Explicit $\gamma - \gamma$ Angular Correlations. II. Polarization Correlations

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The γ-γ angular correlation function for the case where one or both of the γ-ray detectors is able to discriminate between plane polarization states of a γ-ray is given in terms of the corresponding γ-γ directional correlation. The formulas given are directly applicable to one-three type correlations in a triple cascade, and also to direction-plane polarization correlations in which the unpolarized particle is not necessarily a γ-ray, e.g., β-polarized γ. Correlations for the case where the detectors are sensitive to circular polarizations are not treated.

1. INTRODUCTION

THE methods explained in a previous article[1] give the angular correlation function of two successive nuclear radiations as a finite series in the three-dimensional rotation group functions $d^{(\lambda)}(\theta)_{\mu'\mu}$, which reduce to Legendre polynomials $P_\lambda(\cos\theta)=d^{(\lambda)}(\theta)_{0,0}$ in directional correlations. One possible interpretation of the correlation formula so obtained is to say that one is using two different coordinate systems with which to describe the detectors of the successive radiations and that each detector lies on its own z-axis (the source is at the common origin of the two coordinate systems); the $d^{(\lambda)}(\theta)_{\mu'\mu}$ appear then in the correlation formula when one makes use of the linear relations that hold between sets of nuclear states quantized with respect to the two different coordinate systems. In addition to the simplicity gained by needing only matrix elements for emission of the radiations along the quantization axis, one finds other deep simplicities in the structure of the coefficients of the $d^{(\lambda)}(\theta)_{\mu'\mu}$ in such an expansion of the correlation. One of these, the breaking up of the coefficients into independent factors for the separate transitions of the cascade, has already been brought out in (Ia). In the following another property of the coefficients is made use of: the coefficients break up into a nuclear factor (involving spins and multipole amplitudes) and a polarization factor depending only on the multipole orders and parities of the γ-rays.[2,3] The situation is probably best made clear by offering for inspection Eq. (2), below, for plane polarization-plane polarization γ-γ correlations.

It does not seem that circular-circular γ-γ polarization correlations† can be obtained from considerations as easy as those of Sec. II, and we postpone detailed examination of the circular-circular case until a practical "quarter-wave plate for γ-rays," or some such device, has been developed.

2. POLARIZATION CORRELATIONS

(a) γ-Ray Detectors

We suppose that a γ-ray detector, besides establishing the direction from source to detector as the direction of the γ-ray, establishes also a detector reference plane, or, for short, a *detector plane*, containing the direction of the γ-ray, which has the property that the number of counts per plane polarized γ-ray traversing the detector is $\epsilon+\rho\cos 2\chi$, where χ is the angle the electric vector of the γ-ray makes with the "detector plane." Clearly, ϵ, (≥ 0), is the efficiency of the detector averaged over plane polarizations and $|\rho/\epsilon|$, (≤ 1), is a measure of the ability of the detector to sense the direction of polarization in a plane polarized beam.[4] These efficiencies are functions of the γ-ray energy.

The response of the detector to a beam of γ-rays for which the vector potential is, say, $\mathbf{A}=\mathbf{a}\exp(-ikz+ikct)+$c.c. will in general have the form: (counting rate) $\doteq \mathbf{a}^*\cdot R\cdot\mathbf{a}$ where $R=R_c^*$ is a plane hermitian dyadic. A canonical form (with respect to real rotations in the x, y-plane) for such an R is:

$$R=\epsilon(\mathbf{ii}+\mathbf{jj})+\rho(\mathbf{ii}-\mathbf{jj})+i\sigma(\mathbf{ij}-\mathbf{ji}).$$

