varphi)$ of $f(\varphi)$ do exist according to (2.4). But $D^nf(\varphi)$ is not equal to $g(\varphi)=[f^{(n)}(x)](\varphi)$, since a functional

$$\int_{-\infty}^{+\infty}dxg(x)\varphi(x)$$

associated with a nonsummable function $g(x)$ is not a distribution.[7] For instance, we find for the distribution-derivative of the distribution $\epsilon(\varphi)$ which is associated with the (summable) function $\epsilon(x)$ [$\epsilon(x)=1$ if $x>0$, $=-1$ if $x<0$, not defined for $x=0$]:

$$\epsilon^{(1)}(\varphi)\equiv D\epsilon(\varphi)=-\epsilon(\varphi')=-\int_{-\infty}^{+\infty}dx\epsilon(x)\varphi'(x)$$

$$=[\varphi(x)]_{-\infty}^{0}-[\varphi(x)]_{0}^{\infty}=2\varphi(0)=2\delta(\varphi). \qquad (2.7)$$

Hence we write, for short,

$$\epsilon^{(1)}(x)=2\delta(x), \qquad (2.8)$$

the argument x noting only the fact that ϵ and δ are distributions "in x". On the other hand, the function-derivative $[\epsilon'(x)]$ is equal to zero if $x\neq 0$, but it does

[7] This follows from the theory of Lebesgue-integrals: If $\int|f(x)|dx$ is convergent, then $\int f(x)dx$ is convergent and conversely. Hence, since $\int|g(x)|dx$ is divergent, $\int g(x)dx$ and, therefore, $\int g(x)\varphi(x)dx$ do not exist.

not exist for $x=0$; $[\epsilon'(x)]$ is not a summable function. So we have, as stated above, $\epsilon^{(1)}(\varphi)\neq[\epsilon'(x)](\varphi)$, the right side of this inequality not being a distribution. We demonstrated that, if $f(x)$ is summable, the distribution $f(\varphi)$ defined by (2.1) exists, and also the distribution-derivatives $D^nf(\varphi)$ exist to any order [according to (2.4)], even if the function-derivatives $[f^{(n)}(x)]$ are nonsummable, i.e., even if the distributions $[f^{(n)}(x)](\varphi)$ do not exist. This case will be investigated in detail.

Let us define $f_\pm(x)$ by $f_\pm(x)=f(x)$ if $x>0$, $=0$ if $x<0$, $f_\pm(x)$ being not explained for $x=0$. Then we consider the summable function $f_+(x)=(1/\alpha x^\alpha)_+$ (i.e., $=0$ if $x<0$, $=1/\alpha x^\alpha$ if $x>0$, not defined for $x=0$) α being a number with $0<\alpha<1$. The distribution $f_+(\varphi)$ exists, according to (2.1), together with its distribution-derivatives $D^nf_+(\varphi)$, which are defined by $(-1)^nf_+(\varphi^{(n)})$ [Eq. (2.4)]. At first we treat $Df_+(\varphi)=-f_+(\varphi')$. This is a distribution associated with the function f_+ by means of φ', and, therefore, defined according to (2.1) by

$$f_+(\varphi')=\int_{-\infty}^{+\infty}dxf_+(x)\varphi'(x)=\int_0^{+\infty}dxf(x)\varphi'(x).$$

Since f_+ is not defined for $x=0$, the integral on the right hand of this equation has to be determined in the usual manner by

$$\lim_{\epsilon\to0}\int_\epsilon^\infty dxf(x)\varphi'(x);$$

that is, $-f_+(\varphi')$ is defined by

$$-f_+(\varphi')=\lim_{\epsilon\to0}\int_\epsilon^\infty dx(1/\alpha x^\alpha)\varphi'(x). \tag{2.9}$$

As is well known from the theory of Lebesgue-integrals, partial integration is allowed, which yields

$$-f_+(\varphi')=\lim_{\epsilon\to0}\left\{\int_\epsilon^\infty dx[(1/\alpha x^\alpha)']\varphi(x)+(1/\alpha\epsilon^\alpha)\varphi(\epsilon)\right\},$$

$[(1/\alpha x^\alpha)']$ being the usual function-derivative $-1/x^{\alpha+1}$. Since $\varphi(\epsilon)=\varphi(0)+O(\epsilon)$, we may write

$$D\{f_+(\varphi)\}=D(1/\alpha x^\alpha)_+(\varphi)=-f_+(\varphi')$$

$$=-\lim_{\epsilon\to0}\left\{\int_\epsilon^\infty dx(1/x^{\alpha+1})\varphi(x)-(1/\alpha\epsilon^\alpha)\varphi(0)\right\}. \tag{2.10}$$

This is by definition the explicit form of the distribution-derivative of $f_+(\varphi)=(1/\alpha x^\alpha)(\varphi)$, expressed in terms of φ. *For abbreviation* we write

$$\lim_{\epsilon\to0}\left\{\int_\epsilon^\infty dxx^{-\alpha-1}\varphi(x)-(1/\alpha)\epsilon^{-\alpha}\varphi(0)\right\}$$

$$\equiv\mathrm{Pf}\int_0^\infty dxx^{-\alpha-1}\varphi(x)=\mathrm{Pf}\int_{-\infty}^\infty dx(x^{-\alpha-1})_+\varphi(x). \tag{2.11}$$

It is convenient to write symbolically

$$\mathrm{Pf}\int_{-\infty}^{+\infty}dx(x^{-\alpha-1})_+\varphi(x)\equiv\mathrm{Pf}\{(x^{-\alpha-1})_+(\varphi)\}$$
$$\equiv\{\mathrm{Pf}(x^{-\alpha-1})_+\}(\varphi). \tag{2.12}$$

Hence the distribution-derivative Df_+ of $f_+=(1/\alpha x^\alpha)$ is given by

$$(1/\alpha x^\alpha)_+{}^{(1)}(\varphi)\equiv D\{(1/\alpha x^\alpha)_+(\varphi)\}$$
$$=-\{\mathrm{Pf}(x^{-\alpha-1})_+\}(\varphi). \tag{2.13}$$

The definition (2.11, 12) is a very suggestive one since with the use of the Pf symbol the Eq. (2.13) is very similar to Eq. (2.6); the latter holds if f_+ and $[f_+']$ both are summable. It is seen that

$$f_+{}^{(1)}(\varphi)=(1/\alpha x^\alpha)_+{}^{(1)}(\varphi)\equiv D\{(1/\alpha x^\alpha)_+(\varphi)\}$$

is not equal to $[f_+'](\varphi)=-(x^{-\alpha-1})_+(\varphi)$ as one would expect by adopting (2.6) without any care, since $(x^{-\alpha-1})_+(\varphi)$ does not exist as a distribution.[7] Only distributions are allowed to be considered if we want to operate with singular quantities correctly. Hence the distribution associated with $(x^{-\alpha-1})_+$ is given by $\mathrm{Pf}(x^{-\alpha-1})_+(\varphi)$ but not by $(x^{-\alpha-1})_+(\varphi)$. It should be stated once more that the "Pf integral" is only an abbreviation for a well-defined quantity. An interesting remark about the notation Pf will be made in Appendix II.

Now we compute the distribution-derivatives $D^nf_+(\varphi)$, which exist according to (2.4) since $f_+(\varphi)$ does so. (Note that $[f_+{}^{(n)}](\varphi)$ does not exist!) For the second derivatives we obtain by legitimate integrations by parts,

$$D^2\{f_+(\varphi)\}=f_+(\varphi'')=\lim_{\epsilon\to0}\int_\epsilon^\infty dx(1/\alpha x^\alpha)\varphi''(x)$$

$$=\lim_{\epsilon\to0}\left\{(1+\alpha)\int_\epsilon^\infty dxx^{-\alpha-2}\varphi(x)\right.$$
$$\left.-\epsilon^{-\alpha-1}\varphi(\epsilon)-(1/\alpha)\epsilon^{-\alpha}\varphi'(\epsilon)\right\},$$

on account of $\varphi^{(n)}(\pm\infty)\equiv0$. With $\varphi(\epsilon)=\varphi(0)+\epsilon\varphi'(0)+O(\epsilon)$, $\varphi'(\epsilon)=\varphi'(0)+O(\epsilon)$, we find easily

$$(1+\alpha)^{-1}D^2\{f_+(\varphi)\}=\lim_{\epsilon\to0}\left\{\int_\epsilon^\infty dxx^{-\alpha-2}\varphi(x)\right.$$
$$\left.-\frac{\varphi(0)}{(1+\alpha)\epsilon^{\alpha+1}}-\frac{1}{\alpha\epsilon^\alpha}\varphi'(0)\right\}. \tag{2.14}$$

The right side of this equation will be abbreviated as

$$\mathrm{Pf}\int_{-\infty}^{+\infty}dx(x^{-\alpha-2})_+\varphi(x)\equiv\{\mathrm{Pf}(x^{-\alpha-2})_+\}(\varphi)$$
$$\equiv\mathrm{Pf}\{(x^{-\alpha-2})_+(\varphi)\}. \tag{2.15}$$

So we have

$$D^2\{(1/\alpha x^\alpha)_+(\varphi)\}=(1+\alpha)\ \mathrm{Pf}\{(1/x^{\alpha+2})_+\}(\varphi). \quad (2.16)$$

Taking into account (2.13), the left hand of (2.16) is equal to $-D\{\mathrm{Pf}(x^{-\alpha-1})_+(\varphi)\}$. Hence

$$D\{\mathrm{Pf}(x^{-\alpha-1})_+(\varphi)\}=-(1+\alpha)\ \mathrm{Pf}(x^{-\alpha-2})_+(\varphi). \quad (2.17)$$

