energy yields $\Gamma_n/\Gamma \approx \frac{1}{3}$ on the statistical model. With the same estimate of $|\bar{J}|^2$ as for O^{16}, this gives $|F|^2 \sim \frac{1}{4}N$ at the peak of the giant resonance. The shoulder at lower energy, however, has the same order of $|F|^2$ as for N^{14} and O^{16}. Incoherent ED excitation need not be affected by partial breakdown of the coherent mechanism.

It should perhaps be remarked that the matrix elements $|\bar{J}|^2$ are exceptionally large. They are computed on the assumption that the wave functions of the initial and final states in the photoelectric transition overlap perfectly. Any deviations from this correspondence would in general reduce the magnitude of $|\bar{J}|^2$, which could not be tolerated in the face of the measured cross sections. Although such perfect overlap may seem reasonable in low energy γ-transitions considered in the shell model, it is rather a surprise to find one-particle wave functions so much alike for ground and highly excited states.

It is a pleasure to record a stimulating conversation with Professor L. Katz.

PHYSICAL REVIEW VOLUME 88, NUMBER 4 NOVEMBER 15, 1952

Solar "Enhanced Radiation" and Plasma Oscillations

HARI K. SEN
National Bureau of Standards, Washington, D. C.
(Received July 3, 1952)

The dispersion relation for a plasma oscillating in a static magnetic field is derived by the Laplace transform method. The plasma oscillations are found to be unstable in frequency bands around multiples of the gyrofrequency. A numerical application to spot magnetic fields at coronal distances indicates sufficient amplification to make plausible the theory of the origin of solar "enhanced radiation" in plasma oscillations of electrons gyrating round the magnetic field of sunspots.

THE sun is known to emit radiofrequency radiation in the meter range of wavelengths which maintains a high but variable level for periods of hours or days.[1] This "enhanced radiation" proceeds from the direction of sunspots and shows circular polarization. An attempt is made in this paper to ascribe this phenomenon to motion of solar material (prominence or corpuscles) in the magnetic field of sunspots.[2] In the highly conductive corona, the prominence (or corpuscles) will move along the lines of magnetic force. The transverse component of any tendency to oblique motion of the material will set the electrons (and ions) gyrating round the lines of force with a peaked velocity distribution.

Malmfors[3] was the first to point out the similarity between the conditions in the solar corona and in a trochotron and ascribe the origin of solar noise storms to plasma oscillations of electrons in the spot magnetic field. When the current exceeds a certain limit, noise and negative current are observed in the trochotron. Ordinary collision processes cannot explain the effect. The physical picture seems to be that any distortion of the distribution of the electrons is repeated with the gyro-period of the electrons. The space charge electric field varies with the same period, achieving an effect similar to what obtains in a cyclotron.

In a later paper, Malmfors[4] has given a more detailed treatment by the hydrodynamic equation of motion of the states of oscillation in a system of electrons moving with uniform speed in circular paths perpendicular to a magnetic field, and has found that the system is unstable.

Plasma oscillations in electric and magnetic fields have been treated by Bailey[5] with Maxwell's transfer equations. The Maxwell transfer equations (the so-called "hydrodynamic approximation") are easier to apply than the Boltzmann equation, and it is fortunate that they lead in most cases to qualitatively correct conclusions (particularly at long wavelengths). However, to obtain accurate results one must apply the more general Boltzmann equation. One may thus obtain not only quantitatively more accurate results, but also qualitatively new ones, e.g., the heavy damping at wavelengths near the Debye length.

Gross[6] by a kinetic theory treatment has obtained a dispersion relation different from Malmfors'.[7] Gross

[1] J. P. Wild, Australian J. Sci. Res. A4, 36 (1951).
[2] A thermal origin cannot satisfactorily explain this "enhanced radiation" which must be distinguished from the slowly varying component, at frequencies of 600 Mc/sec and above, that is closely correlated with sunspot area. See J. H. Piddington and H. C. Minnett, Australian J. Sci. Res. A4, 131 (1951).
[3] H. Alfven *et al.*, "Theory and application of trochotrons," Kgl. Tekniska Hogskolans Handlingar No. 22 (1948).
[4] K. G. Malmfors, Arkiv. fys. 1, 569 (1950).
[5] V. A. Bailey, Phys. Rev. 83, 439 (1951).
[6] E. P. Gross, Phys. Rev. 82, 232 (1951).
[7] In the opinion of the author, the reason for the discrepancy between Malmfors' and Gross' results in the following. Malmfors (see reference 3) has used the same vectorial angle ϕ in his Fig. 1, which refers to velocity space, and in his equation of perturbation, in Sec. 3, of the x coordinate, which refers to ordinary space. Making correction for this error (changing ϕ into $\frac{1}{2}\pi+\phi$ in Malmfors' continuity equation), the author finds that the two results check.

finds from his dispersion equation the existence of gaps in the spectrum at frequencies that are approximately multiples of the gyrofrequency. The magnitude of the gap depends on the temperature of the gas, being proportional to it for long wavelengths. This leads him to the prediction of selective reflection of waves impinging on a plasma with frequency in the forbidden range.