Here, $\mathbf{i}$ and $\mathbf{j}$ are unit vectors along the x and y axes, respectively, and ϵ, ρ, and σ are all real. The σ-term in this expression for $\mathbf{R}$ is invariant under real rotations in the x, y-plane, and, just as $|\rho/\epsilon|$ measures the sensitivity of the detector to plane polarizations, the independent coefficient $|\sigma/\epsilon|$, (≤ 1), measures the sensitivity of the detector to circular polarizations. It is assumed in the following that $\sigma=0$. If $\sigma\neq 0$ were the case one would have to go back and calculate the angular correlation with the quantum field quantized according to "normal modes" of the detector; i.e., setting $\rho=\tau\cos 2v$, $\sigma=\tau\sin 2v$, one would use the two elliptical polarization states: $\mathbf{e}_1=\mathbf{i}\cos v+i\mathbf{j}\sin v$, $\mathbf{e}_2=i\mathbf{i}\sin v+\mathbf{j}\cos v$, $(\mathbf{e}_1^*\cdot\mathbf{e}_1=\mathbf{e}_2^*\cdot\mathbf{e}_2=1,\ \mathbf{e}_1^*\cdot\mathbf{e}_2=0)$, as the (complex) independent polarization states of a z-axis quantum. (There is a different set of $\mathbf{e}$'s for each detector, of course.) In terms of these the response dyadic R of the detector is again diagonal: $R=\epsilon(\mathbf{e}_1^*\mathbf{e}_1+\mathbf{e}_2^*\mathbf{e}_2)+\tau(\mathbf{e}_1^*\mathbf{e}_1-\mathbf{e}_2^*\mathbf{e}_2)$, so that one needs only probabilities (and not amplitudes) for emission into the independent states $\mathbf{e}_1$ and $\mathbf{e}_2$.

In the angular correlation, allowing $\sigma\neq 0$ would bring in terms

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[1] S. P. Lloyd, Phys. Rev. **85**, 904 (1952), referred to hereafter as (Ia). The notation of the present article is mostly that of (Ia).

[2] See G. Racah, Phys. Rev. **84**, 910 (1951).

[3] This is not quite true for β-decay, where one can have several distinct interfering "multipoles" of the same angular momentum and parity in a given transition. The notation and results of (Ia) can be modified easily to cover the β-ray case. The formulas for correlations measured with β-polarimeters will be rather more complicated than the ones given in the following for γ-polarimeters. The author has not investigated the point in detail. See D. L. Falkoff and G. E. Uhlenbeck, Phys. Rev. **79**, 334 (1950) for detailed discussion of β-ray directional correlations; also, M. Fuchs, thesis, University of Michigan, 1951 (unpublished).

† There are no direction—circular or plane—circular correlations.

[4] See M. Deutsch and F. Metzger, Phys. Rev. **74**, 1542 (1948), Fig. 1. If the detector plane is taken to be the plane through crystals A and B of the polarimeter, the parameter ρ is negative, i.e., $\rho/\epsilon=-(D-1)/(D+1)$ in terms of the parameter D of Deutsch and Metzger. For their $D=2.1$, $\rho/\epsilon=-0.35$.

involving the $d^{(\lambda)}(\theta)_{\mu'\mu}$ for which the index λ is odd[2]; in particular, odd Legendre polynomials in circular—circular polarization correlations. In the plane polarization correlations of the present article, the pure multipole $\lambda=$odd terms vanish identically, and the multipole mixture $\lambda=$odd terms vanish as a result of the reality property of the multipole scalar amplitudes.

(b) Polarization-polarization Correlations

Let the directional correlation between the γ-rays emitted in the nuclear cascade $J_1(\gamma)J(\gamma)J_2$ be written as

$$W(\theta)=1+\sum_{L_1\leq L_1'}\sum_{L_2\leq L_2'}\times\sum_\lambda A_\lambda(L_1L_1';L_2L_2')P_\lambda(\cos\theta). \quad (1)$$

The coefficients $A_\lambda(L_1L_1';L_2L_2')$, where L_1 and L_1' are the orders of the interfering multipoles in the $J_1(\gamma)J$ transition and L_2 and L_2' those for $J(\gamma)J_2$, have been given in some detail in (Ia),‡ and tabulations of the lowest multipole coefficients are to be found in (I).[5] The coefficients $A_\lambda(L_1L_1';L_2L_2')$ of Eq. (1) depend also on nuclear angular momenta, and contain as factors the appropriate scalar relative amplitudes $(J_f\,{:}L{:}\,J_i)$ for 2^L-pole multipole emission, which are defined in (Ia).