The distribution $\mathrm{Pf}(x^{-\alpha-2})_+(\varphi)$ appears to be attributed to the nonsummable function $(x^{-\alpha-2})_+$. By continuing in this manner for the computation of $f_+^{(n)}(\varphi)$, i.e., by successively integrating by parts the distribution

$$(-1)^n \lim_{\epsilon\to 0}\int_\epsilon^\infty dx(1/\alpha x^\alpha)\varphi^{(n)}(x),$$

and using

$$\varphi(\epsilon)=\sum_{r=0}^{\infty}(\epsilon^r/r!)\varphi^{(r)}(0),$$

we come in a direct manner to the definition of the distribution $\mathrm{Pf}(x^{-m})_+(\varphi)$ associated with $(x^{-m})_+$, *viz.*,

$$\mathrm{Pf}(x^{-m})_+(\varphi)=\lim_{\epsilon\to 0}\left\{\int_\epsilon^\infty dx x^{-m}\varphi(x)+\sum_{\mu=0}^{[m-1]}\frac{\varphi^{(\mu)}(0)\epsilon^{\mu+1-m}}{\mu!(\mu+1-m)}\right\}, \quad (2.18)$$

where m is not an integer $\geqslant 1$ and $[m-1]$ is the highest integer which is $\leqslant m-1$. If m is an arbitrary number with $-\infty<m<1$, the sum in (2.18) must be omitted; i.e., the Pf symbol is superfluous, since in this case $(x^{-m})_+(\varphi)$ is a distribution associated with a summable function according to (2.1). Distributions of the form (2.18) will be called "pseudofunctions." We can see by generalizing Eq. (2.17) that, if m is an arbitrary number, but not an integer $\geqslant 1$, the following relation holds:

$$D\ \mathrm{Pf}(x^{-m})_\pm(\varphi)=-m\ \mathrm{Pf}(x^{-m-1})_\pm(\varphi), \quad (2.19)$$

and, since $\mathrm{Pf}(x^{-m})(\varphi)=\mathrm{Pf}(x^{-m})_+(\varphi)+\mathrm{Pf}(x^{-m})_-(\varphi)$,

$$D\ \mathrm{Pf}x^{-m}(\varphi)=-m\ \mathrm{Pf}x^{-(m+1)}(\varphi), \quad (2.20)$$

i.e., the distribution-derivative of the pseudofunction $\mathrm{Pf}x^{-m}(\varphi)$ (m not being an integer $\geqslant 1$) is obtained by the ordinary differentiation rule.

Now we study the case $(1/x^m)_+$ when m is an integer $\geqslant 1$, say $m=n$. For that purpose we consider the summable function $f_+(x)=(\log|x|)_+$, whose associated distribution is given according to (2.1) by

$$f_+(\varphi)=\lim_{\epsilon\to 0}\int_\epsilon^\infty dx \log|x|\cdot\varphi(x). \quad (2.21)$$

As is well known, the definition by a limiting process of the integral involving $\log|x|$ is necessary since $\log|x|$ is undetermined for $x=0$. Since the distribution-derivative $D\{f_+(\varphi)\}$ defined by $-f_+(\varphi')$ exists, we have

$$D\{f_+(\varphi)\}=-f_+(\varphi')=-\lim_{\epsilon\to 0}\int_\epsilon^\infty dx\log|x|\cdot\varphi'(x). \quad (2.22)$$

By performing a legitimate partial integration, taking account of $\varphi(\epsilon)=\varphi(0)+O(\epsilon)$, we find

$$D\{f_+(\varphi)\}=\lim_{\epsilon\to 0}\left\{\int_\epsilon^\infty dx x^{-1}\varphi(x)+\log\epsilon\cdot\varphi(0)\right\}. \quad (2.23)$$

If we define for abbreviation

$$\mathrm{Pf}\{(x^{-1})_+(\varphi)\}=\lim_{\epsilon\to 0}\left\{\int_\epsilon^\infty dx x^{-1}\varphi(x)+\log\epsilon\cdot\varphi(0)\right\}, \quad (2.24)$$

we have

$$D\{(\log|x|)_+(\varphi)\}=\mathrm{Pf}\{(x^{-1})_+(\varphi)\}. \quad (2.25)$$

The definition (2.24) is contained in (2.18) if we prescribe that, if m is an integer $\geqslant 1$, say $m=n$, the term $\mu=m-1$ in (2.18), i.e., $\epsilon^0/0$, is to be replaced by $\log\epsilon$. Hence, for n an integer, we define $\mathrm{Pf}(x^{-n})_+(\varphi)$ by

$$\mathrm{Pf}(x^{-n})_+(\varphi)=\mathrm{Pf}\int_{-\infty}^{+\infty}dx(x^{-n})_+\varphi(x)$$
$$=\lim_{\epsilon\to 0}\left\{\int_\epsilon^\infty dx x^{-n}\varphi(x)+\sum_{\mu=0}^{n-2}\frac{\varphi^{(\mu)}(0)}{\mu!}\times\frac{\epsilon^{\mu+1-n}}{\mu+1-n}+\frac{\varphi^{(n-1)}(0)}{(n-1)!}\log\epsilon\right\}. \quad (2.26)$$

In the case $n=1$, the first sum on the right hand has to be omitted according to (2.24). $\mathrm{Pf}(x^{-n})_+(\varphi)$ is the distribution associated with $(x^{-n})_+$. To get the distribution associated with the function $(x^{-n})_-$ we define the distribution $\tilde{T}$ obtained from T by reflection in the origin as $\tilde{T}(\varphi(x))=T(\varphi(-x))$. Then we easily find (for instance, by considering the integral representations) that $\mathrm{Pf}\{(x^{-n})_-(\varphi)\}=(-1)^n\ \mathrm{Pf}\{(x^{-n})_+(\varphi(-x))\}$. Hence $\mathrm{Pf}(x^{-n})_-(\varphi)$ follows from the definition of $\mathrm{Pf}(x^{-n})_+(\varphi)$ by substituting of $\varphi(x)\to(-1)^n\varphi(-x)$ and $\varphi^{(\mu)}(0)\to(-1)^{n+\mu}\varphi^{(\mu)}(0)$ on the right side of (2.26):

$$\mathrm{Pf}(x^{-n})_-(\varphi)=\mathrm{Pf}\int_{-\infty}^{+\infty}dx(x^{-n})_-\varphi(x)$$
$$=\lim_{\epsilon\to 0}\left\{\int_{-\infty}^{-\epsilon}dx x^{-n}\varphi(x)+\sum_{\mu=0}^{n-2}(-1)^{n+\mu}\times\frac{\varphi^{(\mu)}(0)\epsilon^{\mu+1-n}}{\mu!(\mu+1-n)}-\frac{\varphi^{(n-1)}(0)}{(n-1)!}\log\epsilon\right\}, \quad (2.27)$$

n being an integer $\geqslant 1$. Since $\mathrm{Pf}(x^{-n})(\varphi)=\mathrm{Pf}(x^{-n})_+(\varphi)+\mathrm{Pf}(x^{-n})_-(\varphi)$, we obtain from (2.26, 27)

$$\mathrm{Pf}(x^{-1})(\varphi)=\mathrm{p.v.}\int_{-\infty}^{+\infty} dx x^{-1}\varphi(x)=\mathrm{p.v.}(x^{-1})(\varphi),$$

p.v. being Cauchy's principal value which appears as a direct consequence of the Pf symbol. As we will see in Sec. III, the rule given by (2.19) breaks down if m is an integer $\geqslant 1$, $\mathrm{Pf}(x^{-m})_+(\varphi)$ being defined according to (2.26).

These results can be generalized to the case of more than one variable. For instance, a quantity $\mathrm{Pf}(r^{-1})(\varphi)$ can be obtained,[8] where $r=(x^2+y^2+z^2)^{\frac{1}{2}}$. This will not be explained here, but a result may be noted which demonstrates the efficacy of the pseudofunction. In order to satisfy the Poisson equation $\Delta\Phi=0$ by the function $\Phi=1/r$ the value $r=0$ has to be excluded—a case which is physically important. But distribution-analytically we have $\Delta\{\mathrm{Pf}(r^{-1})\}(\varphi)=-\delta(\varphi)$, a relation which shows that a singularity will correctly be defined by means of the Pf symbol.

Some further properties of the distributions will be reported. For each distribution whose support is the origin, an expansion of the form $T(\varphi)=\sum c_n\delta^{(n)}(\varphi)$ exists, from which we conclude $c_n=0$ if $T=0$. With distributions defined on subspaces we can form direct products, for instance, $\delta_x\cdot\delta_y$, and construct extensions to the entire space. For example, we have (in two dimensions) the general solution $T(x,y)$ of the equation $y^mT=0$ as

$$T=\sum_{\mu=0}^{m-1} T_\mu(x)\delta^{(\mu)}(y),$$

with arbitrary distributions $T_\mu(x)$. The product of an arbitrary distribution T with an indefinitely differentiable function $g(x)$ obeys the usual rules of multiplication and differentiation, and is defined as $(Tg)(\varphi)=T(g\varphi)$. We have $D\{(Tg)(\varphi)\}=(g'T)(\varphi)+g\{DT(\varphi)\}$. For instance, we get $x\delta(\varphi)=\delta(x\varphi)=(x\varphi)_{x=0}=0$ and $(g\delta')(\varphi)=\delta'(g\varphi)=-\delta((g\varphi)'g)=(g(0)\delta'-g'(0)\delta)(\varphi)$.

The problem of division is very important. Division is defined by the inversion of multiplication, as is usual in mathematics. But the division problem of distribution analysis is distinguished from that of classical analysis by the fact that δ-algebra contains divisors of zero. The following theorem holds: If S is an arbitrary distribution, there is an infinite set of distributions T satisfying the equation

$$x\cdot T=S, \tag{2.28}$$

and two distributions of the set are distinguished from each other by an arbitrary multiple of δ. In other words, the general solution T of (2.28) is given by

$$T=S/x+c\delta, \tag{2.29}$$

c being an arbitrary "division constant." The proof of this theorem is given by Schwartz.[9] We confine ourselves to its interpretation. It is seen that the general solution T of (2.28) is the sum of a special solution of this inhomogeneous equation, i.e., S/x, and of the general solution T_0 of the homogeneous equation $x\cdot T_0=0$, i.e., $T_0=c\delta$; the latter contains an arbitrary constant c. Since the division can be defined only by the inversion of the multiplication, even in classical analysis, it must be concluded from this theorem that arbitrary constants together with distributions whose support is the origin appear automatically whenever negative powers of x are introduced by division. At first sight this statement seems a little strange, but it is absolutely correct. It should be pointed out that the first term of the right hand of (2.29) is determined only by the adoption of the Pf symbol. For instance, with $S=x$, the solution of (2.28) is not $T=1$ but $T=1+c\delta$. The uncertainty in c of this solution originates from the indeterminacy at the point $x=0$ of T in (2.28). In the same way one obtains the general solution T of $x^nT=0$ as

$$T=S/x^n+\sum_{\mu=0}^{n-1} c_\mu\delta^{(\mu)}(x),$$

the c_μ being arbitrary complex division constants. These results may be generalized to more than one variable.