On account of certain singularities[8] appearing in Gross' treatment, the author has thought it worth while to reconsider the Boltzmann equation in a magnetic field by the Laplace transform method used by Landau[9] in an analogous problem. The analysis confirms Gross'[6] dispersion equation. But it is found that the system is unstable in frequency bands around multiples of the gyrofrequency.

We consider an electron plasma gyrating with a uniform speed v_0 round a static magnetic field of strength H_0.

Let $f(\mathbf{r}, \mathbf{v}, t)$ be the distribution function of the plasma, satisfying the Boltzmann equation[10]

$$\frac{df}{dt}\equiv\frac{\partial f}{\partial t}+\mathbf{v}\cdot\nabla f+\frac{e}{m}\left(\mathbf{E}+\frac{\mathbf{v}\times\mathbf{H}_0}{c}\right)\cdot\nabla_{\mathbf{v}}f=0. \quad (1)$$

$\mathbf{E}$ is the electric field due to the space charge. We assume the absence of static electric fields and neglect collisions.[11] For oscillations of small amplitude, we may put

$$f(\mathbf{r}, \mathbf{v}, t)=f_0(\mathbf{v})+f_1(\mathbf{r}, \mathbf{v}, t), \quad (2)$$

where $f_0(\mathbf{v})$ is the equilibrium distribution, and $f_1\ll f_0$.

In this approximation the Boltzmann equation is linear and reduces to

$$\frac{\partial f_1}{\partial t}+\mathbf{v}\cdot\nabla f_1-\frac{e}{m}\nabla\phi\cdot\nabla_{\mathbf{v}}f_0+\frac{e}{mc}(\mathbf{v}\times\mathbf{H}_0)\cdot\nabla_{\mathbf{v}}f_1=0, \quad (3)$$

where the electric potential $\phi(\mathbf{r}, t)$ satisfies the Poisson equation[12]

$$\nabla^2\phi=-4\pi e\int f_1 d\tau(d\tau=dv_xdv_ydv_z). \quad (4)$$

[8] Gross' distribution function (f_1) (see reference 5) has singularities at multiples of the gyrofrequency (ω_c). When $\omega=n\omega_c$, f_1 cannot be made periodic in δ (vectorial angle), except for the trivial case of vanishing ac electric field. The Laplace transform method, as we shall see, enables us to work directly with the electric potential which happens to be nonsingular.

[9] L. Landau, J. Phys. (U.S.S.R.), **Vol. X, No. 1**, 25 (1946).

[10] S. Chapman and T. G. Cowling, *The Mathematical Theory of Non-Uniform Gases* (Cambridge University Press, Cambridge, 1939), p. 322, Sec. 18.2.

[11] We suppose the ions to be stationary, on account of their large mass, and smeared into a uniform charge distribution sufficient to cancel the static negative charge of the electrons. We also consider the atmosphere to be rarefied enough to justify the neglect of collisions.

[12] It is true that in a static magnetic field there is a strong coupling of longitudinal and transverse motions and we should, in strictness, use not the Poisson equation but the full set of Maxwell's equations. The use of the Poisson equation is, however, justified for the limiting case of separation of plasma from electromagnetic waves, i.e., when $ck\gg\omega_p$ and $ck\gg\omega_c$, where ω_p and ω_c are, respectively, the plasma and gyrofrequencies, k is the wave number and c the velocity of light (see reference 5).

The equilibrium distribution $f_0(\mathbf{v})$ satisfies the equation

$$\mathbf{v}\times\mathbf{H}_0\cdot\nabla_{\mathbf{v}}f_0=0. \quad (5)$$

A solution of (5) is $f_0(\mathbf{v})=f_0((v_x^2+v_y^2)^{\frac{1}{2}}, v_z)$, where $\mathbf{H}_0$ is assumed to be in the Z direction. The Maxwellian distribution can therefore be an equilibrium one in a static magnetic field.[13]

Assume that Eqs. (3) and (4) have solutions of the form

$$f_1=f_1(\mathbf{v}, t)e^{jkx}, \quad \phi=\phi(t)e^{jkx}. \quad (6)$$

Then (3) and (4) reduce to

$$\frac{\partial f_1}{\partial t}+jkv_xf_1+\omega_c\left[v_y\frac{\partial f_1}{\partial v_x}-v_x\frac{\partial f_1}{\partial v_y}\right]-jk\frac{e}{m}\phi\frac{\partial f_0}{\partial v_x}=0, \quad (7)$$

and

$$k^2\phi(t)=4\pi e\int f_1 d\tau. \quad (8)$$

Here,

$$\omega_c=eH_0/mc \quad (9)$$

is the gyrofrequency.

