Meson Theory of Nuclear Forces and Low Energy Properties of the Neutron-Proton System*

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A detailed analysis is made of the neutron-proton interaction yielded by the symmetrical pseudoscalar meson theory, with pseudoscalar coupling, using the Tamm-Dancoff nonadiabatic method which has been extended to include nucleon pair creation and higher order effects in the exchange of mesons. It is found that, in the nonrelativistic region, the second- and fourth-order terms provide the main contribution to the interaction, the remaining part of the potential giving only a small correction. In the relativistic region, little can be said about the convergence of the interaction, but there are indications that it becomes strongly repulsive at distances comparable with the nucleon Compton wavelength ($\hbar/Mc$). The radiative corrections to the potential are calculated in the nonrelativistic limit, using the equation of Bethe and Salpeter, which has been transformed into a one-time equation by means of a method which has been given previously. It is shown that the corrections arising from vertex parts and closed loops in the Feynman diagrams are at most of the order of $(G^2/4\pi)(\mu/2M)^2$ times the term which they correct. There exists, however, a class of finite self-energy terms which give a contribution to the interaction having the same analytical form as the fourth-order potential, times a numerical factor which can be expressed as a power series in $G^2/4\pi$.

The low energy properties of the neutron-proton system are discussed, using the nonrelativistic potential which is calculated in this paper, and replacing the interaction in the relativistic region by a boundary condition prescribing that the wave function tends to zero at a finite distance r_c. It is found that a good agreement with experiment can be achieved by choosing $G^2/4\pi=9.7\pm1.3$ and $r_c=(0.38\pm0.03)(\hbar/\mu c)$. Finally, an investigation of the neutron-proton scattering at 40 Mev shows that the same potential leads to a satisfactory description of the available experimental data.

I. INTRODUCTION

THE pseudoscalar meson field[1] theory has not yet progressed very far in the quantitative description of nuclear forces, mainly because of the two following difficulties:

(1) The calculation of the nucleon-nucleon interaction, to the second order in the coupling constant[2] G, leads to a static approximation which is too strongly singular near the origin to permit the existence of stationary states. It is clear, however, that this singularity has no physical meaning, since the static interaction is no longer valid for distances of the order of the nucleon Compton wavelength ($1/M$).

(2) The high value of $G^2/4\pi$ which is obtained by fitting the simplest low energy properties of the neutron-proton system casts strong doubts on the validity of any perturbation expansion of the interaction. Unfortunately, it appears very difficult to treat the pseudoscalar meson field by other methods ("strong" or "intermediate" coupling), since the nucleons cannot be replaced, in this case, by infinitely massive extended sources, because of the importance of nucleons pair creation.[3]

It should be noted, however, that few serious attempts have been made so far to determine to what extent these difficulties are real.[4] The nonstatic effects might, for example, lower the singularity of the interaction, or even change its sign at very small distances. On the other hand, the creation of virtual nucleon pairs might improve, at least in some region, the convergence of the perturbation expansion, the effective expansion parameter being, in that region, much smaller than $G^2/4\pi$.

It is the purpose of the present paper to investigate critically the above difficulties by means of a term by term analysis of the interaction expansion yielded by a weak coupling treatment of the symmetrical pseudoscalar meson theory, with pseudoscalar coupling. This analysis includes not only the finite parts of the interaction, but also the radiative corrections in which the

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[1] Only the pseudoscalar coupling will be considered in this paper, since it is only in this case that the intrinsic field theoretical infinities can be separated and re-interpreted consistently. See J. C. Ward, Phys. Rev. **84**, 897 (1951); A. Salam, Phys. Rev. **84**, 426 (1951); **86**, 731 (1952).

[2] The physical evidence having considerably reduced in recent years the possibilities of different types of π-meson fields and couplings, it seems useful to replace the conventional nomenclature [see for example, L. Rosenfeld, *Nuclear Forces* (North Holland Publishing Company, Amsterdam, 1948), p. 322] by a simplified system of notations. We denote by G the pseudoscalar coupling constant of a pseudoscalar field (suggesting by this choice that this constant is expected to be big!) and by g the quantity $G(\mu/2M)$, which can be considered as an equivalent pseudovector coupling constant in the cases where the equivalence theorem is valid [see, E. C. Nelson, Phys. Rev. **60**, 830 (1941); F. J. Dyson, Phys. Rev. **73**, 929 (1948); K. M. Case, Phys. Rev. **76**, 14 (1949)]. Both $G^2/4\pi$ and $g^2/4\pi$ are expressed in the same units as the fine structure constant in electrodynamics. A system of units where $\hbar=c=1$ is used in this paper.

[3] It is an essential feature of the γ_5 coupling that the matrix elements which simply create or annihilate a meson are, in the nonrelativistic limit, much smaller than those which, in addition, create or annihilate a pair of nucleons.

[4] The fourth-order interaction and its influence on nuclear forces have been discussed by several authors, for example: H. A. Bethe, Phys. Rev. **76**, 191 (1949); K. M. Watson and J. V. Lepore, Phys. Rev. **76**, 1157 (1949); Y. Nambu, Prog. Theor. Phys. **5**, 614 (1950); G. Wentzel, Phys. Rev. **86**, 802 (1952); etc. Nobody has, however, investigated seriously the behavior of the remaining parts of the static interaction and the effect of the nonstatic corrections.

mass and coupling constant have to be renormalized explicitly. In order to carry out this program, use has been made of the nonadiabatic treatment of Tamm[5] and Dancoff[6] which, in a previous paper,[7] has been extended to include the creation of virtual nucleon pairs and the exchange of an arbitrary number of mesons. In the framework of this formalism, it is necessary to give first the definition of an interaction operator having a general validity. This operator, which is defined in Sec. II, is similar to the "velocity dependent potential" introduced by Wheeler,[8] but has the added feature that it depends explicitly on the total energy of the system.

In Sec. III, the part of the interaction which contains no contribution arising from radiative processes will be analyzed term by term. It will be shown that, in the nonrelativistic region,[9] even if $G^2/4\pi$ is of the order of 10, there exists a finite number of terms (namely, the second-order and the largest part of the fourth-order potential) which can be used as the starting basis of a perturbation treatment, all the remaining portions of the interaction being considered as small corrections. In the relativistic region, the effective expansion parameter is $G^2/4\pi$, and little can be said about the convergence of the interaction. There are, however, some indications that it becomes repulsive at short distances, the lowest order terms of the potential being dominated by the so-called "contact" terms, which actually have a range of order $\hbar/Mc$. On the other hand, the pseudoscalar meson theory should not be expected to give too reliable results in that region, on account of the existence of heavier mesons than π-mesons, isobaric states of nucleons, etc. . . .

The contributions to the interaction arising from the radiative processes will be calculated in Sec. IV, starting from the equation of Bethe and Salpeter[10] and transforming it into a one-time equation by means of a method which has been given previously.[7] It will be found that the correction contributed by vertex parts and closed loops in the irreducible Feynman diagrams are at most of the order of $(G^2/4\pi)(\mu/2M)^2$ times the interaction term which they correct (μ and M are respectively the meson and nucleon masses). There exists, however, a class of finite self-energy terms, which give a contribution to the interaction which is not negligible. In the nonrelativistic region this contribution has the same analytical form as the fourth-order potential, times a numerical factor which can be expressed as a power series in $G^2/4\pi$.

[5] I. Tamm, J. Phys. (USSR) **9**, 449 (1945).
[6] S. M. Dancoff, Phys. Rev. **78**, 382 (1950).
[7] M. Lévy, Phys. Rev. **88**, 72 (1952). This paper will be referred to as (A) in the following.
[8] J. A. Wheeler, Phys. Rev. **50**, 643 (1936).
[9] By "nonrelativistic region" is meant the part of the momentum distribution in which the nuclear recoils can be neglected ($|p|\ll M$). In coordinate space, this part of the interaction coincides with the static potential which still depends, however, on the total energy of the system.
[10] E. Salpeter and H. A. Bethe, Phys. Rev. **84**, 1232 (1951); M. Gell-Mann and F. Low, Phys. Rev. **84**, 350 (1951). This equation will sometimes be referred to as (B.S.) in the following.

The preceding features of the pseudoscalar interaction permit a semiquantitative description of the neutron-proton system at low and intermediate energies. In the nonrelativistic region the exact potential calculated in this paper can be used, the relativistic part of the interaction being arbitrarily replaced by a boundary condition, prescribing that the wave function tends to zero at a finite distance r_c. Considering the relatively large value of the ratio (μ/M), the separation of a nonrelativistic region where the potential can be calculated exactly is, to a certain extent, arbitrary. It can therefore be expected that the corresponding semiquantitative treatment becomes less and less satisfactory as the energy of the system increases. It will be shown, in Sec. V, that a good agreement with the low energy properties of the neutron-proton system can be achieved by choosing $G^2/4\pi\simeq 9.7\pm1.3$ and $r_c=(0.38\pm0.03)(\hbar/\mu c)$. The same constants lead to a good description of the available experimental data on the neutron-proton scattering at 40 Mev.

II. DEFINITION AND GENERAL PROPERTIES OF THE INTERACTION

In the nonadiabatic treatment of the two-body problem,[7] the state vector of the system is defined by means of a set of probability amplitudes $a_\lambda^{(m,n)}$ of the free states where m mesons and n nucleon pairs are present; the two initially interacting nucleons are treated separately, and λ is a variable which specifies the momenta, spins, isotopic spins, etc., of the particular free state which is considered.