When the plane polarization-dependent correlation is worked out from Eqs. (8), (13), and (29) of (Ia) the advantages of the $d^{(\lambda)}(\theta)_{\mu'\mu}$ expansion are apparent: the polarization correlation can be obtained from the corresponding directional correlation by replacing $P_\lambda(\mathbf{\Omega}_1\cdot\mathbf{\Omega}_2)$ by a function $f_\lambda(L_1L_1', L_2L_2'; \mathbf{\Omega}_1\mathbf{e}_1, \mathbf{\Omega}_2\mathbf{e}_2)$ which is independent of nuclear spins and nuclear matrix elements. Instead of giving these functions explicitly, we give the result of combining the correlation function $W(\mathbf{\Omega}_1, \mathbf{e}_1; \mathbf{\Omega}_2, \mathbf{e}_2)$ so obtained with the appropriate detector efficiencies. Let ϵ_{a1}, ρ_{a1} be the efficiencies of detector a for the first γ-ray $J_1(\gamma)J$ of $J_1(\gamma)J(\gamma)J_2$, ϵ_{b1}, ρ_{b1} the efficiencies of detector b for the same γ-ray, etc. Angles φ_a and φ_b, which are the inclinations of the respective detector planes to the (a, b, source) plane, are both positive when measured counterclockwise, looking toward the source. The function $N(\theta;\varphi_a,\varphi_b)$, the theoretical mean coincidence rate in counters a and b due to correlations between successive γ-rays, works out after straightforward calculation to be, apart from normalization factors:

$$N(\theta;\varphi_a,\varphi_b)=1+\sum_{L_1\leq L_1'}\sum_{L_2\leq L_2'}\sum_\lambda A_\lambda(L_1L_1';L_2L_2')\Big\{P_\lambda(\mu)+\frac{(\lambda-2)!}{(\lambda+2)!}\big[\epsilon^{-1}(\rho_{a1}\epsilon_{b2}K_\lambda(L_1L_1')$$
$$+\rho_{a2}\epsilon_{b1}K_\lambda(L_2L_2'))P_\lambda^2(\mu)\cos2\varphi_a+\epsilon^{-1}(\rho_{b1}\epsilon_{a2}K_\lambda(L_1L_1')+\rho_{b2}\epsilon_{a1}K_\lambda(L_2L_2'))P_\lambda^2(\mu)\cos2\varphi_b$$
$$+\epsilon^{-1}(\rho_{a1}\rho_{b2}+\rho_{a2}K_\lambda\rho_{b1})(L_1L_1')K_\lambda(L_2L_2')\{\tfrac{1}{2}d_\lambda(-\mu)\cos2(\varphi_a-\varphi_b)+\tfrac{1}{2}d_\lambda(\mu)\cos2(\varphi_a+\varphi_b)\}\big]\Big\}, \quad (2)$$

where $\epsilon=\epsilon_{a1}\epsilon_{b2}+\epsilon_{b1}\epsilon_{a2}$. Also, $P_\lambda^2(\mu)$ are unnormalized associated Legendre functions (with $\mu=\cos\theta$):

$$P_\lambda^2(\mu)=(1-\mu^2)\frac{d^2}{d\mu^2}P_\lambda(\mu)=\left[\frac{(\lambda+2)!}{(\lambda-2)!}\right]^{\frac{1}{2}}d^{(\lambda)}(\theta)_{2,0}, \quad (\lambda\geq2), \quad (3)$$

and $d_\lambda(\mu)$ is an abbreviation for

$$d_\lambda(\mu)=d^{(\lambda)}(\theta)_{2,-2}=d^{(\lambda)}(\pi-\theta)_{2,2}$$
$$=\sum_{\kappa=0}^{\lambda-2}\frac{(-1)^\kappa(1-\mu)^{\kappa+2}(1+\mu)^{\lambda-\kappa-2}}{\kappa!(\kappa+4)!((\lambda-2-\kappa)!)^2}\cdot\frac{(\lambda-2)!(\lambda+2)!}{2^\lambda}. \quad (4)$$

The coefficients $K_\lambda(LL')$ are explicitly:

$$K_\lambda(LL')=s(LL')k_\lambda(LL'),$$

with

$$s(LL')=1 \quad \text{for el } L\text{, el or mag } L',$$
$$s(LL')=-1 \text{ for mag } L\text{, el or mag } L', \quad (5)$$

and (with $\lambda=$even)

$$k_\lambda(LL')=-\left[\frac{(\lambda+2)!}{(\lambda-2)!}\right]^{\frac{1}{2}}\frac{(LL'11|LL'\lambda2)}{(LL'-11|LL'\lambda0)}$$
$$=L'(L'+1)-L(L+1) \quad \text{when} \quad L+L'=\text{odd},$$
$$=\frac{\lambda(\lambda+1)[L(L+1)+L'(L'+1)]-[L'(L'+1)-L(L+1)]^2}{L(L+1)+L'(L'+1)-\lambda(\lambda+1)}$$
$$\text{when} \quad L+L'=\text{even}. \quad (6)$$