The "convolution" of two functions $f(x)$ and $g(x)$, defined by

$$h(x)=f(x)*g(x)=\int_{-\infty}^{+\infty} dt f(x-t)g(t)$$

yields

$$h(\varphi)=\int_{-\infty}^{+\infty} dx h(x)\varphi(x)=\iint_{-\infty}^{+\infty} d\zeta d\eta f(\zeta)g(\eta)\varphi(\zeta+\eta)$$

$$\equiv f_\zeta\{g_\eta[\varphi(\zeta+\eta)]\}.$$

Hence we define the convolution of two distributions S,T by

$$(S*T)_x(\varphi(x))=S_\zeta\{T_\eta[\varphi(\zeta+\eta)]\}=T_\eta\{S_\zeta[\varphi(\zeta+\eta)]\}.$$

δ appears as the unit operator of the convolution:

$$(\delta*T)_x(\varphi)=T_\zeta\{\delta_\eta[\varphi(\zeta+\eta)]\}=T_\zeta[\varphi(\zeta)]=T(\varphi).$$

It is easily to be seen that $\delta'*T=DT$. So we are led to Dirac's original definition of δ. In the space of the so-called tempered distributions, Fourier transforms will be defined by starting with the Parseval relation,

$$\int dy V(y)v(y)=\int dx U(x)u(-x),$$

or, symbolically written,

$$V_y[v(y)]=U_x[u(-x)],$$

where $V(y)=FU(x)$, $v(y)=Fu(x)$ and F implies the ordinary Fourier transformation ($\bar{F}$ is its reciprocal).

[8] L. Schwartz, reference 3, distributions I, p. 46.

[9] L. Schwartz, reference 3, distributions I, p. 121.

With u and v playing the role of φ, the Parseval relation defines the Fourier transform V of a distribution U according to

$$(FU)(v)=V_y[v(y)]=U_x[u(-x)]=(1/2\pi)U_x[(Fv)]$$
$$=(1/2\pi)U_x[v_y(\exp(-ixy))].$$

For instance, we have

$$(F\delta_x')_y(v_y)=(1/2\pi)\delta_x'[Fv]=(1/2\pi)\delta_x'[v_y(\exp(-ixy))]$$
$$=(-1/2\pi)d[v_y(\exp(-ixy))/dx]_{x=0}=(1/2\pi)iy(v_y),$$
$$\text{i.e., } F\delta_x'=(1/2\pi)iy.$$

This outline of the theory of distributions will be sufficient to cover the applications in the following sections.

III. DISTRIBUTION-ANALYTICAL TREATMENT OF CLOSED LOOP PROCESSES

The formalism sketched in the preceding section will give the basis for the treatment of quantum-theoretical problems. Without any restriction we confine ourselves for simplicity to the discussion of processes of the closed loop type. It should be pointed out that most of the quantities appearing in the following have to be considered as distributions, for instance, the self-charge and the matrix element M, even if the symbol φ is not written explicitly.

We start with the proof of Eq. (1.2), which reads, for $n=1$,

$$\mathrm{Pf}[\delta(x)y^{-1}+\delta(y)x^{-1}]$$
$$=-\tfrac{1}{2}\int_{-1}^{+1}da\delta'(x(1+a)/2+y(1-a)/2). \quad (3.1)$$

This equation does not hold, e.g., for $x=y$ as is easily seen from the evaluation of the right-hand side. However, the case $x=y$ is physically important. In order to prove (3.1), the relation

$$[\delta(x)y^{-1}+\delta(y)x^{-1}]=[\delta(x)-\delta(y)](y-x)^{-1} \quad (3.2)$$

is used in the conventional formalism by representing the right-hand side of this equation by Fourier integrals in a well-known manner which yields the right-hand side of (3.1). The relation (3.2) is no identity, and it is necessary to verify it explicitly. Both sides of (3.2) exist as distributions only by adopting the Pf symbol. Since $(y-x)$ appears in the denominator of the right-hand side of (3.2), the left-hand side of this equation can only be explained as the solution of an equation like (2.28). The division of $[\delta(x)-\delta(y)]$ by $(y-x)$ is defined only by the inversion of the multiplication. Therefore, we must look for the general solution T of the equation

$$[(y-x)T(x,y)](\varphi(x,y))=[\delta(x)-\delta(y)](\varphi(x,y)). \quad (3.3)$$

A special solution is found to be

$$S(x,y)=\mathrm{Pf}[\delta(x)y^{-1}+\delta(y)x^{-1}],$$

since we have

$$[(y-x)S](\varphi)=S[(y-x)\varphi]$$
$$=\mathrm{Pf}\int\int dxdy[\delta(x)y^{-1}+\delta(y)x^{-1}](y-x)\varphi(x,y)$$
$$=\int dy\varphi(0,y)-\int dx(x,0)=[\delta(x)-\delta(y)](\varphi(x,y)).$$

But according to the theorem stated in Sec. II [see Eqs. (2.28, 29)], the general solution T is given by the sum of the special solution S of the inhomogeneous Eq. (3.3) and of the general solution $S_0(x,y)$ of the homogeneous equation $[(y-x)S_0](\varphi)=0$; the latter is $S_0(x,y)=f(x)\delta(x-y)$, $f(x)$ being an arbitrary function. So we have, instead of (3.2),

$$\mathrm{Pf}[\delta(x)y^{-1}+\delta(y)x^{-1}]$$
$$=[\delta(x)-\delta(y)]\,\mathrm{Pf}(y-x)^{-1}+f(x)\delta(y-x). \quad (3.4)$$

The uncertainty in $f(x)$ of the second term on the right-hand side of (3.4) originates from the indeterminacy in the point $x=y$ of the first term on the right-hand side. By multiplying Eq. (3.4) by $(y-x)$ its correctness may be verified directly. Equations of this kind, containing arbitrary constants or functions, are important, for example, for $\delta_\pm$-functions and scattering problems.[10] Hence Eq. (3.1) has to be supplied by $f(x)\delta(x-y)$. The same result is obtained by evaluating the left side of (3.1) by a transformation of the variables of the integrals represented [see Eqs. (42, 43) in the paper of Pauli-Villars[1]] and symmetrization; by the latter process an arbitrary function is induced again. By generalizing this result a term of the form

$$\sum_{i<k}T_i(x_0,\cdots,x_{k-1},x_{k+1},\cdots,x_n)\delta(x_i-x_k), \quad (3.5)$$

with $x_i=k_i^2+m^2$, has to be added to the right-hand side of (1.2), the T_i being arbitrary distributions with respect to the variables x_k $(k\neq i)$ and arbitrary functions with respect to x_i. Carrying out the transformations of Sec. II, we find that the matrix elements obtained contain arbitrary distributions which modify observable effects in a quite uncertain way.

For the further analysis of the matrix elements M_n, the derivatives of $\mathrm{Pf}(x^{-n})_\pm$ are needed. As we have shown in Sec. II, the distributions associated with the nonsummable functions $(x^{-n})_\pm$ are given by $\mathrm{Pf}(x^{-n})_\pm(\varphi)$, Eqs. (2.26, 27). There it was stated that their distribution-derivatives cannot be equal to distributions associated with the respective function-derivatives. According to (2.21) the distribution $f_+(\varphi)=(\log|x|)_+(\varphi)$ exists together with its distribution-derivatives, the first of which is given by (2.22). The nth derivative

[10] See P. A. M. Dirac, *Die Prinzipien der Quantenmechanik* (Teubner, Leipzig, 1930), first edition, Chap. 10.

$D^n f_+(\varphi)$ is defined according to (2.4) by

$$D^n(f_+(\varphi))=(-1)^n(\log|x|)_+(\varphi^{(n)}(x))$$
$$=(-1)^n \lim_{\epsilon\to 0}\int_\epsilon^\infty dx \log|x|\cdot\varphi^{(n)}(x). \quad (3.6)$$

To rewrite the right-hand side of this equation in a form which contains φ explicitly, we perform a legitimate successive integration by parts:

$$D^n(f_+(\varphi))$$
$$=(-1)^{n-1}\lim_{\epsilon\to 0}\left\{\int_\epsilon^\infty dxx^{-1}\varphi^{(n-1)}(x)+\log\epsilon\cdot\varphi^{(n-1)}(\epsilon)\right\}$$
$$=\cdots=(-1)^{n-1}\lim_{\epsilon\to 0}\left\{(n-1)!\int_\epsilon^\infty dxx^{-n}\varphi(x)+\varphi^{(n-1)}(\epsilon)\log\epsilon-\sum_{\mu=1}^{n-1}(\mu-1)!\epsilon^{-\mu}\varphi^{(n-\mu-1)}(\epsilon)\right\},$$

since $\varphi^{(n)}(\pm\infty)\equiv 0$. Taking into account

$$\varphi(\epsilon)=\sum_{r=0}^{N}(\epsilon^r/r!)\varphi^{(r)}(0)+O(\epsilon),$$

and evaluating the resulting double sum, we obtain

$$D^n[f_+(\varphi)]$$
$$=(-1)^{n-1}(n-1)!\lim_{\epsilon\to 0}\left\{\int_\epsilon^\infty dxx^{-n}\varphi(x)+\sum_{\mu=0}^{n-2}(\varphi^{(\mu)}(0)/\mu!)\frac{\epsilon^{\mu+1-n}}{\mu+1-n}+\frac{\varphi^{(n-1)}(0)}{(n-1)!}\log\epsilon\right\}$$
$$-\sum_{\mu=1}^{n-1}(1/\mu)\delta^{(n-1)}(\varphi). \quad (3.7)$$