The set $[a_\lambda^{(m,n)}]$ satisfies a system of simultaneous integral equations which has been discussed in detail in (A). If, in particular, one eliminates all the amplitudes except $a_\lambda^{(0,0)}$ by means of successive substitutions, one obtains the general equation (A, 6) which we now write in the center-of-mass system as follows:

$$(W-2E_p)a^{(0,0)}(\mathbf{p},-\mathbf{p}) = (2\pi)^{-3}\int K(\mathbf{p},\mathbf{p}';W)a^{(0,0)}(\mathbf{p}',-\mathbf{p}')d\mathbf{p}',\quad (1)$$

in which W is the total energy of the system and $E_p=(p^2+M^2)^{\frac{1}{2}}$. The kernel $K(\mathbf{p},\mathbf{p}';W)$ is expressed as a power series of G^2:

$$K(\mathbf{p},\mathbf{p}';W=\textstyle\sum_n G^{2n}K_{2n}(\mathbf{p},\mathbf{p}';W).\quad (2)$$

Multiplying both sides of (1) by $(W+2M)^{-1}(W+2E_p)$ and defining $\epsilon=W-2M$ as the binding energy (or, for an unbound state, the total nonrelativistic energy in the center-of-mass system), we obtain the equation

$$\left(\frac{p^2}{M\rho}-\epsilon\right)u^{(0,0)}(\mathbf{p}) = -(2\pi)^{-3}\int\frac{[(2E_p+W)(2E_{p'}+W)]^{\frac{1}{2}}}{4M\rho}\times K(\mathbf{p},\mathbf{p}';W)u^{(0,0)}(\mathbf{p}')d\mathbf{p}',\quad (3)$$

where we have set $\rho=1+\epsilon/4M$ and introduced the amplitude[11]

$$u^{(0,0)}(\mathbf{p})=\left[\frac{4M\rho}{2E_p+W}\right]^{\frac{1}{2}}a^{(0,0)}(\mathbf{p},-\mathbf{p}). \quad (4)$$

Transforming Eq. (3) into coordinate space, we get

$$\left(\frac{\nabla^2}{M\rho}+\epsilon\right)\phi^{(0,0)}(\mathbf{r})=\int U(\mathbf{r},\mathbf{r}';W)\phi^{(0,0)}(\mathbf{r}')d\mathbf{r}', \quad (5)$$

where

$$\phi^{(0,0)}(\mathbf{r})=(2\pi)^{-\frac{3}{2}}\int u^{(0,0)}(\mathbf{p})e^{i\mathbf{pr}}d\mathbf{p} \quad (6)$$

is the Fourier transform of $u^{(0,0)}(\mathbf{p})$ and $U(\mathbf{r},\mathbf{r}';W)$ the interaction operator

$$U(\mathbf{r},\mathbf{r}';W)=\sum_n G^{2n}U_{2n}(\mathbf{r},\mathbf{r}';W)$$
$$=\sum_n\frac{G^{2n}}{(2\pi)^3}\int\frac{[(2E_p+W)(2E_{p'}+W)]^{\frac{1}{2}}}{4M\rho}\times K_{2n}(\mathbf{p},\mathbf{p}';W)e^{i\mathbf{pr}-i\mathbf{p}'\mathbf{r}'}d\mathbf{p}d\mathbf{p}'. \quad (7)$$

Equation (5) is an ordinary Schrödinger equation for a particle of reduced mass $M\rho$ and energy ϵ, and, therefore, the interaction operator of the right hand side is unambiguously defined. By writing, on the right-hand side of (7), $\mathbf{pr}-\mathbf{p}'\mathbf{r}=\frac{1}{2}[(\mathbf{p}+\mathbf{p}')(\mathbf{r}-\mathbf{r}')+(\mathbf{p}-\mathbf{p}')(\mathbf{r}+\mathbf{r}')]$, one sees easily that, for a given value of $|\mathbf{r}+\mathbf{r}'|$, $U(\mathbf{r},\mathbf{r}';W)$ is a function of $|\mathbf{r}-\mathbf{r}'|$ of the Gaussian type, the width of which becomes more and more narrow when $|\mathbf{r}+\mathbf{r}'|$ increases. We have in fact, for $|\mathbf{r}+\mathbf{r}'|\gg(1/M)$:

$$U_{2n}(\mathbf{r},\mathbf{r}';W)\rightarrow\delta\left(\frac{\mathbf{r}-\mathbf{r}'}{2}\right)V_{2n}\left(\frac{|\mathbf{r}+\mathbf{r}'|}{2}\right). \quad (8)$$

Consequently, when *r or r′* is large compared with the nucleon Compton wavelength, the interaction operator reduces to a usual potential.

For small values of r, the analysis of the interaction becomes a relatively complicated problem,[12] since its value at a given point $\mathbf{r}$ cannot be separated from that of the wave function in a certain volume around $\mathbf{r}$. It is clear, however, that the main contribution to the right-hand side of (5) arises, in this case, from the part of the right-hand side of (3) in which both the momentum $|\mathbf{p}|$ and the momentum transfer $|\mathbf{p}-\mathbf{p}'|$ are large compared with the meson mass μ. In the case where $\phi^{(0,0)}(\mathbf{r})$ is finite at (and does not vary too rapidly in the neighborhood of) the origin, the extreme relativistic behavior of the right-hand side of (7) can therefore be expressed as follows:

$$\int U(\mathbf{r},\mathbf{r}';W)\phi^{(0,0)}(\mathbf{r}')d\mathbf{r}'$$
$$\rightarrow\frac{\phi^{(0,0)}(0)}{(2\pi)^3}\int\left(\frac{2E_p+W}{4M\rho}\right)^{\frac{1}{2}}K(\mathbf{p},0;W)e^{i\mathbf{pr}}d\mathbf{p}. \quad (9)$$

In the case where $\phi^{(0,0)}(\mathbf{r})$ is not finite at the origin (in particular, when it tends to zero on account of a strong repulsion at short distances), Eq. (9) still corresponds to a good approximation, if $\phi^{(0,0)}(0)$ is replaced by a suitable constant of the form $\int F(\mathbf{p}')u^{(0,0)}(\mathbf{p}')d\mathbf{p}'$, provided that the wave function is mainly concentrated in the nonrelativistic region, which it is likely to be in this particular case.

III. ANALYSIS OF THE INTERACTION (WITHOUT RADIATIVE CORRECTIONS)

In this section, a detailed study of the interaction will first be made in the nonrelativistic region (with the exception of the radiative corrections which will be considered in the next section). A discussion of the behavior of the interaction in the relativistic region will follow.

A. Nonrelativistic Region

As was seen in the preceding section, the interaction reduces in this case, to an ordinary potential $V(r)$, which is obtained[13] by setting, in the expression of $K(\mathbf{p},\mathbf{p}';W)$, as well as on the right-hand side of (7): $E_p=E_{p'}=M$. Consequently, $u^{(0,0)}(\mathbf{p})$ and $a^{(0,0)}(\mathbf{p},-\mathbf{p})$ are, in this case, identical, and $\phi^{(0,0)}(\mathbf{r})$ is simply the Fourier transform of $a^{(0,0)}(\mathbf{p},-\mathbf{p})$. Furthermore, the total energy of the system W wil be equaled to $2M$, since only the low energy properties of the nucleon-nucleon system are considered in this paper. A remark will however be made at the end of this section on the influence of the energy dependence of the interaction on the high energy nucleon-nucleon scattering.

1. *Second-order potential*

The first term in the expansion (2) can be written [see Eq. (A, 7)]:

$$K_2(\mathbf{p},\mathbf{p}';W)=-G^2(\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)\frac{(M+E_p)(M+E_{p'})}{4E_pE_{p'}}\times\frac{(\boldsymbol{\sigma}_1\cdot\mathbf{R})(\boldsymbol{\sigma}_2\cdot\mathbf{R})}{\omega(\mathbf{p},\mathbf{p}')[\omega(\mathbf{p},\mathbf{p}')+E_p+E_{p'}-W]}, \quad (10)$$

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[11] This new amplitude is introduced in order to preserve the symmetry of the interaction operator with respects to the variables $\mathbf{r}$ and $\mathbf{r}'$.

[12] Except in the particular case where the kernel $K(\mathbf{p},\mathbf{p}';W)$ can be broken into a product of two functions of p and p', respectively.

[13] In the calculation of the different terms of the kernel $K_n(\mathbf{p},\mathbf{p}';W)$ there will occur integrations over some intermediate momenta, such as $\mathbf{p}''$. In principle, these integrations must be carried out exactly; it can easily be shown, however, that, by setting also in those terms $E_p''=M$, only corrections of order $(\mu/M)^2$ are neglected, except when the corresponding integral becomes divergent if this approximation is made. In this case, the terms involving $\mathbf{p}''$ must evidently be left untouched.