Following the Laplace transform method,[14] we define the function $f_p(\mathbf{v})$ by

$$f_p(\mathbf{v})=\int_0^\infty f_1(\mathbf{v}, t)e^{-pt}dt, \quad (10)$$

where

$$f_1(\mathbf{v}, t)=\frac{1}{2\pi j}\int_{-j\infty+\gamma}^{+j\infty+\gamma} f_p(\mathbf{v})e^{pt}dp. \quad (11)$$

We suppose on physical grounds that $f_1(\mathbf{v}, t)$ has a continuous derivative and that $|f_1(\mathbf{v}, t)|<Ke^{ct}$, where K and c are positive constants. Then γ in (11) $>c$.

Multiply both sides of Eqs. (7) and (8) by e^{-pt} and integrate over t. This yields

$$(p+jkv_x)f_p+\omega_c\left[v_y\frac{\partial f_p}{\partial v_x}-v_x\frac{\partial f_p}{\partial v_y}\right]-jk\frac{e}{m}\phi_p\frac{\partial f_0}{\partial v_x}=g(\mathbf{v}), \quad (12)$$

and

$$k^2\phi_p=4\pi e\int f_p d\tau, \quad (13)$$

where

$$g(\mathbf{v})\equiv f_1(\mathbf{v}, 0). \quad (14)$$

On transformation into cylindrical coordinates,

$$v_x=\rho\cos\delta, \quad v_y=\rho\sin\delta, \quad v_z=v_z, \quad (15)$$

Equation (12) reduces to

$$(p+jk\rho\cos\delta)f_p-\omega_c\frac{\partial f_p}{\partial\delta}-jk\frac{e}{m}\phi_p\frac{df_0}{d\rho}\cos\delta=g(\mathbf{v}). \quad (16)$$

On account of the axial symmetry, we shall henceforth suppose the dependence on z of the physical variables to have been taken out by integration over z, and shall consider only the polar variables (ρ, δ).

[13] See Appendix I.

[14] H. S. Carslaw and J. C. Jaeger, *Operational Methods in Applied Mathematics* (Oxford University Press, London, 1941), p. 72.

If we regard (16) as a linear differential equation in (δ), the solution is

$$f_p(\rho,\delta)=A(\rho)e^{(p\delta+jk\rho\sin\delta)/\omega_c}-e^{(p\delta+jk\rho\sin\delta)/\omega_c}\int_0^\delta e^{-(p\delta+jk\rho\sin\delta)/\omega_c}\times\frac{g(\mathbf{v})+jk(e/m)\phi_p(df_0/d\rho)\cos\delta}{\omega_c}d\delta,\quad(17)$$

where $A(\rho)$ is an arbitrary function of ρ, which must be determined from the physical consideration that f_p must be periodic in δ with the period 2π.

Let the initial perturbation

$$g(\mathbf{v})\equiv\textstyle\sum_l A_l(\rho)e^{jl\delta}.\quad(18)$$

We use the expansion

$$e^{jz\sin\delta}=\sum_{n=-\infty}^{+\infty}J_n(z)e^{jn\delta},\quad(19)$$

and set (18) into (17). Thus we obtain

$$f_p(\rho,\delta)=e^{(p\delta+jk\rho\sin\delta)/\omega_c}\Bigg[A(\rho)-\frac{1}{\omega_c}\sum_l\sum_n A_l(\rho)\times J_n\left(-\frac{k\rho}{\omega_c}\right)\frac{e^{[-p/\omega_c+j(l+n)]\delta}-1}{-p/\omega_c+j(l+n)}+\sum_n J_n\left(-\frac{k\rho}{\omega_c}\right)jk\frac{e}{2m\omega_c}\phi_p\frac{df_0}{d\rho}\left\{\frac{e^{[-p/\omega_c+j(n+1)]\delta}-1}{-p/\omega_c+j(n+1)}+\frac{e^{[-p/\omega_c+j(n-1)]\delta}-1}{-p/\omega_c+j(n-1)}\right\}\Bigg].\quad(20)$$

The condition that f_p given by (20) must be periodic in δ with period 2π gives

$$A(\rho)=-\frac{1}{\omega_c}\sum_l\sum_n A_l(\rho)J_n\left(-\frac{k\rho}{\omega_c}\right)\left\{\frac{1}{-p/\omega_c+j(l+n)}\right\}-\sum_n jk\frac{e}{2m\omega_c}\phi_p\frac{df_0}{d\rho}J_n\left(-\frac{k\rho}{\omega_c}\right)\left\{\frac{1}{-p/\omega_c+j(n+1)}+\frac{1}{-p/\omega_c+j(n-1)}\right\}.\quad(21)$$