According to (2.26), the first term of the right side of (3.7) is identical with $(-1)^{n-1}(n-1)!\ \mathrm{Pf}[(x^{-n})_+(\varphi)]$. Thus $(n\geqslant 2)$

$$D^n[f_+(\varphi)]=(-1)^{n-1}(n-1)!\ \mathrm{Pf}[(x^{-n})_+(\varphi)]-\sum_{\mu=1}^{n-1}(1/\mu)\delta^{(n-1)}(\varphi). \quad (3.8)$$

This equation can also be proved by induction (see Appendix III). On the other hand, starting with

$$f_-(\varphi)=(\log|x|)_-(\varphi)=\lim_{\epsilon\to 0}\int_{-\infty}^{-\epsilon} dx\log|x|\ \varphi(x),$$

we find

$$D[f_-(\varphi)]=-f_-(\varphi')=-\lim_{\epsilon\to 0}\int_{-\infty}^{-\epsilon} dx\log|x|\ \varphi'(x)$$
$$=\lim_{\epsilon\to 0}\left\{\int_{-\infty}^{-\epsilon}dxx^{-1}\varphi(x)-\log\epsilon\cdot\varphi(0)\right\}=\mathrm{Pf}(x^{-1})_-(\varphi). \quad (3.9)$$

In the same manner as above, we obtain the formula $(n\geqslant 2)$

$$D^n[f_-(\varphi)]$$
$$=(-1)^{n-1}(n-1)!\lim_{\epsilon\to 0}\left\{\int_{-\infty}^{-\epsilon}dxx^{-n}\varphi(x)+\sum_{\mu=0}^{n-2}[\varphi^{(\mu)}(0)/\mu!]\frac{\epsilon^{\mu+1-n}}{\mu+1-n}+\frac{\varphi^{(n-1)}(0)}{(n-1)!}\log\epsilon\right\}$$
$$+\sum_{\mu=1}^{n-1}(1/\mu)\delta^{(n-1)}(\varphi), \quad (3.10)$$

which will be proved in Appendix III. Hence

$$D^{n+1}[f_\pm(\varphi)]=(-1)^n n!\ \mathrm{Pf}[(x^{-n-1})_\pm(\varphi)]\mp\sum_{\mu=1}^{n}(1/\mu)\delta^{(n)}(\varphi). \quad (3.11)$$

But, from (2.4) and (3.8, 10), we get

$$D^{n+1}[f_\pm(\varphi)]=D\{D^n[f_\pm(\varphi)]\}=(-1)^{n-1}(n-1)!\times D\{\mathrm{Pf}(x^{-n})_\pm(\varphi)\}\mp\sum_{\mu=1}^{n-1}(1/\mu)\delta^{(n)}(\varphi). \quad (3.12)$$

By equating (3.11) and (3.12), the important result

$$D\{\mathrm{Pf}[(x^{-n})_\pm(\varphi)]\}$$
$$=-n\ \mathrm{Pf}[(x^{-n-1})_\pm(\varphi)]\pm[(-1)^n/n!]\delta^{(n)}(\varphi) \quad (3.13)$$

is obtained. This rule holds if n is an integer $\geqslant 0$; if $n=0$, the Pf symbol is superfluous. The appearance of $\delta^{(n)}$ on the right side of (3.13), resulting from the rigorous definition of $(x^{-n})_\pm$ for $x=0$ by the Pf symbol, is very reasonable. Omitting for convenience the symbol φ in the following, we conclude from (3.13) since $\mathrm{Pf}(x^{-n})=\mathrm{Pf}(x^{-n})_++\mathrm{Pf}(x^{-n})_-$ that

$$D\{\mathrm{Pf}[(x^{-n})(\varphi)]\}=-n\ \mathrm{Pf}[(x^{-n-1})(\varphi)]. \quad (3.14)$$

Further, we obtain from (3.13) since

$$\mathrm{Pf}(x^{-n})_+-\mathrm{Pf}(x^{-n})_-=\epsilon(x)\ \mathrm{Pf}(x^{-n}),$$
$$D\{\epsilon(x)\ \mathrm{Pf}(x^{-1})(\varphi)\}$$
$$=-[\mathrm{Pf}(x^{-2})_+-\mathrm{Pf}(x^{-2})_-](\varphi)-2\delta'(\varphi)$$
$$=-\epsilon(x)\ \mathrm{Pf}(x^{-2})(\varphi)-2\delta'(\varphi). \quad (3.15)$$

Generally the following relation holds:

$$D^m[\mathrm{Pf}(x^{-n})_\pm]=(-1)^m\prod_{r=0}^{m-1}(n+r)\ \mathrm{Pf}(x^{-(m+n)})_\pm\pm\frac{(-1)^n}{(n-1)!}\sum_{r=0}^{m-1}(n+r)^{-1}\delta^{(m+n-1)}(x), \quad (3.16)$$

which will be proved in Appendix IV. Hence we get

$$D^m[\epsilon(x)\ \mathrm{Pf}x^{-n}]=(-1)^m\prod_{r=0}^{m-1}(n+r)\epsilon(x)\ \mathrm{Pf}x^{-(m+n)}+\frac{2(-1)^n}{(n-1)!}\sum_{r=0}^{m-1}(n+r)^{-1}\delta^{(m+n-1)}(x). \quad (3.17)$$

Now we are prepared for a criticism of the formula (1.3), which is used extensively in field theory. By investigating the solutions of the Laplace equation.[11] Schwartz has shown that the following formula holds (C being Euler's constant):

$$\bar{F}[\epsilon(z)\ \mathrm{Pf}z^{-1}](\varphi(x))\equiv\mathrm{Pf}\left[\int_{-\infty}^{+\infty}dz\epsilon(z)z^{-1}\exp(izx)\right](\varphi)=-2[\log|x|+C+\log(2\pi)](\varphi), \quad (3.18)$$

or, reciprocally written,

$$F(\log|x|)\equiv(1/2\pi)\int_{-\infty}^{+\infty}dx\log|x|\ \exp(-izx)=-\tfrac{1}{2}\epsilon(z)\cdot\mathrm{Pf}(z^{-1})-[C+\log(2\pi)]\delta(z).$$

However, the integral in (3.18) is not invariant with respect to the transformation $z\rightarrow\lambda z$, i.e., it shows a lack of dilatation invariance. The correct formulas are

$$\left[\mathrm{Pf}\int_{-\infty}^{+\infty}dz\epsilon(z)z^{-1}\exp(izx)\right](\varphi)=-2[\log|x|+C+\log(2\pi\lambda)](\varphi), \quad (3.19)$$

$$F(\log|x|)(\varphi)=[-\tfrac{1}{2}\epsilon(z)\cdot\mathrm{Pf}(z^{-1})-\{C+\log(2\pi\lambda)\}\delta(z)](\varphi), \quad (3.20)$$

where λ is an arbitrary finite constant connected with the dilatation transformation named above. This constant may be called a "normalization constant." In the sense of classical analysis, the integral on the left-hand side of (3.19) is a divergent one, but it is convergent in the sense of distribution analysis as a result of a strict definition of $\epsilon(z)$ Pf(z^{-1}) for $z=0$. As stated by Schwartz, the pseudofunction

$$\mathrm{Pf}\int_0^\alpha dx\varphi(x)x^{-n}$$

can be interpreted as the analytical continuation $F(0)$ of an ordinary integral

$$F(z)=\int_0^\alpha dx\varphi(x)x^{z-n}$$

if n is not an integer, or of $F(z)-1/z$ if n is an integer. $F(0)$ is invariant with respect to variable transformations if $n\neq1$, but if $n=1$ we have $F(0)\rightarrow F(0)-\varphi(0)\log\lambda$ by the dilatation transformation $x\rightarrow\lambda x$. This lack of dilatation invariance originates from the compactness of the support of the φ's, and seems to be characteristic for particles with a finite nonvanishing rest mass [see Eqs. (4.5, 6)]. This problem may be investigated on the basis of the theory of differential forms of distributions; the latter lies beyond the object of this paper.

By differentiating (3.20) with respect to x taking account of $F(DU(x))=izF(U(x))$, one can deduce the relations (n being an integer $\geqslant1$)

$$\mathrm{Pf}x^{-n}(\varphi)=(1/2i^n(n-1)!)\times\int_{-\infty}^{+\infty}dz\epsilon(z)z^{n-1}\exp(izx)(\varphi), \quad (3.21)$$

$$\mathrm{Pf}\int_{-\infty}^{+\infty}dxx^{-n}\exp(-izx)(\varphi)=(\pi z^{n-1}\epsilon(z)/i^n(n-1)!)(\varphi).$$

Further, we find, using (3.17, 20), that

$$F(x(\log|x|-1))=(Fx)*(F(\log|x|-1))=i\delta'(z)*F(\log|x|-1)=id[F(\log|x|-1)]/dz=\tfrac{1}{2}i\epsilon(z)\ \mathrm{Pf}(z^{-2})-i[C+\log(2\pi\lambda')]\delta'(z).$$

Hence (λ' being a normalization constant),

$$\mathrm{Pf}\int_{-\infty}^{+\infty}dz\epsilon(z)z^{-2}\exp(izx)=-2ix[\log|x|-1+C+\log(2\pi\lambda')]. \quad (3.22)$$

Finally, it can be shown by investigating Fresnel's integrals that the formula

$$\mathrm{Pf}\int_{-\infty}^{+\infty}d^4k\exp(izk^2)(\varphi)=i\pi^2\epsilon(z)\ \mathrm{Pf}(z^{-2})(\varphi) \quad (3.23)$$

holds. By differentiating (3.23) with respect to z according to (3.17) [noting that the result of differentiation of the left-hand side of (3.23) reads effectively

$$\mathrm{Pf}\int_{-\infty}^{+\infty}d^4k(ik^2)^n\exp(izk^2)(\varphi)$$

according to (2.20)], we easily obtain the important relation

$$\mathrm{Pf}\int_{-\infty}^{+\infty}d^4k(k^2)^n\exp(izk^2)=\pi^2i^{n+1}(n+1)!\epsilon(z)\ \mathrm{Pf}(z^{-n-2})+2\pi^2i^{1-n}\sum_{\mu=2}^{n+1}(1/\mu)\delta^{(n+1)}(z). \quad (3.24)$$

[11] L. Schwartz, reference 3, distributions II, p. 114.