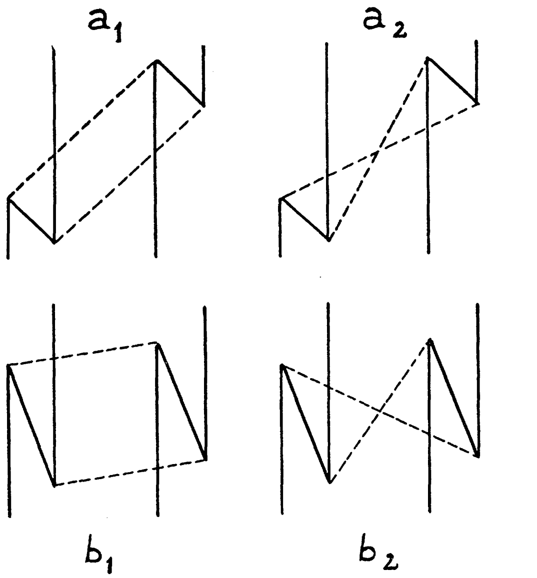

FIG. 1. The virtual processes which give the main contribution to the fourth-order interaction, and which involve the creation of two nucleon pairs in the intermediate states.

where use has been made of the notation:

$$\omega=[|\mathbf{p}-\mathbf{p}'|^2+\mu^2]^{\frac{1}{2}},$$

and

$$\mathbf{R}=\mathbf{p}'/(M+E_p')-\mathbf{p}/(M+E_p). \tag{11}$$

For distances which are large compared with $(1/M)$, one obtains, using (7) and (8):

$$V_2(r)=-\frac{G^2}{(2\pi)^3}\frac{\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2}{(2M)^2}\int\frac{(\boldsymbol{\sigma}_1\cdot\mathbf{k})(\boldsymbol{\sigma}_2\cdot\mathbf{k})}{\omega_k{}^2}e^{i\mathbf{k}\mathbf{r}}d\mathbf{k}, \tag{12}$$

which gives the well-known expression:

$$V_2(r)=\tfrac{1}{3}(\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)\frac{G^2}{4\pi}\left(\frac{\mu}{2M}\right)^2 \times\left\{(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)+S_{12}\left[1+\frac{3}{\mu r}+\frac{3}{(\mu r)^2}\right]\right\}\frac{e^{-\mu r}}{r}, \tag{13}$$

where $S_{12}=3(\boldsymbol{\sigma}_1\cdot\mathbf{r})(\boldsymbol{\sigma}_2\cdot\mathbf{r})/r^2-(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)$. On the right-hand side of (13) we have omitted the so-called "contact" term: $\frac{1}{3}(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)(\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)(2M)^{-2}\delta(\mathbf{r})$, which gives no contribution in the region under consideration. We shall return to this term in paragraph (*B*) of this section.

2. *Fourth-Order Potential*

Because of the special properties of the matrix element of γ_5 (see footnote 3), the main contribution to a given order of the interaction arises, in the nonrelativistic region, from the virtual processes which involve the creation and annihilation of the maximum number of pairs possible to that order. The main part of the fourth order potential will therefore be provided by the virtual processes involving the presence of two nucleon pairs in the intermediate states. Another feature of the fourth-order potential, which is, however, purely accidental, is that the contribution from the processes involving one nucleon pair in the intermediate states vanishes in the nonrelativistic region, in the case of the symmetrical theory.[14] Finally, the part of the potential which is contributed by the processes involving an ordinary two-mesons exchange (and no pairs) is of the order of $(g^2/4\pi)^2=(G^2/4\pi)^2(\mu/2M)^4$ and can be neglected. The part of the interaction which contains all the contributions up to the order of $(G^2/4\pi)^2(\mu/2M)^3$ has the following expression:

$$V_4(r)=-3\left(\frac{G^2}{4\pi}\right)^2\left(\frac{\mu}{2M}\right)^2\frac{1}{\mu r^2} \times\left\{\frac{2}{\pi}K_1(2\mu r)+\frac{\mu}{2M}\left[\frac{2}{\pi}K_1(\mu r)\right]^2\right\}, \tag{14}$$

where $K_n(x)$ is the nth-order Hankel function of imaginary argument.[15]

a. Two-pair terms.—These terms are provided by the virtual processes illustrated in diagrams a_1, a_2, b_1, b_2 of Fig. 1 (plus those which are symmetrical with respect to particles 1 and 2). Diagrams a_1 and a_2, which involve a maximum number of two mesons but only one pair at a given time, give the main contribution

$$V_4{}^{(a)}=-\frac{G^4}{(2\pi)^6}\frac{\tau_\lambda{}^{(1)}\tau_\mu{}^{(1)}[\tau_\lambda{}^{(2)}\tau_\mu{}^{(2)}+\tau_\mu{}^{(2)}\tau_\lambda{}^{(2)}]}{(2M)^2} \times\int\frac{\exp[-i(\mathbf{k}_1+\mathbf{k}_2)\mathbf{r}]}{4\omega_1\omega_2(\omega_1+\omega_2)}d\mathbf{k}_1d\mathbf{k}_2 +\text{symmetrical term in (1) and (2)}, \tag{15}$$

and consequently $V_4{}^{(a)}=-3(G^2/4\pi)^2(2M)^{-2}J_1(r)$, where $J_1(r)$ is a function calculated in the Appendix [Eq. (10a)]. Diagrams b_1 and b_2, where a maximum number of two pairs is present at a given time, give a contribution which is smaller by a factor $(\mu/2M)$:

$$V_4{}^{(b)}=-\frac{G^4}{(2\pi)^6}\frac{\tau_\lambda{}^{(1)}\tau_\mu{}^{(1)}[\tau_\lambda{}^{(2)},\tau_\mu{}^{(2)}]_+}{(2M)^3}\int\frac{\exp[-i(\mathbf{k}_1+\mathbf{k}_2)\mathbf{r}]}{4\omega_1\omega_2}d\mathbf{k}_1d\mathbf{k}_2,$$

and, therefore, $V_4{}^{(b)}=-3(G^2/4\pi)^2(2M)^{-3}(2/\pi)[I_0(r)]^2$, where $I_n(r)$ is defined by Eq. (5a) of the Appendix. In Eqs. (15) and (16) we have set $\omega_i=(k_i{}^2+\mu^2)^{\frac{1}{2}}$. Adding the expressions of $V_4{}^{(a)}$ and $V_4{}^{(b)}$ gives Eq. (14).

b. One-pair terms.—These processes (plus those which are symmetrical with respect to (1) and (2)) are illustrated in Fig. 2. Diagrams ($2a_1$) and ($2a_2$) give

$$V_4{}^{(a')}=\frac{G^4}{(2\pi)^6}(2M)^{-3}\int\{\tau_\lambda{}^{(1)}\tau_\mu{}^{(1)}[\tau_\lambda{}^{(2)},\tau_\mu{}^{(2)}]_+(\boldsymbol{\sigma}_1\cdot\mathbf{k}_1)(\boldsymbol{\sigma}_2\cdot\mathbf{k}_2) +\tau_\lambda{}^{(2)}\tau_\mu{}^{(2)}[\tau_\lambda{}^{(1)},\tau_\mu{}^{(1)}]_+(\boldsymbol{\sigma}_2\cdot\mathbf{k}_1)(\boldsymbol{\sigma}_2\cdot\mathbf{k}_2)\} \times\frac{\exp[-i(\mathbf{k}_1+\mathbf{k}_2)\mathbf{r}]}{4\omega_1{}^2\omega_2{}^2}d\mathbf{k}_1d\mathbf{k}_2$$
$$=-\frac{3G^4}{(2\pi)^6}(2M)^{-3}\int\frac{\mathbf{k}_1\cdot\mathbf{k}_2\exp[-i(\mathbf{k}_1+\mathbf{k}_2)\mathbf{r}]}{\omega_1{}^2\omega_2{}^2}d\mathbf{k}_1d\mathbf{k}_2. \tag{17}$$

[14] This result is only true in the symmetrical theory. The neutral theory, for example, gives the repulsive potential: $(G^2/4\pi)^2 \times(\mu/2M)^3(\mu r^2)^{-1}[1+(\mu r)^{-1}]^2e^{-2\mu r}$.

[15] For the properties of these functions, see G. N. Watson, *Theory of Bessel Functions* (Cambridge University Press, Cambridge, 1922).