Thus we have

$$f_p(\rho,\delta)=-\frac{e^{jk\rho\sin\delta/\omega_c}}{\omega_c}\times\Bigg[\sum_l\sum_n A_l(\rho)J_n\left(-\frac{k\rho}{\omega_c}\right)\frac{e^{j(l+n)\delta}}{-p/\omega_c+j(l+n)}+\sum_n J_n\left(-\frac{k\rho}{\omega_c}\right)jk\frac{e}{2m}\phi_p\frac{df_0}{d\rho}\left\{\frac{e^{j(n+1)\delta}}{-p/\omega_c+j(n+1)}+\frac{e^{j(n-1)\delta}}{-p/\omega_c+j(n-1)}\right\}\Bigg].\quad(22)$$

We set (22) back into (13) and obtain

$$\phi_p=\frac{4\pi e}{\omega_c}\frac{\displaystyle\sum_l\sum_n\int\frac{A_l(\rho)J_n(-k\rho/\omega_c)e^{j[(l+n)\delta+k\rho\sin\delta/\omega_c]}}{p/\omega_c-j(l+n)}d\tau}{\displaystyle k^2+jk\frac{2\pi e^2}{m\omega_c}\sum_n\int\frac{df_0}{d\rho}\left\{\frac{e^{j(n+1)\delta}}{-p/\omega_c+j(n+1)}+\frac{e^{j(n-1)\delta}}{-p/\omega_c+j(n-1)}\right\}J_n\left(-\frac{k\rho}{\omega_c}\right)e^{jk\rho\sin\delta/\omega_c}d\tau},\quad(23)$$

where, in our coordinate system (15), $d\tau=\rho d\rho d\delta$.

We use in (23) the following properties of the Bessel functions of integral order:

$$J_{-n}(z)=\frac{1}{2\pi}\int_0^{2\pi}e^{jn\delta}e^{jz\sin\delta}d\delta,$$
$$J_n(-z)=(-1)^nJ_n(z),\quad J_{-n}(z)=(-1)^nJ_n(z),\quad(24)$$

and obtain

$$\phi_p=\frac{8\pi^2e}{\omega_c}\frac{\displaystyle\sum_l\sum_n\frac{(-1)^l}{p/\omega_c-j(l+n)}\int_0^\infty A_l(\rho)J_n\left(\frac{k\rho}{\omega_c}\right)J_{l+n}\left(\frac{k\rho}{\omega_c}\right)\rho d\rho}{\displaystyle k^2-jk\frac{4\pi^2e^2}{m\omega_c}\sum_n\frac{\int_0^\infty J_n\left(\frac{k\rho}{\omega_c}\right)\left\{J_{n+1}\left(\frac{k\rho}{\omega_c}\right)+J_{n-1}\left(\frac{k\rho}{\omega_c}\right)\right\}\frac{df_0}{d\rho}\rho d\rho}{-p/\omega_c+jn}}.\quad(25)$$

We shall suppose, on physical grounds, that the functions $A_l(\rho)$ that determine the initial perturbation and the function $f_0(\rho)$ that determines the unperturbed distribution are such that the integrals in (25) are convergent. When p is an integral multiple of $j\omega_c$, both the numerator and the denominator of ϕ_p in (25) have singularities that cancel out. Thus ϕ_p is a meromorphic function of the complex variable p with poles at the values of p that make the denominator of (25) vanish. In other words, the poles of ϕ_p are the roots p_1, p_2, $\cdots p_r$, $\cdots$ of the equation (in p)

$$4\pi^2\frac{je^2}{mk\omega_c}\sum_n \frac{\displaystyle\int_0^\infty J_n\left(\frac{k\rho}{\omega_c}\right)\left\{J_{n+1}\left(\frac{k\rho}{\omega_c}\right)+J_{n-1}\left(\frac{k\rho}{\omega_c}\right)\right\}\frac{df_0}{d\rho}\rho d\rho}{\displaystyle -\frac{p}{\omega_c}+jn}=1. \tag{26}$$

We shall suppose again on physical grounds that $\phi(t)$ has a continuous derivative and that $|\phi(t)|<Ke^{ct}$, where K and c are positive constants. Then, by (11), we have

$$\phi(t)=\frac{1}{2\pi j}\int_{-j\infty+\gamma}^{+j\infty+\gamma}\phi_p e^{pt}dp, \tag{27}$$

where $\gamma>c$.

Now ϕ_p in (25) is $O(p^{-1})$. Hence the line integral in (27) can be replaced by the integral over any circle C, center the origin, that includes all the poles of ϕ_p.[15] Thus, by Cauchy's residue theorem,

$$\phi(t)=\textstyle\sum_r b_r e^{p_r t}, \tag{28}$$

where p_r is a root of Eq. (26), and b_r is the residue of ϕ_p at the pole p_r.

From (6), the potential ϕ is given by

$$\phi=\textstyle\sum_r b_r e^{jkx+p_r t}. \tag{29}$$

Equation (26) is therefore the required dispersion equation.

The transform of the electron distribution function $f_1(\mathbf{v}, t)$ is the meromorphic function $f_p(\rho, \delta)$ given by (22), which has poles for $p=jn\omega_c$, besides the poles of ϕ_p given by (26). The poles at $p=jn\omega_c$ contribute frequencies that are multiples of the gyrofrequency to the distribution function $f_1(\mathbf{v}, t)$ of the electrons.