From this the formula (1.3) is shown to be incorrect, except for $n=1$.

Now we can compute the S-matrix element M_n [Eq. (1.1)] correctly. Neglecting temporarily the arbitrary distributions resulting from (3.5), the performance of the k-integration according to (3.24) yields the term

$$h(z)=2\pi^2 i^{1-n}[\sum_{\mu=2}^{n+1}(1/\mu)]\delta^{(n+1)}(z),$$

which must be added to the integrand of the z-integral of (1.4), say $\epsilon(z)z^{n-j-2}\rightarrow\epsilon(z)z^{n-j-2}+h(z)$. From this it follows that some matrix elements, for instance the transition element of the decay of a scalar τ-meson into two pseudoscalar π-mesons, will be modified [see Eq. (4.10)]. Such additive quantities appear only for elements with $j=n_0$ and $n\leqslant 3$. By the substitution of the Fourier representations of the $\bar{S}$-functions a distribution-analytical division problem is induced, i.e., by $\mathrm{Pf}(k^2+m^2)^{-1}$, which is apparently eliminated by the integral representation of $\mathrm{Pf}(k^2+m^2)^{-1}$ by (1.2). This division problem appears again in the form of negative powers of z in performing the k-integration according to (3.24) by which, as we have demonstrated in the preceding section, a distribution of the form

$$\sum_{j=0}^{N} c_j^{(n)}\delta^{(j)}(z),$$

($N=1, 0, 0$ for $n=1, 2, 3$, respectively, $c_j^{(n)}=0$ if $n\geqslant 4$, $c_j^{(n)}=$arbitrary constants), with the origin as support is produced in the integrand of the z-integral. There is no counterpart of this division problem in classical analysis. This result may also be proved explicitly by verifying carefully each step of the calculation [as with Eqs. (3.1–4)]. After some calculations we find, instead of (1.4), for M_n:

$$\begin{aligned}M_n(x)=&(G/4(2\pi)^{4n+2})\sum_{j=0}^{n_0}\prod_{k=1}^{n}\left(\int_{-\infty}^{+\infty}d^4p_k\bar{U}_k(p_k)\right.\\&\times\left.\int_{-1}^{+1}da_k\right)A_j\alpha_n i^{n+j+1}(j+1)!\exp(-ixp_0)\\&\times\mathrm{Pf}\int_{-\infty}^{+\infty}dz\exp\left[iz\left(m^2+\sum_{k=0}^{n}\beta_k\lambda_k^2\right)\right]\\&\times\{z^{n-j-2}\epsilon(z)+c_0^{(1)}\delta_{n,1}\delta(z)+\delta_{j,n_0}[R(-1)^{n_0}\\&\times[(n_0+2-n)/(n_0-1)!]+c_j^{(n)}]\\&\times\delta^{(n_0-n+1)}(z),\quad(3.25)\end{aligned}$$

from which it follows that

$$\begin{aligned}M_1(x)=&(G/4(2\pi)^{10})\int_{-1}^{+1}da_1\{2A_1[m^2-(1-a_1^2)\Box/4]\\&\times[\log|m^2-(1-a_1^2)\Box/4|+C+\log(2\pi\lambda')\\&-2+c_0^{(1)}/2]-A_0[\log|m^2-(1-a_1^2)\Box/4|\\&+C+\log(2\pi\lambda)+c_1^{(1)}]\}U_1(x),\quad(3.26)\end{aligned}$$

and for $n\geqslant 2$,

$$\begin{aligned}M_n(x)=&((-1)^nG/2(2\pi)^{4n+2})\\&\times\prod_{j=1}^{n}\left(\int_{-\infty}^{+\infty}d^4p_j\bar{U}_j(p_j)\int_{-1}^{+1}da_j\right)\alpha_n\\&\times\exp(-ixp_0)\left\{\sum_{k=0}^{n_0}A_k(k+1)!(n-k-2)!\right.\\&\times\left(m^2+\sum_0^n\beta_i\lambda_i^2\right)^{k+1-n}-A_{n-1}n!\\&\times B\left[\log|m^2+\sum_0^n\beta_i\lambda_i^2|+C+D\right.\\&\left.\left.+\log(2\pi\lambda)-c_{n-1}^{(n)}/2\right]\right\}.\quad(3.27)\end{aligned}$$

Here $B=1$ if $n=2, 3$; $B=0$ for $n>3$; $D=\frac{1}{2}$ if $n=2$, $D=-\frac{5}{6}$ if $n=3$, $D=0$ for $n>3$, and A are given as in (1.4). These matrix elements, however, are seen to be convergent, and contain arbitrary division constants $c_1^{(1)}$, $c_{n-1}^{(n)}$ and normalization constants λ, λ' if $n\leqslant 3$.

IV. TWO- AND THREE-FIELD PROBLEMS

In this section the formalism explained above will be applied to special interaction problems. First we study the self-energy of the photon. The current δj_μ induced in the vacuum is obtained from M_1 and $U_1=A_\nu^{\text{ext}}$, $\Gamma_0=\gamma_\nu$, $\Gamma_1=\gamma_\mu$, as

$$\delta j_\mu(x)=M_1(x)=-4e^2\int_{-\infty}^{+\infty}d^4x_1K_{\mu\nu}(x-x_1)A_\nu^{\text{ext}}(x_1),$$

where $K_{\mu\nu}$ is given by

$$K_{\mu\nu}(x_0-x_1)=\mathrm{Sp}\sum S^{(1)}(0,1)\gamma_\mu\bar{S}(1,0)\gamma_\nu.\quad(4.1)$$

Introducing a new normalization constant λ'' by $\log\lambda'=\log\lambda+\frac{1}{2}\log\lambda''$, and using (3.26), the Fourier transform $K_{\mu\nu}(p)$ of $K_{\mu\nu}(x)$ is found to be

$$\begin{aligned}K_{\mu\nu}(p)=&[\tfrac{1}{16}(2\pi)^2]\int_{-1}^{+1}da\{(1-a^2)(p_\mu p_\nu-\delta_{\mu\nu}p^2)\\&\times[\log|(1-a^2)(p^2/4)+m^2|+C+\log(2\pi\lambda)+c_1]\\&+\delta_{\mu\nu}[(1-a^2)(p^2/4)+m^2](1-\log\lambda''+c_2)\},\quad(4.2)\end{aligned}$$

where c_1 and c_2 are arbitrary division constants. This expression, i.e., δj_μ, is not gauge-invariant. The gauge-invariant result,

$$\begin{aligned}K_{\mu\nu}(p)=&(p^2\delta_{\mu\nu}-p_\mu p_\nu)[\tfrac{1}{8}(2\pi)^2]\int_{-1}^{+1}da((1-a^2)/4)\\&\times\left\{\mathrm{Pf}\int_{-\infty}^{+\infty}dz\epsilon(z)z^{-1}\exp iz[(1-a^2)(p^2/4)+m^2]+c\right\},\end{aligned}\quad(4.3)$$

($c=-c_1/2$), is obtained either by suitable choice of c_2 (e.g., neglecting the lack of dilatation invariance by using Eq. (3.18), i.e., $\lambda=\lambda'=\lambda''=1$, the choice $c_2=-1$ is sufficient) or, letting c_2 be zero, by the choice $\log\lambda''=1$. This conclusion is shown to be unique [provided that arbitrary distributions (3.5) are neglected], for expanding the left hand of $p_\mu K_{\mu\nu}(p)=0$ in powers of m, expressions of the form $\alpha\delta(z)+\beta\delta'(z)$ must vanish [see Eq. (4.7a)] yielding $\alpha=\beta=0$ (see Sec. II). Therefore, the vanishing of the self-energy of interacting photons is guaranteed exactly, but, according to the arbitrary normalization constant λ and the division constant c we cannot conclude that the unobservable (finite) mass of the bare photon is definite. With

$$K_{\mu\nu}(p)=(p_\mu p_\nu-\delta_{\mu\nu}p^2)K(p^2),$$

$$\log|m^2+(1-a^2)(p^2/4)| = \log m^2-\sum_{n=1}^{\infty}[(a^2-1)/4m^2]^n(p^2)^n/n!,$$

we find from (4.3), making use of (3.19), that

$$K(p^2)=(1/48\pi^2)\log(m^2\lambda^*)+(m/4\pi)^2\sum_{n=1}^{\infty}K_n\cdot(p^2)^n, \tag{4.4}$$

where

$$K_n=(1/n!)\int_{-1}^{+1}da[(a^2-1)/4m^2]^{n+1}, \tag{4.4a}$$

$$\log\lambda^*=C+\log(2\pi\lambda)-c/2.$$

The observable quantities are in agreement with those of Schwinger.[5] The renormalization of charge reads

$$\delta e=4e^2K(0)=(\alpha/3\pi)\log(m^2\lambda^*),\ \alpha=e^2/4\pi=1/137, \tag{4.5}$$

which is convergent but indeterminate in the normalization and division constants.

Generally it is seen from (3.25–27) that, according to their common origin, the normalization constants appear simultaneously with the division constants. Since both constants are arbitrary, one of them may be omitted in the results. Thus it can be concluded from the distribution-analytical formulation of field theory that an arbitrary constant corresponds to each quantity which is divergent in the conventional theory [see, for instance, Eq. (4.5)]. Some consequences of this fact will be drawn in what follows.