The contribution of $(2b_1)$ and $(2b_2)$ is, on the other hand,

$$V_4^{(b')}=-\frac{3G^4}{(2\pi)^6}(2M)^{-3}\int\frac{\exp[-i(\mathbf{k}_1+\mathbf{k}_2)\mathbf{r}]\mathbf{k}_1\cdot\mathbf{k}_2}{\omega_1^2\omega_2(\omega_1+\omega_2)}d\mathbf{k}_1d\mathbf{k}_2, \quad (18)$$

and that of diagrams $(2c_1)$ and $(2c_2)$:

$$V_4^{(c')}=-\frac{3G^4}{(2\pi)^6}(2M)^{-3}\int\frac{\exp[-i(\mathbf{k}_1+\mathbf{k}_2)\mathbf{r}]\mathbf{k}_1\cdot\mathbf{k}_2}{\omega_1\omega_2^2(\omega_1+\omega_2)}d\mathbf{k}_1d\mathbf{k}_2. \quad (19)$$

It can easily be verified that $V_4^{(a')}+V_4^{(b')}+V_4^{(c')}=0$.

c. No-pair terms.—These are the customary two meson exchange terms (with or without crossing of the meson lines) which have been discussed in detail in (*A*, Sec. 2.22). In the case of the pseudoscalar interaction, they can easily be brought, in the nonrelativistic region, into the following form:

$$\begin{aligned}V_4''=&-\frac{G^4}{(2\pi)^6}\frac{(\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)^2}{(2M)^4}\\&\times\int\frac{(\boldsymbol{\sigma}_1\cdot\mathbf{k}_1)(\boldsymbol{\sigma}_2\cdot\mathbf{k}_1)(\boldsymbol{\sigma}_1\cdot\mathbf{k}_2)(\boldsymbol{\sigma}_2\cdot\mathbf{k}_2)\exp[-i(\mathbf{k}_1+\mathbf{k}_2)\mathbf{r}}{\omega_1^3\omega_2^2}\\&\times d\mathbf{k}_1d\mathbf{k}_2-\frac{3G^4}{(2\pi)^6}(2M)^{-4}\int\frac{\boldsymbol{\sigma}_1\cdot[\mathbf{k}_1\times\mathbf{k}_2]\boldsymbol{\sigma}_2\cdot[\mathbf{k}_1\times\mathbf{k}_2]}{\omega_1^2\omega_2(\omega_1+\omega_2)}\\&\times\left(\frac{2}{\omega_1}+\frac{1}{\omega_2}\right)\exp[-i(\mathbf{k}_1+\mathbf{k}_2)\mathbf{r}]d\mathbf{k}_1d\mathbf{k}_2. \quad (20)\end{aligned}$$

This part of the fourth-order potential is of the order $(G^2/4\pi)^2\times(\mu/2M)^4$ and can be neglected for the time being. It contains, however, a "contact" term which will be discussed in paragraph (*B*) of this section.

We conclude this paragraph by a few remarks on the potential of Eq. (14), which is spin and isotopic spin independent and attractive in all states. The first term of the right-hand side is $G^2/4\pi$ times bigger than $V_2(r)$, which, if $G^2/4\pi$ is of the order of 10, is of the same order as the second term of (14). However, $V_4(r)$ has a range $(1/2\mu)$, since, the asymptotic value of $K_n(x)$ being $(\pi/2x)^{\frac{1}{2}}e^{-x}$, its behavior for large distances is defined by the following expression:

$$V_4(r)\sim-3\left(\frac{G^2}{4\pi}\right)^2\left(\frac{\mu}{2M}\right)\frac{1}{\mu r^2}\times\left[1+\frac{\mu}{M}(\pi\mu r)^{-\frac{1}{2}}\right](\pi\mu r)^{-\frac{1}{2}}e^{-2\mu r}. \quad (21)$$

For small values of x, $K_1(x)$ is roughly equal to $1/x$, and therefore, the two terms of Eq. (14) behave respectively like r^{-3} and r^{-4} at short distances.

It is finally not without interest to note the connection of the results of this paragraph with the remarks made by Lepore[16] and Wentzel,[17] who have shown that, by means of the canonical transformation used by Dyson,[18] the pseudoscalar interaction term of the nucleon-meson Hamiltonian can be brought into the following form:

$$-iG\bar{\psi}\gamma_5\tau_\alpha\psi\phi_\alpha\rightarrow-\tfrac{1}{2}i(G/M)\bar{\psi}'\tau_\beta\psi'\boldsymbol{\sigma}\cdot\nabla\phi_\beta-\tfrac{1}{2}(G^2/M)\bar{\psi}'\psi'\phi_\beta^2, \quad (22)$$

where only terms of order $(G/2M)^2$ have been neglected. A weak coupling calculation of the nucleon-nucleon potential corresponding to each of the two terms of the right-hand side of (22) yields, in first approximation, $V_2(r)$ and the first part of $V_4(r)$, respectively.

[16] J. V. Lepore, Phys. Rev. **87**, 209 (1952).
[17] G. Wentzel, Phys. Rev. **86**, 802 (1952).
[18] F. J. Dyson, Phys. Rev. **73**, 929 (1948).

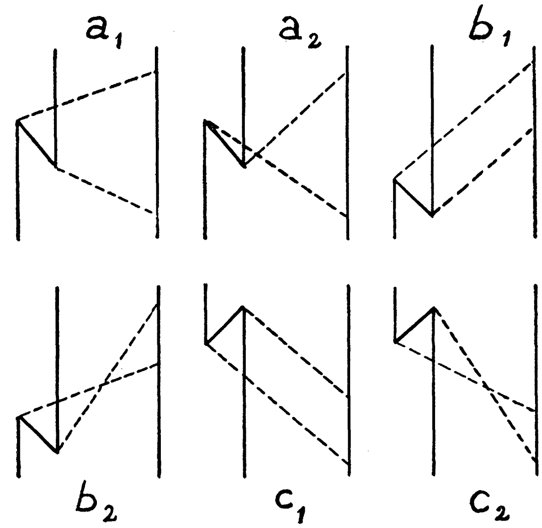

FIG. 2. The virtual processes which involve the creation of one virtual pair in the intermediate states, the contributions of which cancel each other in the nonrelativistic region, in the case of the symmetrical theory.

3. *Higher-Order Terms of the Potential*

These terms have the following general properties:

a. Strength.—The interaction is characterized, in the non-relativistic region, by two expansion parameters: $\alpha_1=(G^2/4\pi)(\mu/2M)^2$, which corresponds to the exchange of one meson without pair creation;

$$\alpha_2=(G^2/4\pi)^2(\mu/2M)^2,$$

which corresponds to a double mesonic exchange with the creation of two pairs. To the order of G^{4n}, the leading term of the nonrelativistic potential is proportional to α_2^n, whereas to the order of G^{4n+2}, it is proportional to $\alpha_2^n\alpha_1$, and therefore much smaller. If $G^2/4\pi$ is of the order of 10, the expansion parameters have the following numerical values: $\alpha_1\sim0.06$ and $\alpha_2\sim0.6$.

b. Range.—A term of order G^{2n} corresponds to a potential of range $(1/n\mu)$. Since the nonstatic effects become predominant at a distance of order $(2/M)\sim(1/3\mu)$, the reduction due to the range becomes effective, in the nonrelativistic region, for the terms of the sixth order in G; for the eighth and higher order terms, it influences the convergence of the interaction expansion in a more important way than the reduction due to the strength.

c. Singularity.—If it were possible to approximate at short distances the Hankel functions by the first term of their expansion in powers of the argument, the singularity of the potential would increase rapidly with the order of the terms, and the expansion parameters should rather be taken as $\alpha_1'=(G^2/4\pi)(2Mr)^{-2}$ and $\alpha_2'=(G^2/4\pi)^2(2Mr)^{-2}$. However, since the function $K_m(n\mu r)$ can be replaced by $\frac{1}{2}(m-1)!\,(2n\mu r)^{-m}$ only for

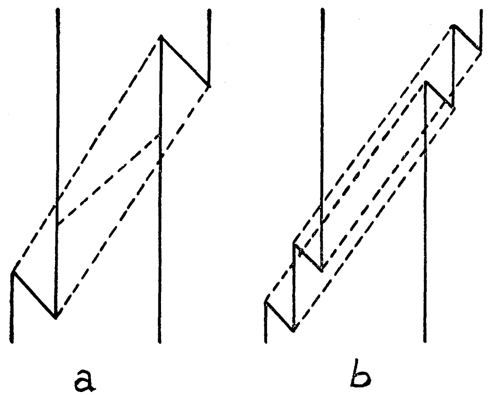

FIG. 3. Two typical diagrams of the virtual processes which give the leading contributions to the sixth- and eighth-order potential, respectively (in the nonrelativistic region).

$r \ll (m-1)^{\frac{1}{2}}(2n\mu)^{-1}$ (for $m>1$) and $r \ll (1/n\mu)$ (for $m=0$, 1), the range in which this approximation is possible is rapidly displaced into the relativistic region, where the static approximation is no longer valid.[19]

The above properties will be illustrated by estimating the leading terms of the sixth- and eighth-order potentials. The magnitude of the latter will, in particular, provide an essential test for the convergence of the interaction expansion, since its strength involves α_2 only, and its "apparent" singularity (in the sense explained in paragraph (c) above) is r^{-5}.