The original $W(\mathbf{\Omega}_1\mathbf{e}_1, \mathbf{\Omega}_2\mathbf{e}_2)$ from which Eq. (2) was obtained can be recovered by setting $\epsilon_{a1}=\epsilon_{b2}=\rho_{a1}=\rho_{b2}=1$, $\epsilon_{a2}=\epsilon_{b1}=\rho_{a2}=\rho_{b1}=0$, and then putting $\varphi_a=\varphi_1$, $\varphi_b=\varphi_2$. One has then $N(\theta;\varphi_1,\varphi_2)=W(\mathbf{\Omega}_1\mathbf{e}_1, \mathbf{\Omega}_2\mathbf{e}_2)$, where φ_1 and φ_2 are the angles the electric vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ of the successive quanta make with the plane containing directions $\mathbf{\Omega}_1$ and $\mathbf{\Omega}_2$ of the quanta, and where $\mu=\cos\theta=\mathbf{\Omega}_1\cdot\mathbf{\Omega}_2$.

(c) Direction-Polarization Correlations

Evidently, direction-polarization correlations can be obtained from Eq. (2) by setting the appropriate detector efficiencies equal to zero. For example, if detector b is not a polarimeter one puts $\rho_{b1}=\rho_{b2}=0$, leaving only the polarization terms involving $\cos2\varphi_a$.

‡ There is an obvious misprint (due to ms error) in the expression for $a_\lambda(LL')$, Eq. (23) of (Ia): the first factorial in the denominator which now reads $(\frac{1}{2}(\lambda+L+L'))!$ should read $(\frac{1}{2}(\lambda+L-L'))!$.

[5] S. P. Lloyd, Phys. Rev. **83**, 716 (1952). The coefficients in the multipole mixture terms should have signs opposite to those given in (I), as explained in footnote (22) of (Ia).

TABLE I. $P_\lambda^2(\mu)$. The $P_\lambda^2(\mu)$ are even in μ: $P_\lambda^2(-\mu)=P_\lambda^2(\mu)$.

μ	$P_2^2(\mu)$	$P_4^2(\mu)$	$P_6^2(\mu)$
0.0	3.00	−7.500	13.125
0.1	2.97	−6.905	10.698
0.2	2.88	−5.184	+ 4.193
0.3	2.73	−2.525	− 4.213
0.4	2.52	+0.756	−11.413
0.5	2.25	4.219	−14.150
0.6	1.92	7.296	−10.107
0.7	1.53	9.295	+ 0.691
0.8	1.08	9.396	14.160
0.9	0.57	6.655	20.128
1.0	0	0	0

Furthermore one sees from the derivation that Eq. (2) gives direction-polarization correlations whatever the unpolarized radiation. Thus the β-(plane polarized γ) correlation for $J_1(\beta)J(\gamma)J_2$ is obtained from Eq. (2) as follows: (1) For the coefficients $A_\lambda(L_1L_1';L_2L_2')$ in Eq. (2) use the coefficients of the $J_1(\beta)J(\gamma)J_2$ directional correlation, which are assumed to be known; (2) if detector a is the β-counter, put $\rho_{a1}=\rho_{a2}=0$, corresponding to the fact that the β-counter is not sensitive to polarizations in either the β-ray or the γ-ray beam; (3) put $\rho_{b1}=0$, since the γ-polarimeter is not supposed to measure polarization in the β-ray beam.§ If the γ-counter is shielded against β-rays, so that also $\epsilon_{b1}=0$, the efficiency ϵ_{a1} of the β-counter drops out of the correlation, leaving ρ_{b2}/ϵ_{b2} as the only parameter needed for γ-polarimeter b.