To analyze the problem of a finite nonvanishing self-energy of the photon we consider for simplicity Eqs. (43) and (44) of the paper of Pauli-Villars.[1] Here the integral

$$\int_{-\infty}^{+\infty}dz\epsilon(z)d[\exp(izx)\ \mathrm{Pf}(z^{-1})]/dz$$

has to be evaluated. This integral is not equal to

$$-2\int_{-\infty}^{+\infty}dz\delta(z)\ \mathrm{Pf}(z^{-1})\exp(izx)$$

obtained by ordinary integration by parts, as usually stated, corresponding to the quadratic divergence of the photon self-energy if—as it is done in the ordinary formalism—the Pf symbol is omitted. Instead of this we find, by differentiating $\epsilon(z)\exp(izx)\ \mathrm{Pf}(z^{-1})$ according to (3.17),

$$\epsilon(z)d[\exp(izx)\ \mathrm{Pf}(z^{-1})]/dz = d[\epsilon(z)\exp(izx)\ \mathrm{Pf}(z^{-1})]/dz+2\delta'(z)\exp(izx).$$

The first term on the right-hand side of this equation vanishes when integrated over z. From the second expression a finite self-energy is obtained. But we have

$$2\delta'(z)\exp(izx)=[-2\delta(z)\ \mathrm{Pf}(z^{-1})+c\delta(z)]\exp(izx)$$

[since from $z\delta(z)=0$, $\delta(z)+z\delta'(z)=0$ it follows that $\delta'(z)=\delta(z)\ \mathrm{Pf}(z^{-1})+c\delta(z)$ according to (2.28, 29)] where the first term in the bracket corresponds to the quadratic divergence if the Pf symbol is omitted, i.e., if no use is made of distribution analysis. This explains the discrepancies between the classical quadratic divergence of the photon self-energy, the finite result of Wentzel,[4] and the vanishing observable mass.

The connection of the normalization constant λ with Feynman's cut-off factor may be exhibited as follows: By going over from $\delta(x_\nu^2)$ to $f(x_\nu^2)$ in Feynman's formalism, a convergence factor

$$C(k^2)=(\mu^2-\lambda_0^2)/(k^2-\lambda_0^2)$$

is introduced into the integrand of integrals over the momentum k of virtual particles, like

$$\int_{-\infty}^{+\infty}d^4k(k^2-\mu^2)^{-1}\rightarrow\int_{-\infty}^{+\infty}d^4k(k^2-\mu^2)^{-1}C(k^2) = -\int_{-\infty}^{+\infty}d^4k\int_{\mu^2}^{\lambda_0^2}dL(k^2-L)^{-2},$$

$\lambda_0\rightarrow\infty$, which yields a term $\log(\lambda_0^2\mu^{-2})$. On the other hand, $(k^2-\mu^2)^{-1}$ is shown to be equivalent to $-\delta'(k^2-\mu^2)$ when integrated on k. But by distribution analysis we obtain

$$-\int_{-\infty}^{+\infty}d^4k\delta'(k^2-\mu^2)\sim\mathrm{Pf}\int_{-\infty}^{+\infty}dz\epsilon(z)z^{-1} \times\exp(iz\mu^2)\sim\log(\lambda\mu^{-2}),$$

with the normalization constant λ, by omitting a division constant. So we can conclude that the normalization constant (and the division constants) appearing by application of distribution analysis may be interpreted as a cut-off factor, although its mathematical meaning is quite different from an actual cutoff. We see again that the division constant is equivalent to the normalization constant.

Distribution-analytically, the efficacy of the regulator of Pauli-Villars appears in a new light. As can be shown by considering (3.25) for various processes, the ambiguous terms to be eliminated from the S-matrix generally have the form $[C_1\delta(z)+C_2\delta'(z)]\exp(izm^2)$ in the integrand of the z-integral of (3.25). By introducing the regulator $R(z)=\int d\kappa\rho(\kappa)\exp(iz\kappa)$ into the field-theoretical formalism we get, therefore,

$$[C_1\delta(z)+C_2\delta'(z)]R(z)=[C_1R(0)-C_2R'(0)]\delta(z)+C_2R(0)\delta'(z),$$

which yields the regularization conditions $R(0)=R'(0)=0$. On the other hand, by the distribution calculus there appears an expression $[c_1\delta(z)+c_2\delta'(z)]\exp(izm^2)$ in (3.25) which is able to eliminate the ambiguous terms by the choice of $c_i=-C_i$. From that we conclude that only such terms have to be subjected to the regulator whose the "m-factors" are induced by $\exp(izm^2)$ coming from $\mathrm{Pf}(k^2+m^2)^{-1}$ in $\bar{S}$, without any consideration of the powers of m contained in the traces. This explains the various regularization alternatives,[12] but such doubtful limiting methods are not necessary in the new formalism which is mathematically correct as well as more powerful, even in nonrenormalizable theories.

The discrepancies contained in the relations between S_F-functions and $\bar{S}$- and $S^{(1)}$-functions are removed in the same way. For instance, by calculating the self-energy of the electron in Dyson's formalism, the real part of the one-electron-zero-photon term of the second-order S-matrix, which is not equal to zero in the ordinary formalism, disappears in distribution calculus by suitable choice of division constants (provided arbitrary distributions are put to be identical zero).

Further, an evaluation of $\Delta^{(1)}(x)$ by a distribution-analytical modification of Schwinger's computation[5] shows that

$$\partial\Delta^{(1)}(x)/\partial x_\nu=-x_\nu[\pi^{-2}\,\mathrm{Pf}\zeta^{-2}+(m/2\pi)^2\,\mathrm{Pf}\zeta^{-1}+\text{reg. in }(\zeta=-x_\nu^2)]$$

vanishes identically on the light cone—as a consequence of the Pf symbol—yielding correctly in this manner the continuity relation $\partial K_{\mu\nu}(x)/\partial x_\mu=0$. A strict definition of $\mathrm{Pf}(k^2+m^2)^{-1}$ will not be given here. Some difficulties appear with regard to the properties of symmetry and reality of $\bar{S}$ and $S^{(1)}$, leading to a lack of invariance with respect to reflection in the origin of the light cone, corresponding to the dipole properties of the latter.

[12] S. Tomonaga *et al.*, Prog. Theoret. Phys. **4**, 477 (1949); S. Ozaki *et al.*, Prog. Theoret. Phys. **4**, 524 (1949); **5**, 25, 165 (1950); P. T. Matthews, reference 1; J. Steinberger, reference 1.

To compute the self-energies of the electron and nucleon the same method can be used as for the self-energy of the photon. The essential part of the self-energy of an electron reads:

$$\int_{-1}^{+1}da[i\gamma p(1-a)/2+2m]\,\mathrm{Pf}\int_{-\infty}^{+\infty}dz[\epsilon(z)z^{-1}+c\delta(z)]\times\exp[izp^2(1+a)^2/4].$$

Joining the division constant c with the normalization constant [see Eq. (3.19)]—which is always possible—to get a new constant λ_0 (for instance, neglecting the normalization constant, i.e., putting $\lambda=1$, and using c alone), we find, according to (3.19),

$$\delta m=(3m\alpha/2\pi)[\log(\lambda_0/m)+\tfrac{5}{6}]. \qquad (4.6)$$

Examining the calculations of Géhéniau-Villars,[13] the values of the anomalous magnetic moments of the electron and nucleon are shown to remain unchanged provided arbitrary distributions, which appear here as in (3.5), are neglected. But it should be pointed out that there is in general no reason for neglecting such arbitrary quantities, especially in the case of the nucleon moment. By the same methods the self-stress of the electron is found to be

$$\langle\bar{T}_{11}\rangle_0=[e^2m/8(2\pi)^2]\int_{-1}^{+1}da(3-a)(4-c)/2.$$

This is equal to zero, as demanded by the theory of relativity, if we choose $c=4$. Therefore, to obtain a vanishing self-stress no admixture of vector fields is necessary.[14]

Some meson processes may now be discussed. The decay of a boson into a fermion and the processes $\pi\rightarrow\mu+\nu$, $\pi\rightarrow e+\nu$ via the nucleon field are described by M_1. The divergence theorem, including gauge-invariance, is shown to be fulfilled formally by the types of coupling $(\Gamma_0,\Gamma_1)=(v,v)$, (t,t), (v,pt) and the equivalence theorem by $M_{ps}=(pv,ps)$, $M_{pv}=(pv,pv)$. Evaluating the traces we find from (3.26), including $c_0^{(1)}$ and $c_1^{(1)}$ into λ, λ' temporarily, with $\log\lambda'=\log\lambda+\frac{1}{2}$, that

$$\begin{aligned}K(p)=-(4\pi)^{-2}\int_{-1}^{+1}da\,\mathrm{Sp}\{&((1-a^2)/4)\\&\times[p_\alpha p_\beta+\tfrac{1}{2}\delta_{\alpha\beta}p^2](\gamma_\alpha\Gamma_1\gamma_\beta\Gamma_0)\\&+\tfrac{1}{2}m^2(\gamma_i\Gamma_1\gamma_i\Gamma_0+2\Gamma_1\Gamma_0)\\&-\tfrac{1}{2}imp_\alpha([\gamma_\alpha,\Gamma_1]_-\Gamma_0)\\&\times[\log|m^2+\{(1-a^2)/4\}p^2|\\&+C+\log(2\pi\lambda)]\},\end{aligned} \qquad (4.7)$$

[13] J. Géhéniau and F. Villars, Helv. Phys. Acta **23**, 178 (1950).
[14] F. Rohrlich, Phys. Rev. **77**, 357 (1950); F. Villars, Phys. Rev. **79**, 122 (1950).

where

$$K(x)=F[K(p)]=\mathrm{Sp}\sum S^{(1)}(0,1)\Gamma_1\bar{S}(1,0)\Gamma_0,$$

$$x=x_0-x_1,\quad M_1(x_0)=G\int_{-\infty}^{+\infty}d^4x_1U_1(x_1)K(x).$$

From this convergent formula, for instance, the equivalent elements M_{ps} and M_{pv} are calculated, and it is shown that the specified relation $\log\lambda'=\log\lambda+\frac{1}{2}$ between λ and λ' is the necessary and sufficient one for all processes which are subjected to the postulates of gauge invariance, equivalence, and to the divergence theorem. When there are no general prescriptions of this kind, it is not possible to get results which are free of arbitrary constants. In addition, even in the case where there are general rules, no assertion is made about the meaning of the arbitrary distributions (3.5) to be added to M_n. We have seen that the normalization constant is contained only in quantities which are not observable [see (4.5, 6)], i.e., in renormalization factors. Therefore we can say that a distribution-analytical treatment of field theory leads directly to a renormalization. On the other hand, the normalization constant was shown to appear at the same place as the cut-off factor of the conventional theory. Therefore, we no longer have in principle any distinction between renormalizable and nonrenormalizable formalisms. Having "normalized" the matrix elements by λ and c so as to satisfy the general rules, those quantities which are not dilatation invariant, i.e., which contain a normalization constant and therefore do not correspond to observable effects, may be neglected in the results; they may be included in observable masses and coupling constants.