Sixth-order potential.—The leading term, which is proportional to $\alpha_2\alpha_1$, arises from virtual processes which can be illustrated by joining the nucleon lines of the diagrams (a_1) and (a_2) of Fig. 2 by means of a single meson line in the region where there are already two mesons (and no nucleon pair) present [an example is given in Fig. 3(a)]. By adding the contributions of all possible graphs, the following expression is obtained:

$$V_6(r) = -\left(\frac{G^2}{4\pi}\right)^3 \frac{(\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)}{(2M)^4}(2\pi^2)^{-3}\int \frac{\boldsymbol{\sigma}_1\cdot\mathbf{k}_3\ \exp[-i(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)\mathbf{r}]}{2\omega_1\omega_2\omega_3(\omega_1+\omega_2)(\omega_1+\omega_2+\omega_3)} \times\left(\frac{3\boldsymbol{\sigma}_2\cdot\mathbf{k}_3}{\omega_1+\omega_2}+\frac{\boldsymbol{\sigma}_2\cdot\mathbf{k}_1}{\omega_2+\omega_3}\right)d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3$$

$$+\text{symmetrical term with respect to (1) and (2).} \quad (23)$$

Eighth-order potential.—The main contribution to this potential comes from processes involving the exchange of four mesons and the creation of four nucleon pairs (not more than one being present at a given time). One of them is illustrated in Fig. 3(b). All the others can be obtained by allowing the meson lines to cross each other in all possible ways, and by interchanging the role of particles (1) and (2). The resulting expression is given by

$$V_8(r) = -3\left(\frac{G^2}{2M}\right)^4(2\pi)^{-12}\int \frac{\exp[-i(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4)\mathbf{r}]}{\omega_1\omega_2\omega_3\omega_4(\omega_1+\omega_2)} \times\left(\frac{3}{(\omega_1+\omega_2)(\omega_3+\omega_4)}+\frac{1}{(\omega_1+\omega_3)(\omega_2+\omega_4)}\right)d\mathbf{k}_1\cdots d\mathbf{k}_4. \quad (24)$$

In order to estimate the influence of this potential, we remark that the second term between brackets, on the right-hand side, gives roughly one-third of the contribution of the first term, so that $V_8(r)$ can be approximated by

$$V_8(r) \sim -12\left(\frac{G^2}{2M}\right)^4(2\pi)^{-12}\int \frac{\exp[-i(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4)\mathbf{r}]}{\omega_1\omega_2\omega_3\omega_4(\omega_1+\omega_2)^2(\omega_3+\omega_4)} = -12(G^2/2M)^4(2\pi)^{-6}J_1(r)J_2(r). \quad (25)$$

Using the definitions of the functions $J_n(r)$, calculated in the Appendix [see Eqs. (10a) and (12a)], one obtains easily:

$$V_8(r) \sim -12\left(\frac{G^2}{4\pi}\right)^4\left(\frac{\mu}{2M}\right)^4\left(\frac{2}{\pi}\right)^3\frac{1}{\mu^3 r^4}K_1(2\mu r)\int_{\mu r}^{+\infty}[K_0(x)]^2 x dx. \quad (26)$$

This potential is plotted as a dotted line in Fig. 5 for $G^2/4\pi = 9.7$ (value which will be obtained in Sec. V, from the low energy properties of the neutron-proton system). One sees that its influence is only sensible very near the edge of the relativistic region. Its inclusion in the central part of the total interaction has been found to modify the value of the singlet neutron-proton scattering length at zero energy by about 1 percent.

B. Relativistic Region

An examination of the expansion of the kernel $K(\mathbf{p}, \mathbf{p}'; W)$ shows that, for high values of $|\mathbf{p}|$ and $|\mathbf{p}-\mathbf{p}'|$ (and $G^2/4\pi=1$), all its terms are of the same order of magnitude and that the effective expansion parameter is therefore $G^2/4\pi$. Under these circumstances, no definite conclusion can be drawn from a term by term analysis of this kernel. However, even if its magnitude cannot be determined with any certainty in this region, its sign is of crucial importance, because a repulsive interaction, no matter how strong and singular, would still insure the existence of stationary states, since the potential in the nonrelativistic region is sufficiently attractive. What we would like to do here is to show, by means of a rapid discussion of the second- and fourth-order terms of the expansion of $K(\mathbf{p}, \mathbf{p}'; W)$, that there are some indications that this interaction becomes repulsive for distances of order $(1/M)$.

1. *Second-Order Relativistic Interaction*

We first separate out of the kernel defined by Eq. (10) the part which corresponds, in the nonrelativistic region, to the "contact" term which has been omitted in Eq. (13). For this purpose, we write $K_2(\mathbf{p}, \mathbf{p}', W)$ as follows:

$$K_2(\mathbf{p}, \mathbf{p}'; W) = -G^2(\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)\left\{\frac{1}{3}\frac{(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)}{4E_pE_{p'}} \times\left[1-\frac{\mu^2+(E_p-E_{p'})^2+\omega(E_p+E_{p'}-W)}{\omega(\omega+E_p+E_{p'}-W)}\right] +\frac{(M+E_p)(M+E_{p'})}{4E_pE_{p'}}\left[\frac{(\boldsymbol{\sigma}_1\cdot\mathbf{R})(\boldsymbol{\sigma}_2\cdot\mathbf{R})}{R^2} -\frac{1}{3}(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)\right]\frac{R^2}{\omega(\omega+E_p+E_{p'}-W)}\right\}. \quad (27)$$

The term of the third and fourth lines, on the right-hand side, is the relativistic generalization of the tensor force. The first term of the first and second lines gives the

[19] For a process involving the exchange of m mesons, the static interaction itself becomes less singular at distances of order $(m/2M)$, since the quantity $m\omega_k$ cannot then be neglected in comparison with $2M$.

"contact" interaction, which has the following form in co-ordinate space [see Eq. (5a) of the Appendix]:

$$U_c^{(2)}(\mathbf{r},\mathbf{r}')=-\frac{G^2}{3}(\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)\frac{M^2}{rr'}K_1(Mr)K_1(Mr'). \quad (28)$$

Assuming that the integral $\beta=\int r'^{-1}K_1(Mr')\phi^{(0,0)}\times(\mathbf{r}')d\mathbf{r}'$ has a finite value, one sees therefore that the interaction (28) is equivalent to a potential which behaves like r^{-2} at short distances and like $(r^{-\frac{3}{2}}e^{-Mr})$ for large values of r (compared with $1/M$). This interactions acts only if the wave function is spherically symmetrical (S states) and is repulsive in singlet and triplet spin states.

We now investigate the complete behavior of the second-order interaction for small values of r, in order to see if the repulsive operator (28) is not compensated by the contribution arising from the attractive part of (27). Using Eq. (9), we may write, for $r\ll 1/\mu$,

$$\int U_2(\mathbf{r},\mathbf{r}';W)\phi^{(0,0)}(\mathbf{r}')d\mathbf{r}'\sim -\beta\frac{G^2}{(2\pi)^3}(\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)\int\frac{(\boldsymbol{\sigma}_1\cdot\mathbf{p})(\boldsymbol{\sigma}_2\cdot\mathbf{p})e^{i\mathbf{p}\mathbf{r}}d\mathbf{p}}{2E_p(M+E_p)\omega(\omega+E_p-M)}, \quad (29)$$

which can also be written

$$\int U_2(\mathbf{r},\mathbf{r}';W)\phi^{(0,0)}(\mathbf{r}')d\mathbf{r}'\sim M\beta\left(\frac{G^2}{4\pi}\right)\frac{(\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)}{3}\times\left\{(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)\left[F''(Mr)+\frac{2F'(Mr)}{Mr}\right]+S_{12}\left[F''(Mr)-\frac{F'(Mr)}{Mr}\right]\right\}, \quad (30)$$

where only terms of order $(\mu/M)^2$ have been neglected and where

$$F(Mr)=\frac{1}{2\pi r}\int_0^\infty\left(1-\frac{p}{M+E_p}\right)\frac{\sin prdp}{pE_p}. \quad (31)$$

By expanding $F(Mr)$ as a power series in the neighborhood of the origin, it is not difficult to verify that the central force is still repulsive and that the singularity of both the central and the tensor forces is now $r^{-1}\log(Mr)$. Since the effective coupling constant in the relativistic region is $G^2/4\pi$, the distance at which this repulsion takes place can roughly be estimated as of the order of $(M\mu)^{-\frac{1}{2}}$.

2. *Fourth-Order Relativistic Interaction*

The kernel $K_4(\mathbf{p},\mathbf{p}';W)$ being very complicated in the relativistic region, we shall only show that the fourth-order potential contains a repulsive δ-function and give its actual expression at short distances.

In the nonrelativistic fourth-order potential which has been calculated in paragraph (A2) of this section, the only term which contains a "contact" interaction is V_4'' given by Eq. (20), where the first term of the right-hand side can be written as

$$V_4''^{(1)}=\frac{G^2}{2\pi^2}\left(\frac{\mu}{2M}\right)^2\frac{(\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)}{3}\{(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)[3K_0(\mu r)-K_2(\mu r)]+S_{12}K_2(\mu r)\}V_2(r); \quad (32)$$

consequently, the contact term has the form

$$(G^2/4\pi)^2(4/3\pi)(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)^2(\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)^2(2Mr)^{-2}[\delta(\mathbf{r})/4M^2];$$

it is always repulsive but acts only in S states. Its actual expression, in the relativistic region, is proportional to $(G^2/4\pi)^2(Mr^2)^{-1}K_1(Mr)e^{-Mr}$. It behaves like r^{-3} at very short distances and its range is $(1/2M)$.

Remark on the energy dependence of the interaction.—In the case of high energy nucleon-nucleon scattering, the dependence of the potential on W tends to increase the range of the interaction, as can be seen by considering the second-order potential (which is predominant at large distances):

$$V_2(r,\epsilon)=\frac{G^2}{4\pi}\left(\frac{\mu}{2M}\right)^2(\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)(\boldsymbol{\sigma}_1\cdot\nabla)(\boldsymbol{\sigma}_2\cdot\nabla)\frac{Y(r,\epsilon)}{\mu^2}, \quad (33)$$

where $\epsilon=W-2M$ and

$$Y(r,\epsilon)=\frac{1}{2\pi^2}\int\frac{\exp(i\mathbf{k}\mathbf{r})d\mathbf{k}}{\omega_k(\omega_k-\epsilon)}=\frac{1}{2\pi^2}\int\frac{\exp(i\mathbf{k}\mathbf{r})d\mathbf{k}}{k^2+\mu^2-\epsilon^2}\left(1+\frac{\epsilon}{\omega_k}\right). \quad (34)$$

The first term of the right-hand side is a Yukawa potential of range $\chi^{-1}=(\mu^2-\epsilon^2)^{-\frac{1}{2}}$. The second term, which is more complicated, behaves roughly like $r^{-\frac{1}{2}}e^{-\chi' r}$ with $\chi'\sim(\chi^2\mu)^{\frac{1}{3}}$. This increase of the potential range with the energy might be useful in interpreting the experimental neutron-proton scattering data, since the customary static potentials have a tendency to predict too small D-phase shifts. When $\epsilon\geq\mu$ (which, in the laboratory system, corresponds to a neutron energy of about 280 Mev), the potential (33) begins to oscillate at large distances. For such energies, however, the static approximation should not be considered too seriously.