One can also use Eq. (2) as it stands for one-three polarization correlations in triple cascades;[6] one requires only that the coefficient of $P_\lambda(\mu)$ in the corresponding directional correlation be broken up into a sum of contributions from each interfering multipole pair L, L' in each transition. Specialized forms of Eq. (2) have been given by Falkoff,[7] Hamilton,[8] and Zinnes.[9]

TABLE II. $d_\lambda(\mu)=d^{(\lambda)}(\cos^{-1}\mu)_{2,-2}$.

μ	$d_2(\mu)$	$d_4(\mu)$	$d_6(\mu)$
−1.0	1.0000	1.0000	+1.0000
−0.9	0.9025	+0.3339	−0.1845
−0.8	0.8100	−0.0972	−0.4135
−0.7	0.7225	−0.3396	−0.2188
−0.6	0.6400	−0.4352	+0.0717
−0.5	0.5625	−0.4219	0.2791
−0.4	0.4900	−0.3332	0.3379
−0.3	0.4225	−0.1986	0.2559
−0.2	0.3600	−0.0432	+0.0835
−0.1	0.3025	+0.1119	−0.1117
0.0	0.2500	0.2500	−0.2656
+0.1	0.2025	0.3584	−0.3315
0.2	0.1600	0.4288	−0.2893
0.3	0.1225	0.4569	−0.1491
0.4	0.0900	0.4428	+0.0513
0.5	0.0625	0.3906	0.2537
0.6	0.0400	0.3088	0.3933
0.7	0.0225	0.2099	0.4173
0.8	0.0100	0.1108	0.3093
0.9	0.0025	0.0324	0.1195
1.0	0	0	0

§ Correlations involving electron polarimeters would bring in $\cos\varphi$ instead of $\cos 2\varphi$, anyway.

[6] Biedenharn, Arfken, and Rose, Phys. Rev. 83, 586 (1951).

3. TABLES

(a) Power Series and Legendre Series

If

$$A_0+A_2P_2(\cos\theta)+A_4P_4(\cos\theta)+A_6P_6(\cos\theta)+A_8P_8(\cos\theta)=Q+R\cos^2\theta+S\cos^4\theta+T\cos^6\theta+U\cos^8\theta,$$

then

$$A_0=Q+(1/3)R+(1/5)S+(1/7)T+(1/9)U$$
$$A_2=(2/3)R+(4/7)S+(10/21)T+(40/99)U$$
$$A_4=(8/35)S+(24/77)T+(48/143)U$$
$$A_6=(16/231)T+(64/495)U$$
$$A_8=(128/6435)U$$

and

$$Q=A_0-(1/2)A_2+(3/8)A_4-(5/16)A_6+(35/128)A_8$$
$$R=(3/2)A_2-(30/8)A_4+(105/16)A_6-(1260/128)A_8$$
$$S=(35/8)A_4-(315/16)A_6+(6930/128)A_8$$
$$T=(231/16)A_6-(12012/128)A_8$$
$$U=(6435/128)A_8.$$

TABLE III. The $k_\lambda(LL)$.

$\lambda \backslash L$	1	2	3	4	5
2	−12	12	8	120/17	30/3
4	—	−30	120	40	30
6	—	—	−56	−840	140

(b) The Angular Functions

The first few of the functions of Eqs. (3) and (4) are given in Tables I and II. They are explicitly:

$$P_2^2(\mu)=3(1-\mu^2)$$
$$P_4^2(\mu)=(15/2)(1-\mu^2)(7\mu^2-1)$$
$$P_6^2(\mu)=(105/8)(1-\mu^2)(33\mu^4-18\mu^2+1)$$

and

$$d_2(\mu)=(1/4)(1-\mu)^2$$
$$d_4(\mu)=(1/4)(1-\mu)^2(7\mu^2+7\mu+1)$$
$$d_6(\mu)=(1/64)(1-\mu)^2(495\mu^4+660\mu^3+90\mu^2-108\mu-17).$$

(c) The Coefficients $k_\lambda(LL')$

The multipole mixture coefficients for the usual case $L'=L+1$ are $k_\lambda(L\,L+1)=2(L+1)$; the pure multipole coefficients for correlation up to $P_6(\cos\theta)$ and $L'=L\leq 5$ are given in Table III.

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[7] D. L. Falkoff, Phys. Rev. 73, 518 (1948).

[8] D. R. Hamilton, Phys. Rev. 74, 782 (1948).

[9] I. Zinnes, Phys. Rev. 80, 386 (1950).