Before we can apply the dispersion equation (26), we must specify the unperturbed distribution function $f_0(\rho)$. For a system of electrons gyrating round the magnetic field (Z axis) with a uniform speed v_0, we may write

$$f_0(\rho)=(n_0/2\pi v_0)\delta(\rho-v_0), \tag{30}$$

where n_0 is the electron density and δ is the Dirac δ-function whose derivative has the integral property

$$\int_{-\infty}^{\infty} f(x)\delta'(x-a)dx=-f'(a). \tag{31}$$

Using (30) and (31), we find that the dispersion equation (26) reduces to the form[16]

$$1=-\frac{\omega_0^2}{2\omega_c^2}\frac{1}{\lambda}\frac{d}{d\lambda}\left[\lambda\sum_{-\infty}^{\infty}\frac{J_n(\lambda)\{J_{n-1}(\lambda)+J_{n+1}(\lambda)\}}{n-\omega/\omega_c}\right], \tag{32}$$

where

$$\lambda=kv_0/\omega_c, \tag{33}$$

and ω_0 is the plasma frequency given by

$$\omega_0^2=4\pi n_0 e^2/m. \tag{34}$$

For the Maxwellian distribution function

$$f_0=n_0\left(\frac{m}{2\pi\kappa T}\right)^{\frac{3}{2}}\exp\left[-\frac{m}{2\kappa T}(v_z^2+\rho^2)\right], \tag{35}$$

the dispersion equation (26) reduces to

$$\omega=4\omega_0^2\frac{\omega_c^3}{k^4}\left(\frac{m}{2\kappa T}\right)^2\int_0^\infty \lambda e^{-\lambda^2/2\mu}d\lambda\sum_{n=-\infty}^{+\infty}\frac{n^2J_n^2(\lambda)}{n+\omega/\omega_c}, \tag{36}$$

where

$$\lambda=k\rho/\omega_c \quad\text{and}\quad 1/\mu=(m/\kappa T)(\omega_c^2/k^2). \tag{37}$$

The relation (36), which was obtained by Gross,[6] is further considered in Appendix II.

Gross[6] has considered the dispersion relation (32) for small λ ($\lambda\ll 1$) and has shown that in this case there are gaps in the spectrum at approximate multiples of the gyrofrequency. He believes that "the same type of consideration would seem to hold for all values of λ." He has not, however, undertaken a complete study of the dispersion relation (32) (i.e., for large λ).

The author has thought fit to consider the dispersion equation (32) for $\lambda>1$, and finds that it can be satisfied by complex values of ω, leading to unstable oscillations growing exponentially in time.

We set

$$K=(\omega_0/\omega_c)^2=4\pi N_0 mc^2/H_0^2,\quad \omega/\omega_c=\alpha+j\beta, \tag{38}$$

in the dispersion equation (32), and, equating real and

[15] See reference 14, p. 76.

[16] Gross obtained the dispersion equation in this form from a direct solution of the Boltzmann equation (see reference 5). As Gross assumed the oscillating quantities to vary as $e^{j(kx-\omega t)}$, we have replaced p in (26) by $-j\omega$.

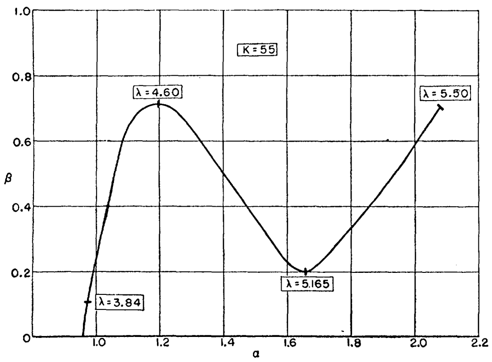

FIG. 1. Curve showing the variations of β (amplification) with α (frequency in units of the gyrofrequency).

imaginary parts, obtain

$$\sum_{n=1}^{+\infty}\frac{n(n^2-\alpha^2+\beta^2)}{(n^2-\alpha^2+\beta^2)^2+4\alpha^2\beta^2}(J_{n+1}{}^2-J_{n-1}{}^2)=\frac{1}{K}$$
$$\sum_{n=1}^{+\infty} n\frac{J_{n+1}{}^2-J_{n-1}{}^2}{(n^2-\alpha^2+\beta^2)^2+4\alpha^2\beta^2}=0. \quad (39)$$

A numerical solution of Eq. (39), for different values of K as a parameter, shows that the equation can be satisfied by real positive values of α^2 and β^2;[17] thus, the dispersion equation (32) can be satisfied by complex ω, indicating amplification of random fluctuations with time.

We will state our results for the particular value of $K=55$, which applies to spot magnetic fields ($H_0\simeq 30$ gauss) at coronal distances ($\simeq 10^{10}$ cm above the photosphere) and to the density of prominance material ($n_0=5\times 10^9$ cm^{-3}).