IV. RADIATIVE CORRECTIONS

In order to calculate the contribution of the radiative corrections to the effective pseudoscalar interaction, it is first necessary to separate covariantly the infinite renormalization effects of mass and coupling constant. For this purpose, we start from the equation of Bethe and Salpeter,[10] transformed into a one-time equation by means of the method which has been discussed in detail in (A). It has been shown that the expression of the effective interaction which is obtained in this way is, in the nonrelativistic region, identical term by term with the expression (2) which results from the Tamm-Dancoff treatment.

In the following, we shall further restrict ourselves to the "ladder" approximation,[7,10] since analogous results can be obtained in a straightforward manner by applying the same method to the other terms of the kernel expansion of the B.S. equation. In this approximation, the amplitude $A_{ij}(\mathbf{p},\mathbf{p}_0)$ $(i,j=1,2)$ which is defined by Eq. (A.25) obeys, in the center-of-mass system, the following equation:

$$A_{ij}(\mathbf{p},p_0)=\frac{1}{[\Lambda_i(p)-p_0][\Lambda_j(p)+p_0]}\sum_{k,l}\frac{-iG^2}{(2\pi)^4}\times\int\frac{\Gamma_{ik}^{(1)}(\mathbf{p},\mathbf{p}')\Gamma_{jl}^{(2)}(\mathbf{p},\mathbf{p}')A_{kl}(\mathbf{p}',p_0')d\mathbf{p}'dp_0'}{\Delta p^2+\mu^2},\quad(35)$$

where we have set $\Delta p^2=\omega^2(\mathbf{p},\mathbf{p}')-(p_0-p_0')^2$ and

$$\Gamma_{ik}^{(r)}=\bar{u}_i^{(r)}(\mathbf{p})\gamma_5u_k^{(r)}(\mathbf{p}'),\quad(36)$$

$u_i^{(r)}(\mathbf{p})$ being the amplitude of the Dirac spinor defined by (A, 26). The quantities $\Lambda_i(p)$ are defined by (A, 29).

The radiative corrections are taken into account: (1) By substituting to $\Delta_F(x)$ [the Fourier transform of which appears in the kernel of Eq. (35)] the function $\Delta_F'(x)$, which has been defined by Dyson[20] and which accounts for all closed loop insertions in the meson lines of the irreducible diagrams of the S matrix; (2) by replacing γ_5, in the matrix elements of Eq. (36), by the operator Γ_5, which embodies the contributions arising from the vertex parts of those irreducible diagrams; (3) by introducing in the kernel of the integral equation the remaining contributions of the (finite) irreducible graphs which include self-energy effects.[21]

We shall first calculate the functions Δ_F and Γ_5, up to the second order in G.

A. Calculation of Δ_F'

This calculation has been made by Watson and Lepore,[22] with the following result,

$$\Delta_F'(x)=-\frac{2i}{(2\pi)^4}\int\frac{e^{ikx}d^4k}{k^2+\mu^2}\left[1-\frac{G^2}{2\pi^2}U(k^2)\right],\quad(37)$$

in which terms of order $(\mu/M)^2$ have been neglected and the following definition introduced:

$$U(k^2)=\int_0^1\frac{k^2x(1-2x)dx}{k^2x(1-x)+M^2},\quad(38)$$

the poles of this function being defined as usual by introducing a small negative imaginary part in M.

B. Calculation of $\Gamma_5\ (p,p')$

This calculation will be made without assuming that the nucleons are free in the initial and final states of momenta p and p'. To the second order in G, we write

$$\Gamma_5(p,p')=\gamma_5\{1-G^2[C_1(p,p')+C_2(p,p')]\},\quad(39)$$

where, after some elementary transformation, the functions C_i are found to have the following expressions:

$$C_1(p,p')=-\frac{2i}{(2\pi)^4}\int\frac{k^2d^4k}{[(p+k)^2+M^2][(p'+k)^2+M^2](k^2+\mu^2)},\quad(40)$$

$$C_2(p,p')=-\frac{2i}{(2\pi)^4}\int\frac{[-i\gamma k(i\gamma p'+M)+(-i\gamma p+M)i\gamma k+(-i\gamma p+M)(i\gamma p'+M)]d^4k}{[(p+k)^2+M^2][(p'+k)^2+M^2](k^2+\mu^2)}.\quad(41)$$

$C_2(p,p')$ is a finite integral which vanishes if the initial and final states are free, $C_1(p,p')$, which is a logarithmically divergent integral can be written as

$$C_1(p,p')=-\frac{2i}{(2\pi)^4}\int\frac{d^4k}{(k^2+M^2)^2}+\frac{2i}{(2\pi)^4}\int\frac{\mu^2d^4k}{[(p+k)^2+M^2][(p'+k)^2+M^2](k^2+\mu^2)}+C_1'(p,p').\quad(42)$$

The first term on the right corresponds to a coupling constant renormalization; the second term will be incorporated to $C_2(p,p')$; finally, by using Feynman integral representation of product denominators, $C_1'(p,p')$ is found equal to

$$C_1'=-\frac{2i}{(2\pi)^4}\int_0^1\int\frac{\Delta p^2x(1-x)[2(k^2+M^2)+\Delta p^2x(1-x)]d^4kdx}{(k^2+M^2)^2[k^2+M^2+\Delta p^2x(1-x)]^2}=\frac{1}{4\pi^2}U(\Delta p^2).\quad(43)$$

In the same way, after some calculations, $C_2(p,p')$ is found such that

$$\gamma_5C_2(p,p')=\frac{1}{4\pi^2}\int_0^1\int_0^1\frac{ududvK(p,p';u,v)}{\Delta^2(p,p';u,v)},\quad(44)$$

where the second term of $C_1(p,p')$ has been incorporated and the following definitions introduced:

$$K(p,p';u,v)=(1-u)(i\gamma p+M)\gamma_5(i\gamma p'+M)+Mu[\gamma_5(i\gamma p'+M)+(i\gamma p+M)\gamma_5]-u\gamma_5[(1-v)(p'^2+M^2)+v(p^2+M^2)]-\mu^2\gamma_5,\quad(45)$$

$$\Delta^2(p,p';u,v)=u^2[v(1-v)\Delta p^2+M^2]+\mu^2(1-u)+u(1-u)[(p^2+M^2)v+(p'^2+M^2)(1-v)].\quad(46)$$

If the expression (39) of Γ_5 is substituted to γ_5 in Eq.

[20] F. J. Dyson, Phys. Rev. **75**, 486 (1949).

[21] The corrections to the interaction due to self-energy effects included in the $S_F'(x)$ function (see reference 20) are very small in the nonrelativistic region, since they vanish when the initial and final states of the nucleons are supposed to be free.

[22] K. M. Watson and J. V. Lepore, Phys. Rev. **76**, 1157 (1949).

(36), the matrix elements $\Gamma_{ik}^{(r)}$ are transformed into

$$\Gamma_{ik}'^{(r)}=\Gamma_{ik}^{(r)}\left\{1-\frac{G^2}{4\pi^2}U(\Delta p^2)+\frac{G^2}{4\pi^2}\int_0^1\int_0^1\frac{ududv}{\Delta_r^2}\right.$$
$$\times F_{ik}^{(r)}(p,p';u,v)-\sum_{ik}{}^{(r)}\frac{G^2}{4\pi^2}$$
$$\times\int_0^1\int_0^1\frac{u^2dudv}{\Delta_r^2}M[\Delta p_0+E_k(p')-E_i(p)], \quad (47)$$

where $F_{ik}^{(r)}$ and Δ_r^2 are, respectively, equal to

$$F_{ik}^{(r)}(p,p';u,v)$$
$$=(1-v)[p_0-\epsilon_r\Lambda_i(p)][p_0'-\epsilon_r\Lambda_k(p')]+\mu^2$$
$$+u\{(1-v)(\mathbf{p}'^2+M^2-p_0^2)+v(\mathbf{p}^2+M^2-p_0^2)$$
$$-(W^2/4)+W\epsilon_r[(1-v)p_0'+vp_0]\}, \quad (48)$$

$$\Delta_r^2=u^2[M^2+v(1-v)\Delta p^2]+(1-u)\mu^2$$
$$+u(1-v)\{v(E_p^2-p_0^2)+(1-v)(E_{p'}^2-p_0'^2)$$
$$+W\epsilon_r[vp_0+(1-v)p_0']-(W^2/4), \quad (49)$$

ϵ_r being defined by $\epsilon_1=+1$, $\epsilon_2=-1$. $\sum_{ik}{}^{(r)}$ is the matrix element of $-\rho_2^{(r)}=\gamma_5^{(r)}\gamma_4^{(r)}$, namely

$$\sum_{ik}{}^{(r)}=\bar{u}_i^{(r)}(\mathbf{p})\gamma_5^{(r)}\gamma_4^{(r)}u_k^{(r)}(\mathbf{p}'). \quad (50)$$

C. Corrections to the Interaction Arising from Vertex Parts and Closed Loops

In order to avoid an excessive use of algebra, we shall only give here some details on the main contribution to the interaction arising from these radiative effects, namely the corrections to the terms $V_4^{(a)}$ and $V_4^{(b)}$ of the potential, defined by Eqs. (15) and (16).