The efficacy of the division constants—which are closely connected with the normalization constants—in eliminating unphysical terms may be demonstrated for M_{ps} and M_{pv}. Computing these elements from (3.25), both expressions can be split up into two parts, $M_{ps}{}^a+M_{ps}{}^b$ and $M_{pv}{}^a+M_{pv}{}^b$, respectively, in such a way that only $M_{ps}{}^b$ and $M_{pv}{}^b$ contain arbitrary constants, say c_1, c_2 and c_1', c_2', respectively. The equivalence theorem is then shown to be valid for $M_{ps}{}^a$ and $M_{pv}{}^a$, so that the relation $ipM_{pv}{}^b+2mM_{ps}{}^b=0$ has to be demanded. Effectively this implies that

$$p_\mu[c_1'\delta(z)+(c_2'-1)\delta'(z)]+2m[c_1\delta(z)+c_2\delta'(z)]=0,\tag{4.7a}$$

yielding $c_1=c_2=c_1'=0$, $c_2'=1$ according to Sec. II. This leads to $M_{ps}{}^b=M_{pv}{}^b=0$, giving the equivalent elements $M_{ps}{}^a$ and $M_{pv}{}^a$ in a unique way.

The calculation of the transitions[15] $\pi\rightarrow\mu+\nu$, $\pi\rightarrow e+\nu$ via the nucleon field is easily performed by means of M_1 with $\Gamma_0=\Gamma_1=1$. Both elements are finite—in contrast to those obtained by the ordinary formalism—containing arbitrary normalization and division constants, a suitable choice of which may give well $\tau_{\pi\rightarrow\mu+\nu}\sim10^{-8}$ sec in agreement with experiments. The ratio $\tau_{\pi\rightarrow\mu+\nu}/\tau_{\pi\rightarrow e+\nu}$, however, is found to be in accordance with experiment ($<10^{-2}$) only if one assumes a dependence of the ratio of masses of the decay products on the arbitrary constants (e.g., λ).

In the frame of interaction of three fields described by M_2, we discuss in particular the decay of a τ-meson into two π-mesons and the decay of neutral π-mesons into two photons.[16] General theorems, although formally fulfilled in coordinate space—even by using distribution analysis—are mostly destroyed by calculating the matrix elements in momentum space according to the ordinary formalism. Distribution-analytically from (3.27) with $n=2$ the characteristic form K of

$$M_2(x_0)=G\iint_{-\infty}^{+\infty}d^4xd^4x_2U_1(x_1)U_2(x_2)K(x_0,x_1,x_2)$$

reads

$$K=(1/2^{15}\pi^{10})\iint_{-1}^{+1}da_1da_2(1-a_1)\iint_{-\infty}^{+\infty}d^4p_1d^4p_2$$
$$\times\{2A(m^2+\sum_0^2\beta_i\lambda_i^2)+2B-ic$$
$$-4D[\log|m^2+\sum_0^2\beta_i\lambda_i^2|-\tfrac{1}{2}+C+\log(2\pi\lambda)]\}$$
$$\times\exp[i\sum_{j=1}^2p_j(x_0-x_j)],\tag{4.8}$$

where A, B, and D are products of spur terms with momentum variables whose explicit form will not be given here; c is a division constant, and λ the normalization constant. Considering the decay $\pi\rightarrow2\gamma$ ($\Gamma_0=1$, $\Gamma_1=\gamma_\mu$, $\Gamma_2=\gamma_\nu$) of a scalar π-meson with scalar coupling, the term $B=-4m\delta_{\mu\nu}$ represents a nongauge invariant term which will be eliminated in a unique manner by choosing $c=-2iB$, with the result $M_2=(G/2m\pi^2)F_{\mu\nu}{}^2(F_{\mu\nu}=\partial_\nu A_\mu-\partial_\mu A_\nu)$; this result was obtained by Steinberger and Fukuda-Miyamoto[16] by the

[15] L. I. Schiff, Phys. Rev. **76**, 303, 1266 (1949); J. Steinberger, reference 1.

[16] Leighton, Wanlass, and Alford, Phys. Rev. **83**, 843 (1951); J. Steinberger, reference 1; H. Fukuda *et al.*, reference 1; S. Ozaki *et al.*, reference 1; A. G. Carlson *et al.*, Phil. Mag. **41**, 701 (1950).

use of regulators. The decay of a scalar τ-meson with scalar coupling into a vector π-meson and a scalar π-meson with vector coupling, which is a nonrenormalizable problem and is described by the element analogous to $\pi\rightarrow 2\gamma$, is subjected to the divergence theorem, yielding the same relation between c and B as above in order to fulfill the theorem in a unique way. As an example of equivalence, we studied the process $\pi\rightarrow 2\gamma$ of a pseudoscalar π-meson and the decay of a pseudoscalar τ-meson into two vector π-mesons with vector coupling, using the same methods as described under two-field processes. For instance, the transition $\tau\rightarrow 2\pi$ is represented by

$$M_{ps}(x)\sim(-G/48\pi^2m^3)(\mu_\tau^2+\mu_1^2+\mu_2^2)\times(\partial_\alpha U_\mu)(\partial_\beta U_\nu)\epsilon_{\mu\nu\alpha\beta}, \quad (4.9)$$

(μ_τ mass of τ, μ_i mass of π_i) and this holds for $(-1/2m)\partial_\mu M_{p\nu}$ which is nonrenormalizable. Finally, we find for the decay of a scalar τ-meson with scalar coupling into two pseudoscalar π-mesons with pseudoscalar coupling (with $\Gamma_0=1$, $\Gamma_1=\gamma_5$, $\Gamma_2=\gamma_5$, $\mu_1=\mu_2=\mu_\pi$)

$$M\sim-(G/(2\pi)^2)\{m^{-1}[\mu_\tau^2+\tfrac{2}{3}\mu_\pi^2] -(m/4)[32\{\log(m^2 2\pi\lambda)+C+\tfrac{1}{8}\}+c-16]\}U_1U_2. \quad (4.10)$$

This transition element is divergent in the conventional formalism. According to the theory of distributions it contains arbitrary constants (c, λ). The term -16 in the second bracket comes from Eq. (3.24); it cannot be obtained by the conventional incorrect formula (1.3).

However, the meaning of the arbitrary distributions which were neglected above throughout, is by no means clarified. An evaluation of the respective terms shows that the additive quantities resulting from (3.5) are distinguished from the elements written in the text only by extending the boundaries of the a_i-integrals to $\pm\infty$ and replacing dz by $dzf(z, a_i)$ in the z-integral, $f(z, a_i)$ being an arbitrary distribution.

Numerous examples can be added to show how to treat other questions in field theory by means of distribution analysis in an unambiguous and self-consistent manner. For example, the calculation of radiative corrections can be made by these methods. It is believed that a careful investigation by means of the theory of distributions leads to consistent results in many cases where the conventional methods fail. It is no longer necessary to adopt such doubtful prescriptions as to attach the value zero to certain divergent integrals, as is done, for instance, in the calculation of radiative corrections to the stress tensor of the electron.[17] We are not obliged to go outside the framework of current field theory by introducing unrealistic auxiliary fields so as to compensate undesirable effects. But it should be pointed out that it is not legitimate to consider only the last steps of a calculation by distribution methods. The distribution analysis has to be performed from the beginning, verifying carefully each step of the computation.

A very important question is the degree of uncertainty of the results obtained only from the distribution calculus without new physical assumptions. The situation arising from the appearance of arbitrary constants and functions in the result is quite analogous to that of the theory of linear and partial differential equations. In both cases the arbitrary parameters serve for the adaptation of the result to initial and boundary conditions. Since there are many processes for which no general rules, such as gauge invariance, are known in order to determinate the arbitrary constants, we have to conclude that the fundamental physical assumptions of the field theory are incomplete. In order to determine these arbitrary parameters, especially in meson theory, additive quantities in the interaction Hamiltonian seem to be necessary.

Corresponding to the arbitrary constants, it is not possible to attribute definite values to the parameters of bare particles. The determination of normalization constants by the general rules, for instance by the gauge invariance of a theory with interaction, will exclude the possibility of the simultaneous existence of a theory which is gauge invariant without interaction. The uncertainty of some effects induced by the arbitrariness of normalization constants seems to be a consequence of the idealized localizability of field quantities in space-time, that is, of the finiteness of the masses of particles, which is manifested by a lack of dilatation invariance.

The main problems to be investigated in the future may concern the local and causal structure of invariant propagation functions, which now have to be considered as distributions under the aspect of the Pf symbol. Then the field operators must be replaced by distributions in order to conserve the sense of the commutation relations as distribution equations.

In conclusion, the author should like to express his hearty thanks to Professor Molière (now in Rio de Janeiro) for his kind advice and encouragement throughout this work. He is also indebted to Professor L. Schwartz (Nancy) for his stimulating discussions.

APPENDIX

I. In order to get the definition (2.3) let us consider a displacement operator $\tau_{\pm h}$ acting on $\varphi(x)$ according to $\tau_{\pm h}\varphi(x)=\varphi(x\mp h)$, and define $\tau_{\pm h}T(\varphi)=T(\tau_{\mp h}\varphi)$. Then it is natural to define

$$D[T(\varphi)] \text{ by } \lim_{h\rightarrow 0}[\tau_{-h}T(\varphi)-T(\varphi)]/h.$$

[17] S. Borowitz and W. Kohn, Phys. Rev. 86, 985 (1952).