Figure 1 gives the variation of β with α, and the progression of the λ-values corresponding to the different points.[18] The curve oscillates round multiples of the gyrofrequency, and fairly large amplifications are available over a wide frequency range. The values of β indicate that the amplification will amount to several powers of 10 in a few seconds. We will not quote precise results, as we cannot trust any first-order theory for such large amplitudes. The frequency band width about covers the range of 70–130 Mc/sec in which the Australians[1] have observed "enhanced radiation." Though a nonlinear theory would be required to achieve quantitative accuracy, we believe that the results obtained herein have a qualitative significance, inasmuch as they render plausible the theory of the origin of solar "enhanced radiation" in plasma oscillations of electrons gyrating round the magnetic field of sunspots.

We may note two points in this connection. As the gyrating electrons have a peaked velocity distribution on account of the prominence motion, they do not suffer from the thermodynamic limitation as in the case first treated by Kiepenheuer.[19] Further, amplification is available over a large frequency band width, which will enable the radiation to escape through regions that are normally overdense for the gyrofrequency.[20]

The amplification tends to increase with K as defined by Eq. (38). This tendency may explain the abnormal increases in the enhanced level found at the time of solar flares.[21] The background continuum shows at times short-lived increases to exceptionally high values and of broad band width (tens of megacycles per second). The agency in this case may be the dense corpuscles shot out from sunspots at the time of a solar flare.

The "storm bursts" that occur on the enhanced level[1] have a narrow band width (4 Mc/sec) and seem to owe their origin to a different mechanism that is probably localized in the solar atmosphere, e.g., shock waves due to solar corpuscles moving with supersonic velocity. A physical theory of the propagation of shock waves in an ionized gas subject to a static magnetic field will be relevant to this case.

The author's best thanks are due Miss Loris B. Perry for the numerical solution of Eqs. (39) and the drawing of Fig. 1.

APPENDIX I

The Steady-State Solution of the Boltzmann Equation in a Magnetic Field

Let the distribution function $f=f_0(\mathbf{v}, t)$ be a solution of the Boltzmann equation[22] in the magnetic field H_0 of gyrofrequency ω_c,

$$\frac{\partial f}{\partial t}+\omega_c\left[v_y\frac{\partial f}{\partial v_x}-v_x\frac{\partial f}{\partial v_y}\right]=B\nabla^2{}_{\mathbf{v}}f+3\beta f+\beta\mathbf{v}\cdot\nabla_{\mathbf{v}}f, \quad (1)$$

where

$$B=\beta\kappa T/m, \quad (2)$$

κ being the Boltzmann constant and β the electronic collision frequency.

Put

$$f'=f_0e^{-3\beta t}. \quad (3)$$

Then Eq. (1) reduces to

$$\frac{\partial f'}{\partial t}+\omega_c\left[v_y\frac{\partial f'}{\partial v_x}-v_z\frac{\partial f'}{\partial v_y}\right]=B\nabla^2{}_{\mathbf{v}}f'+\beta\mathbf{v}\cdot\nabla_{\mathbf{v}}f'. \quad (4)$$

[17] For real λ, the quantity within the brackets in Eq. (32) is real. It can be reduced to the form:

$$\sum_1^\infty\frac{4n^2J_n{}^2(\lambda)}{n^2-\alpha^2+\beta^2-2j\alpha\beta}.$$

If this sum is real for any particular α, β, it is real and has the same value for any of the four combinations $\pm\alpha$, $\pm\beta$. Hence the roots of (32) occur in quadruples $\pm\alpha\pm j\beta$.

[18] There seems to be a cut-off near $\lambda=3.84$; see Appendix II.

[19] K. O. Kiepenheuer, Nature **158**, 340 (1946).

[20] M. Ryle, Rept. Prog. Phys. **XIII**, 229 (1950).

[21] Reference 1, p. 42.

[22] S. Chandrasekhar, Revs. Modern Phys. **15**, 35 (1943), Eq. (249).

Change to cylindrical coordinates:

$$v_x = \rho \cos\delta, \quad v_y = \rho \sin\delta, \quad v_z = v_z. \tag{5}$$

Then, Eq. (5) reduces to

$$\frac{\partial f'}{\partial t} - \beta\rho\frac{\partial f'}{\partial \rho} - \omega_c\frac{\partial f'}{\partial \delta} - \beta v_z\frac{\partial f'}{\partial v_z} = B\left[\frac{\partial^2 f'}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial f'}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 f'}{\partial \delta^2} + \frac{\partial^2 f'}{\partial v_z^2}\right]. \tag{6}$$

The left-hand side of (6) equated to zero is a linear, homogeneous first-order partial differential equation, and its solution can be derived from the following Lagrangian subsidiary equations:

$$dt = d\rho/-\beta\rho = d\delta/-\omega_c = dv_z/-\beta v_z. \tag{7}$$

Independent integrals of (7) are

$$\rho e^{\beta t} = c_1, \quad \delta + \omega_c t = c_2, \quad \text{and} \quad v_z e^{\beta t} = c_3. \tag{8}$$