After substituting expressions (37) and (47) into (35), one obtains a three-dimensional equation for the amplitude corresponding to equal times of the two nucleons:

$$a_{ij}(\mathbf{p})=\frac{1}{2\pi i}\int_{-\infty}^{+\infty}A_{ij}(\mathbf{p},p_0)dp_0, \quad (51)$$

by following exactly the method given in A: First, one introduces on the right of (35) the function $A_{kl}^{(1)}(\mathbf{p},p_0)$, in which the dependence on p_0 corresponds to taking only into account the diagrams where the maximum number of mesons present at a given time is one (one of them gives $V_4^{(b)}$):

$$A_{kl}^{(1)}(\mathbf{p},p_0)=-2\epsilon_k\Lambda_k(p)\frac{a_{kk}(\mathbf{p})}{\Lambda_k^2-p_0^2}\delta_{kl}. \quad (52)$$

Then, one iterates Eq. (35) another time, in order to include also the diagrams where the maximum number of mesons is two (one of them leads to $V_4^{(a)}$).

(*a*) It is easier to calculate in the first place the corrections arising from Δ_F' and the $G^2C_1(p,p')$ term in Γ_5, which involve only $U(\Delta p^2)$. This function is such that, apart from unimportant terms in $(\mu/M)^2$, the corresponding corrections do not modify the analytical form of the kernel of Eq. (35), but substitute an apparent mass $M[x(1-x)]^{-\frac{1}{2}}$ to the meson mass μ. The correction to order G^2 to the interaction $V_4^{(b)}$ is found to be

$$\Delta V_4^{(b)}=-48\left(\frac{G^2}{4\pi}\right)^3\left(\frac{\mu}{2M}\right)^4\frac{\delta(\mathbf{r})}{\mu^3r}K_1(\mu r)$$
$$+\mathcal{O}[(G^2/4\pi)^3(\mu/2M)^5], \quad (53)$$

whereas the correction to $V_4^{(a)}$ is given by

$$\Delta V_4^{(a)}=-32\left(\frac{G^2}{4\pi}\right)^3\left(\frac{\mu}{2M}\right)^4\frac{\delta(\mathbf{r})}{\mu^3r}\frac{2}{\pi}K_1(\mu r)$$
$$+\mathcal{O}[(G^2/4\pi)^3(\mu/2M)^5]. \quad (54)$$

The leading terms, in both expressions, vanish in the nonrelativistic region. This result might appear a little surprising, especially for the Γ_5-correction, since the corresponding vertex part has been inserted in a transition from positive to negative energy states, where the four-dimensional Δp is of the order of M. It can however be interpreted as meaning that the pairs which are created in the intermediate states are moving rather slowly and that the three-dimensional momentum transfer $\Delta\mathbf{p}$ is small compared to the corresponding transfer of energy.

(*b*) The leading term of the corrections to $V_4^{(a)}$ arising from $G^2C_2(p,p')$ in Γ_5 can be found, after elaborate calculations, to be equal to

$$\Delta V_4'^{(a)}=+\frac{64}{9}\left(\frac{G^2}{4\pi}\right)^3\left(\frac{\mu}{2M}\right)^5\log\left(\frac{\mu}{2M}\right)\frac{1}{\mu r^2}\frac{2}{\pi}K_1(2\mu r). \quad (55)$$

The correction to $V_4^{(b)}$ has an analogous form.

D. Finite Self-Energy Terms[23]

These terms are contributed by a series of irreducible diagrams, the first one of which is illustrated in Fig. 4. (All the others can be obtained by allowing an arbitrary number of mesons to be emitted by one nucleon *before* k_1 and k_2, and to be reabsorbed by the same nucleon *after* k_1 and k_2 have been emitted.) In order to calculate the contribution of these diagrams, it is convenient to suppose that the initial and finite states of the interacting nucleons are free. This assumption leads actually to a correct value of the interaction in the nonrelativistic

[23] We are very indebted to Professor N. Kroll for directing our attention to these terms, and for an interesting discussion. We also gratefully acknowledge helpful discussions with Dr. Lepore and Dr. Ruderman on this point. The finite self-energy terms can alternatively be calculated by means of the noncovariant Tamm-Dancoff method; the result is, however, rather ambiguous, since the separation of renormalization parts in the self-energy effects by means of this method is not unique.

region. After some elementary transformations, where terms of order $(\mu/M)^2$ have been neglected, the contribution of the diagram of Fig. 4 is found to have the following expression:

$$\Delta G_4^{(1)} \sim -\int \frac{\bar{\psi}_{p'} U \psi_p \bar{\psi}_{q'} \gamma_\nu \psi_q k_\nu d^4k}{[(q-k)^2+M^2][k^2+\mu^2][(k-\Delta p)^2+\mu^2]}, \tag{56}$$

with $\Delta p = p'-p$, U being defined by

$$U=\gamma_\mu \int \left[k_\mu + k_\mu' - 2\frac{k_\mu' k_\lambda'(p_\lambda + k_\lambda)}{k'^2+\mu^2}\right] \frac{d^4k'}{[(p+\Delta p-k')^2+M^2][(p+k-k')^2+M^2][(p-k')^2+M^2]}. \tag{57}$$

The integration over k' yields the expression

$$U=\pi^2[i\gamma_\mu k_\mu A + MB], \tag{58}$$

where the following definitions have been introduced:

$$A=\int_0^1\int_0^1\int_0^1 x dx dy dz \left[-\frac{1+x-xy}{D^2}+\frac{xy}{D'^2} + \frac{2x^2y(1-y)Q_\lambda(p_\lambda+k_\lambda)}{D'^4}\right],$$
$$B=\int_0^1\int_0^1\int_0^1 x dx dy dz \left[-\frac{1}{D^2}+\frac{xy}{D'^2} + \frac{2xy(1-xyz)Q_\lambda(p_\lambda+k_\lambda)}{D'^4}\right], \tag{59}$$

D^2, D'^2 and Q_λ being, respectively, given by

$$\begin{aligned} D^2 &= M^2+x^2y(1-y)(k-\Delta p)^2 + xy(1-x)\Delta p^2 + x(1-x)(1-y)k^2,\\ D'^2 &= M^2(1-xyz)^2+x^2yz(1-y)[(p+k)^2+M^2] + x^2y(1-y)(1-z)(k-\Delta p)^2 \\ &\quad + xy(1-y)(1-z)\Delta p^2 + x(1-y)(1-x)k^2,\\ Q_\lambda &= (1-xyz)p + xy(1-z)\Delta p + x(1-y)k. \end{aligned} \tag{60}$$

The expression for $\Delta G_4^{(1)}$, given by (56) and (58), is now introduced in the equation of Bethe and Salpeter, and the corrections to the interaction are obtained in the same way as in the preceding subsection. In the nonrelativistic region, this leads to the expression

$$\Delta V_4^{(c)} \sim C_1 \left(\frac{G^2}{4\pi}\right)^3 \frac{1}{8\pi^6} \sum_{\mu,\nu} \int \Pi_\nu^{(2)}(\mathbf{k}_1,\mathbf{k}_2) \times [\Pi_\mu^{(1)}(\mathbf{k}_1,\mathbf{k}_2) H_{\mu\nu}^{(1)}(\mathbf{k}_1,\mathbf{k}_2) - iMH_\nu^{(2)}(\mathbf{k}_1,\mathbf{k}_2)] \times \exp[-i(\mathbf{k}_1+\mathbf{k}_2)\mathbf{r}] d\mathbf{k}_1 d\mathbf{k}_2, \tag{61}$$

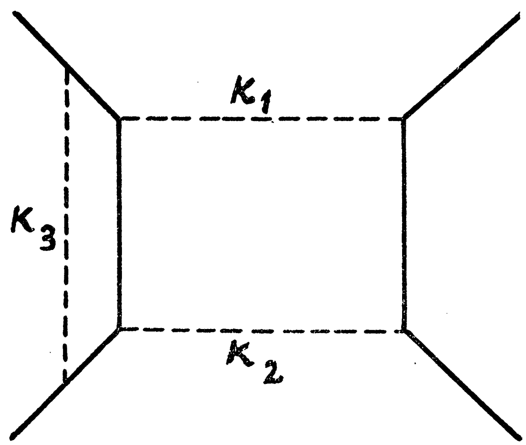

FIG. 4. The lowest order virtual process in which the self-energy of one of the nucleons gives a finite contribution to the interaction in the nonrelativistic region.