Since distributions are linear and continuous, we have

$$D[T(\varphi)]=\lim_{h\to 0}[\tau_{-h}T(\varphi)-T(\varphi)]/h$$
$$=\lim_{h\to 0}[T(\tau_h\varphi)-T(\varphi)]/h=\lim_{h\to 0}T(\tau_h\varphi-\varphi)/h$$
$$=T[\lim_{h\to 0}(\tau_h\varphi-\varphi)/h]=T(-\varphi'(x))=-T(\varphi'),$$

in accordance with (2.3).

II. Let us consider in the frame of classical analysis the divergent integral

$$\int_0^\infty dx(1/x^{\alpha+1})g(x),$$

$0<\alpha<1$, with a continuous function $g(x)$ whose function-derivative is continuous. As is known from the theory of partial differential equations, especially from Cauchy's problem, Hadamard[18] has introduced the notion "finite part of a divergent integral" as follows: Choose a constant c in such a way that the quantity

$$\lim_{z\to 0}\left\{\int_z^\infty dx(1/x^{\alpha+1})g(x)+c/\alpha z^\alpha\right\}$$

exists. This expression can be written in the form

$$\lim_{z\to 0}\left\{\int_z^\infty dx[(g(x)-g(0))/x^{\alpha+1}]\right.$$
$$\left.+g(0)\int_z^\infty dx/x^{\alpha+1}+c/\alpha z^\alpha\right\}$$
$$=\lim_{z\to 0}\left\{\int_z^\infty dx[(g(x)-g(0))/x^{\alpha+1}]+(g(0)+c)/\alpha z^\alpha\right\}.$$

Since the function-derivative $[g'(x)]$ is continuous, i.e., $|g(x)-g(0)|\leqslant M|x|$, the integral in the latter expression is convergent. Therefore, if c is chosen as $c=-g(0)$, the quantity $\lim_{z\to 0}\{\cdots\}$ is a convergent one, and will be explained as the finite part (f.p.) of the divergent integral

$$\int_0^\infty dx(1/x^{\alpha+1})g(x)\ :$$

$$\text{f.p.}\int_0^\infty dx(1/x^{\alpha+1})g(x)$$
$$=\lim_{z\to 0}\left\{\int_z^\infty dx(1/x^{\alpha+1})g(x)-g(0)/\alpha z^\alpha\right\}.$$

[18] J. Hadamard, *Lectures on Cauchy's Problem in Linear Partial Differntial Equations* (Yale University Press, New Haven, 1923).

This definition is a very artificial one, but is useful for Cauchy's problem, especially for the study of the connection of the solutions of the Poisson and wave equations. By choosing $g(x)=\varphi(x)$ we see immediately that the distribution-derivative $Df_+(\varphi)$ of $f_+(\varphi)$ coincides with the finite part of the divergent integral

$$\text{p.f.}\int_{-\infty}^{+\infty} dx[f_+'(x)]\cdot\varphi(x).$$

But this coincidence is quite an accidental one.

III. In order to prove Eq. (3.10) we note that it is true for $n=2$ since, by integration by parts,

$$D^2(f_-(\varphi))=f_-(\varphi'')=\lim_{\epsilon\to 0}\left(\int_{-\infty}^{-\epsilon} dx\log|x|\,\varphi''(x)\right),$$
$$=-\lim_{\epsilon\to 0}\left\{\int_{-\infty}^{-\epsilon} dx(\varphi(x)/x^2)\right.$$
$$\left.-[\varphi(x)/\epsilon+\varphi'(x)\log\epsilon]_{x=-\epsilon}\right\},$$
$$=\lim_{\epsilon\to 0}\left\{\int_{-\infty}^{-\epsilon} dx(\varphi(x)/x^2)-\varphi(0)/\epsilon\right.$$
$$\left.-\varphi'(0)\log\epsilon\right\}-\varphi'(0),$$
$$\equiv-\text{Pf}[(x^{-2})_-(\varphi)]-\varphi'(0)$$
$$=-\text{Pf}[(x^{-2})_-(\varphi)]+\delta'(\varphi).$$

We suppose that (3.10) is true for some fixed n, say $n=k>2$. Then we have, according to (2.4),

$$D^{k+1}[f_-(\varphi)]$$
$$=-D^k[f_-(\varphi')]$$
$$=(-1)^k(k-1)!\,\text{Pf}[(x^{-k})_-(\varphi')]\sum_{\mu=1}^{k-1}(1/\mu)\delta^{(k-1)}(\varphi')$$
$$=(-1)^k(k-1)!\lim_{\epsilon\to 0}\left\{\int_{-\infty}^{-\epsilon} dx[\varphi'(x)/x^k]\right.$$
$$\left.+\sum_{\mu=0}^{k-2}(-1)^{k+\mu}\frac{\varphi^{(\mu+1)}(0)\epsilon^{\mu+1-k}}{\mu!(\mu+1-k)}-\frac{\varphi^{(k)}(0)}{(k-1)!}\log\epsilon\right\}$$
$$-\sum_{\mu=1}^{k-1}(1/\mu)\delta^{(k-1)}(\varphi'),$$

$$=(-1)^k k!\lim_{\epsilon\to 0}\left\{\int_{-\infty}^{-\epsilon}dx[\varphi(x)/x^{k+1}]+(-1)^k\varphi(-\epsilon)/k\epsilon^k+(1/k)\sum_{\mu=1}^{k-1}(-1)^{k+\mu-1}\frac{\varphi^{(\mu)}(0)\epsilon^{\mu-k}}{(\mu-1)!(\mu-k)}-\frac{\varphi^{(k)}(0)}{k!}\log\epsilon\right\}+\sum_{\mu=1}^{k-1}(1/\mu)\delta^{(k)}(\varphi),$$

$$=(-1)^k k!\lim_{\epsilon\to 0}\left\{\int_{-\infty}^{-\epsilon}dx[\varphi(x)/x^{k+1}]+\frac{(-1)^k}{k\epsilon^k}\varphi(0)-\frac{\varphi^{(k)}(0)}{k!}\log\epsilon+(1/k)\sum_{\mu=1}^{k-1}\left[\frac{(-1)^k\epsilon^{\mu-k}}{\mu!}+\frac{(-1)^{k+\mu-1}\epsilon^{\mu-k}}{(\mu-1)!(\mu-k)}\right]\varphi^{(\mu)}(0)\right\}+\frac{(-1)^k}{k}\varphi^{(k)}(0)+\sum_{\mu=1}^{k-1}(1/\mu)\delta^{(k)}(\varphi),$$

$$=-(-1)^{k+1}k!\,\mathrm{Pf}[(x^{-k-1})(\varphi)]+\sum_{\mu=1}^{k}(1/\mu)\delta^{(k)}(\varphi);$$

i.e., (3.10) is true for every n, q.e.d. Eq. (3.8) can be proved in the same manner.

IV. Equation (3.16) is true for $m=1$, n arbitrary, according to (3.13). Assuming (3.16) to be true for $m=k>1$ we have

$$D^{k+1}\,\mathrm{Pf}[(x^{-n})_\pm(\varphi)]=D[D^k\,\mathrm{Pf}(x^{-n})_\pm(\varphi)]$$

$$=(-1)^k\prod_{\nu=0}^{k-1}(n+\nu)D[\mathrm{Pf}(x^{-k-n})_\pm(\varphi)]\pm[(-1)^n/(n-1)!]\sum_{\nu=0}^{k-1}(1/n+\nu)\delta^{(k+n)}(\varphi)$$

$$=(-1)^{k+1}\prod_{\nu=0}^{k}(n+\nu)\,\mathrm{Pf}[(x^{-n-k-1})_\pm(\varphi)]\pm(-1)^n\left[(1/(k+n)!)\prod_{\nu=0}^{k-1}(n+\nu)+(1/(n-1)!)\sum_{\nu=0}^{k-1}(1/n+\nu)\right]\delta^{(k+n)}(\varphi)$$

$$=(-1)^{k+1}\prod_{\nu=0}^{k}(n+\nu)\,\mathrm{Pf}[(x^{-n-k-1})_\pm(\varphi)]\pm[(-1)^n/(n-1)!]\sum_{\nu=0}^{k}(1/n+\nu)\delta^{(k+n)}(\varphi);$$

i.e., (3.16) is true for all n, m.

PHYSICAL REVIEW VOLUME 89, NUMBER 5 MARCH 1, 1953

The Range Correction for Electron Pick-Up

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(Received October 15, 1952)

The extension of positive particle ranges caused by pick-up of electrons at low velocities has been studied in Ilford C2 emulsion. Ranges of Li^8 and B^8 nuclei were measured in emulsion and compared with tracks of helium and hydrogen isotopes of the same velocity. The empirical range-energy relation adduced for light nuclei is: $Z^2R/M=F(T/M)+0.12Z^3$, for $\beta>1.04Z/137$.

I. INTRODUCTION

IT has recently[1] been found possible to collect and analyze physically the products formed when the high energy beam of the 184-inch cyclotron is employed to disintegrate atomic nuclei. A reliable analysis of the products of atomic number greater than two was, however, difficult because the range-energy relations in nuclear track emulsion for multiply charged fragments were uncertain. In the preliminary work in which protons bombarded carbon, no "hammer" tracks indicative of the presence of Li^8 and B^8 were found. With further searching on these plates, a few such tracks have now been found, and hammer tracks are also seen in fair abundance on plates exposed to the disintegration products of various light elements bombarded with alpha-particles or deuterons. In the present experiment about one splinter in two hundred was Li^8, and about one in 5000 was B^8.

The unmistakable appearance of the tracks which they produce make Li^8 and B^8 extremely useful isotopes to employ in studying the range-energy relations for multiply charged ions. At low velocities a positive particle tends to be neutralized by electrons, thus reducing its rate of energy loss. It is the purpose of this experiment to utilize the tracks of Li^8 and B^8 to evaluate the range correction arising from this effect.

II. IDENTIFICATION AND MEASUREMENT OF TRACKS

Tracks are identified by plotting the range, R, *versus* the radius of curvature, ρ, for each track. The tracks of

[1] Walter H. Barkas and J. Kent Bowker, Phys. Rev. **87**, 207 (1952).