Accordingly, to integrate (6), we make the following change of variables:

$$\rho e^{\beta t} = \xi, \quad \delta + \omega_c t = \eta, \quad v_z e^{\beta t} = \zeta, \quad t = t. \tag{9}$$

Equation (6) reduces, in the new variables (ξ, η, ζ, t), to

$$\frac{\partial f'}{\partial t} = B\left[\frac{\partial^2 f'}{\partial \xi^2} + \frac{1}{\xi}\frac{\partial f'}{\partial \xi} + \frac{1}{\xi^2}\frac{\partial^2 f'}{\partial \eta^2} + \frac{\partial^2 f'}{\partial \zeta^2}\right] e^{2\beta t}. \tag{10}$$

The right-hand side of (10) is the Laplacian of f' in the cylindrical coordinates

$$\xi' = \xi \cos\eta, \quad \eta' = \xi \sin\eta, \quad \zeta' = \zeta. \tag{11}$$

With the substitution (11), Eq. (10) reduces to

$$\partial f'/\partial t = Be^{2\beta t}\nabla_{\xi'}^2 f'. \tag{12}$$

We now apply the following lemma:[23]

If $\phi(t)$ is an arbitrary function of time, the solution of the partial differential equation,

$$\partial\chi/\partial t = \phi^2(t)\nabla_\rho^2\chi, \tag{13}$$

which has a source at $\boldsymbol{\varrho} = \boldsymbol{\varrho}_0$ at time $t=0$, is

$$\chi = \frac{1}{\left[4\pi\int_0^t \phi^2(t)dt\right]^{\frac{3}{2}}} \exp\left[-|\boldsymbol{\varrho} - \boldsymbol{\varrho}_0|^2/4\int_0^t \phi^2(t)dt\right]. \tag{14}$$

Applying the above lemma to Eq. (12) and going back to the old variables (ρ, δ, v_z, t) by means of Eqs. (9) and (11), we find that

$$f_0(\mathbf{v}, t) = [(2\pi B/\beta)(1-e^{-2\beta t})]^{-\frac{3}{2}} \exp[-\{\rho\cos(\delta+\omega_c t) - \rho_0\cos\delta_0 e^{-\beta t}\}^2 - \{\rho\sin(\delta+\omega_c t) - \rho_0\sin\delta_0 e^{-\beta t}\}^2 - (v_z - v_0 e^{-\beta t})^2]/[(2B/\beta)(1-e^{-2\beta t})]. \tag{15}$$

[23] See reference 22, p. 34, Lemma I.

As $t\to\infty$, the distribution function given by (15) tends to the Maxwellian one,

$$f_0(\mathbf{v}) = \left(\frac{m}{2\pi\kappa T}\right)^{\frac{3}{2}} \exp\left[-\frac{m}{2\kappa T}|\mathbf{v}|^2\right], \tag{16}$$

on substituting (2).

Note that the presence of the magnetic field does not affect the steady-state solution of the Boltzmann equation. Appendix I gives an alternative proof of this well-known result[10] via the Fokker-Planck diffusion equation (1).

APPENDIX II

A Closed Expression for the Dispersion Equation (32)

The infinite sum in the dispersion equation (32) can be expressed in a closed form as follows:

Let

$$S(\lambda, \alpha) \equiv \lambda \sum_{n=-\infty}^{\infty} \frac{J_n(\lambda)\{J_{n-1}(\lambda) + J_{n+1}(\lambda)\}}{n-\alpha}, \tag{1}$$

where

$$\omega/\omega_c = \alpha. \tag{2}$$

Then

$$S(\lambda, \alpha) = 4\sum_{n=1}^{\infty} \frac{n^2 J_n^2(\lambda)}{n^2-\alpha^2} = 4\sum_1^\infty J_n^2(\lambda) + 4\alpha^2 \sum_1^\infty \frac{J_n^2(\lambda)}{n^2-\alpha^2}. \tag{3}$$

Now,[24]

$$(-1)^n J_n^2(\lambda) = \frac{2}{\pi}\int_0^{\pi/2} J_0(2\lambda\cos\theta)\cos(2n\theta)d\theta, \tag{4}$$

and

$$\sum_1^\infty (-1)^n \frac{\cos 2n\theta}{n^2-\alpha^2} = \frac{1}{2\alpha^2} - \frac{\pi}{2}\frac{\cos(2\alpha\theta)}{\alpha\sin(\pi\alpha)}. \tag{5}$$

Hence,

$$\sum_1^\infty \frac{J_n^2(\lambda)}{n^2-\alpha^2} = \frac{1}{\pi\alpha^2}\int_0^{\pi/2} J_0(2\lambda\cos\theta)d\theta - \int_0^{\pi/2} \frac{J_0(2\lambda\cos\theta)\cos(2\alpha\theta)d\theta}{\alpha\sin(\pi\alpha)} = \frac{1}{2\alpha^2}J_0^2(\lambda) - \frac{\pi J_\alpha(\lambda)J_{-\alpha}(\lambda)}{2\alpha\sin(\pi\alpha)}. \tag{6}$$

Also,

$$\sum_1^\infty J_n^2(\lambda) = \tfrac{1}{2} - \tfrac{1}{2}J_0^2(\lambda). \tag{7}$$

[24] W. Magnus and F. Oberhettinger, *Formulas and Theorems for the Special Functions of Mathematical Physics* (Chelsea Publishers, New York, 1949), p. 28, Sec. 6.

Hence,

$$S(\lambda, \alpha)=2\left\{1-\frac{\pi\alpha}{\sin(\pi\alpha)}J_\alpha(\lambda)J_{-\alpha}(\lambda)\right\}.^{25} \quad (8)$$

From the series expansion[26]

$$J_\alpha(\lambda)J_{-\alpha}(\lambda)=\sum_{m=0}^{\infty}\frac{(-)^m(\frac{1}{2}\lambda)^{2m}(2m)!}{(m!)^2\Gamma(\alpha+m+1)\Gamma(-\alpha+m+1)}, \quad (9)$$

where α is not an integer, we can show that the dispersion equation (32) reduces, when $\lambda \ll 1$, to

$$\frac{2(\alpha^2-4)+3\lambda^2}{2(1-\alpha^2)(4-\alpha^2)}=\frac{1}{K}, \quad (10)$$

where K is given by (38).

Equation (10) gives, for $\lambda=0$, the relation for a static plasma, *viz.*,

$$\omega^2=\omega_0^2+\omega_c^2, \quad (11)$$

where ω_0 is the plasma frequency given by (34).

Setting in Eq. (10),

$$\alpha=\frac{\omega}{\omega_c}=\sqrt{x}+j\sqrt{y}, \quad (12)$$

equating real and imaginary parts, and eliminating x, we have

$$16y^2+8(5+K)y+(K-3)^2+6\lambda^2K=0. \quad (13)$$

Neither root of Eq. (13) can be real and positive. Hence the dispersion equation (32) cannot have complex roots for small $\lambda(\ll 1)$. This result is in conformity with Gross' conclusion[6] and the lower cut-off limit for λ in our Fig. 1.

The Dispersion Relation for a Maxwellian Distribution

The integral in the dispersion relation (36) for the Maxwellian distribution function (35) can be expressed in terms of the generalized hypergeometric function[27] ${}_2F_2$:

$$\int_0^\infty \lambda e^{-\lambda^2/2\mu}d\lambda \sum_{n=-\infty}^{+\infty}\frac{n^2J_n^2(\lambda)}{n+\alpha}$$
$$=\alpha\mu[{}_2F_2(1, \tfrac{1}{2}; 1+\alpha, 1-\alpha; -2\mu)-1], \quad (14)$$

where α is given by (2).

Formula (14) follows from Eq. (8) and the following integral relation:[28]

$$\int_0^\infty \lambda \exp(-a\lambda^2)J_\alpha(\lambda)J_{-\alpha}(\lambda)d\lambda$$
$$=\frac{\sin(\pi\alpha)}{2a\pi\alpha}{}_2F_2\left(1, \tfrac{1}{2}; 1+\alpha, 1-\alpha; -\frac{1}{a}\right). \quad (15)$$

We shall assume that

$$\mu/\alpha^2=(\kappa T/m)(k^2/\omega^2)\ll 1. \quad (16)$$

Then using only the first two terms of the series expansion of ${}_2F_2$ in (14), we can reduce the dispersion equation (36) to

$$\omega^2=\omega_c^2+\omega_0^2-3\mu\omega_0^2/(4-\alpha^2). \quad (17)$$

Equation (17) again reduces to (11) for $T\rightarrow 0$. In the first approximation, Eq. (11) can be set for α in (17), which will lead to the following relation:

$$\omega^2=\omega_c^2+\omega_0^2-\frac{3\omega_0^2(\kappa T/m)k^2}{3\omega_c^2-\omega_0^2}$$
$$=\omega_c^2+\omega_0^2+3\frac{\kappa T}{m}k^2\left(1+3\frac{\omega_c^2}{\omega_0^2}\right), \quad (18)$$

when $\omega_c^2/\omega_0^2\ll 1$.

Equation (18), which leads to the correct limiting forms for zero magnetic field and/or temperature, should be compared with Gross'[6] Eq. (30).

---

[25] Formula (8) is due to Dr. J. J. Freeman.

[26] G. N. Watson, *Theory of Bessel Functions* (Cambridge University Press, Cambridge, 1944), p. 147.

[27] Formula (14) is due to Dr. Fritz Oberhettinger.

[28] See reference 26, p. 396.