where the following definitions have been introduced:

$$\Pi_\mu^{(r)}(\mathbf{k}_1,\mathbf{k}_2) = \bar{u}_1^{(r)}(0)\gamma_\mu^{(r)} u_1^{(r)}(\mathbf{k}_1+\mathbf{k}_2), \tag{62}$$

$$H_{\mu\nu}^{(1)} = \lim_{\epsilon\to 0} \frac{(2i\epsilon)^2}{(2\pi i)^3} \int_{-\infty}^{+\infty} \frac{(p_\mu'-p_\mu)(p_\nu'-p_\nu)A_0 dp_0 dp_0' dp_0''}{[\omega_1^2-(p_0'-p_0)^2][\omega_2^2-(p_0''-p_0')^2](p_0^2+\epsilon^2)(p_0''^2+\epsilon^2)(p_0'-i\epsilon)(p_0'-2M)}, \tag{63}$$

$$H_\nu^{(2)} = \lim_{\epsilon\to 0} \frac{(2i\epsilon)^2}{(2\pi i)^3} \int_{-\infty}^{+\infty} \frac{(p_\nu'-p_\nu)B_0 dp_0 dp_0' dp_0''}{[\omega_1^2-(p_0'-p_0)^2][\omega_2^2-(p_0''-p_0')^2](p_0^2+\epsilon^2)(p_0''^2+\epsilon^2)(p_0'-i\epsilon)(p_0'-2M)}, \tag{64}$$

A_0 and B_0 being obtained from A and B by replacing, in D^2 and D'^2,

$$\begin{aligned} \Delta p^2 &\text{ by } -\Delta p_0^2 = -(p_0''-p_0')^2,\\ k^2 &\text{ by } -k_0^2 = -(p_0'-p_0)^2,\\ (\Delta p - k)^2 &\text{ by } -(p_0''-p_0')^2 = -(\Delta p_0 - k_0)^2, \end{aligned}$$

and by putting $(p+k)^2+M^2 = -(p_0'+i\epsilon)(p_0'+2M)$. The indices μ and ν take the values 1, 2, 3, 4 with $\mathbf{p}'-\mathbf{p}=\mathbf{k}_1$ and $p_4'-p_4 = i(p_0'-p_0)$.

The first term of the right-hand side of (61) yields very small corrections to the iteration of the second order potential, and to the fourth-order potential calculated in Sec. III, 2. The second term of (61) gives, for $\nu=1$, 2, 3, small corrections to the "one-pair terms" of the fourth-order interaction. The main correction comes actually from $H_4^{(2)}$, which can be written

$$H_4^{(2)}=\frac{1}{4\pi M^3}\int_{-\infty}^{+\infty}\frac{dp_0'}{(p_0'^2-\omega_1^2)(p_0'^2-\omega_2^2)}=\frac{-1}{4iM^3\omega_1\omega_2(\omega_1+\omega_2)}. \quad (65)$$

The main part of $V_4^{(c)}$ is therefore proportional to the potential $V_4(r)$ of Eq. (14), the numerical coefficient being of order $(G^2/4\pi)$. The whole series of diagrams analogous to Fig. 4 will similarly contribute corrections proportional to $V_4(r)$, the entire coefficient being expressed as a power series in $G^2/4\pi$. The practical effect of these corrections will be therefore to modify the strength of the fourth-order potential in an unknown fashion. This can be accounted for by replacing G, in Eq. (14), by a new coupling constant G_1, to be determined from experiment.

V. LOW ENERGY PROPERTIES OF THE NEUTRON-PROTON SYSTEM

In order to discuss the low energy properties of the neutron-proton system, it is possible to use, in the nonrelativistic region, the potential which has been calculated in Section (IIIA). At very short distances, it is not advisable to take the second- and fourth-order relativistic interactions which have been discussed in Sec. (IIIB), since the higher order terms of the expansion (as well as the contribution from heavier mesons, isobaric states of nucleons, etc.) would probably modify them seriously. We shall however retain the qualitative features of these interaction terms, which are repulsive and strongly singular, in prescribing that the "wave function" $\phi^{(0,0)}(\mathbf{r})$, defined in Sec. II, must satisfy the boundary condition: $\phi^{(0,0)}(\mathbf{r}_c)=0$, r_c being some finite distance of order $(M\mu)^{-\frac{1}{2}}$. This boundary condition, which is equivalent to assuming that the central force includes an infinitely repulsive core of radius r_c, introduces certainly an over-simplification in the actual behavior of the wave function; it can, however, be justified further by means of the two following arguments.

(a) If the pseudoscalar potential is used in the nonrelativistic region, the only type of relativistic interaction which leaves any possibility of agreement with the low energy experimental data must be strongly repulsive. This argument, which has already been pointed out by Brueckner and Low,[24] is based on the fact that the static pseudoscalar potential, being very singular, leads to a much too small neutron-proton (singlet or triplet) effective range, unless the wave function tends rapidly to zero in the neighborhood of the nucleon Compton wavelength.

[24] K. Brueckner and F. Low, Phys. Rev. 83, 461 (1951).

(b) It can be expected that the low energy properties of the neutron-proton system are not very sensitive to the shape of this short range repulsive interaction, provided that it is sufficiently strong.

A. The Experimental Data

The latest values of the low energy experimental data on the neutron-proton system, as quoted or calculated by Salpeter,[25] are as follows:

Binding energy in the triplet ground state (deuteron):

$$\epsilon=(-2.227\pm0.003)\text{ Mev}.$$

Zero energy singlet scattering length:

$$a_s=(-23.68\pm0.06)10^{-13}\text{ cm}=(-16.91\pm0.38)(1/\mu).$$

Singlet effective range:

$${}^1r_0=(2.7\pm0.5)10^{-13}\text{ cm}=(1.93\pm0.41)(1/\mu).$$

Triplet effective range:

$${}^3r_0=(1.704\pm0.030)10^{-13}\text{ cm}=(1.217\pm0.047)(1/\mu).$$

Electric quadrupole moment:

$$Q=(2.738\pm0.016)10^{-27}\text{ cm}^2.$$

Proportion of D state:[26]

$$0.02\lesssim p_D\lesssim 0.06.$$

B. The Potential

Taking only the main terms of the nonrelativistic interaction calculated in Sec. III, and putting $x=\mu r$, the potential for $x\geq x_c=\mu r_c$ has the following expression:[27]

$$V(x)=\mu[V_c(x)+S_{12}V_t(x)], \quad (66)$$

where the central and tensor forces are respectively defined by

$$V_c(x)=-\frac{G^2}{4\pi}\left(\frac{\mu}{2M}\right)^2\frac{e^{-x}}{x}-3\left(\frac{G^2}{4\pi}\right)^2\left(\frac{\mu}{2M}\right)^2\frac{1}{x^2}\times\left\{\frac{2}{\pi}K_1(2x)+\frac{\mu}{2M}\left[\frac{2}{\pi}K_1(x)\right]^2\right\}, \quad (67)$$

$$V_t(x)=-\frac{G^2}{4\pi}\left(\frac{\mu}{2M}\right)^2\left(1+\frac{3}{x}+\frac{3}{x^2}\right)\frac{e^{-x}}{x}. \quad (68)$$

[25] E. Salpeter, Phys. Rev. 82, 60 (1951).

[26] The nonrelativistic expression of the deuteron magnetic moment $\mu_D=\mu_N+\mu_P-\frac{3}{2}(\mu_N+\mu_P-\frac{1}{2})p_D$ leads to $p_D=0.04$. The relativistic corrections are, however, usually estimated as about 2 percent.

[27] We neglect, for the time being, the influence of the radiative corrections, discussed in Sec. IV, 5, which modify the strength of the fourth order part of the central force, by introducing (in this part of the potential only) a new coupling constant G_1, which is an unknown function of G. Since a good agreement with experiment is obtained by putting $G_1=G$, it is to be expected that a direct determination of G_1 by comparison with experiment would lead to a numerical value very close to that of G.

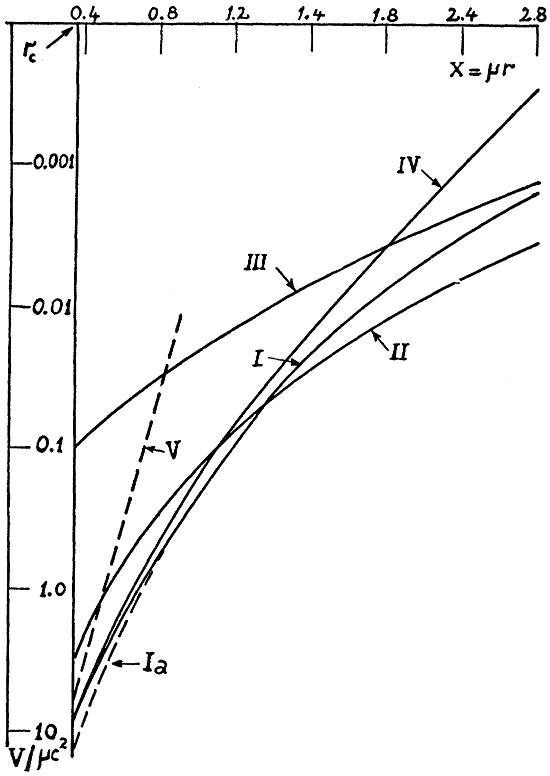

FIG. 5. Plot of the potential corresponding to the constants of Eq. (74) as a function of the distance r. Curve I represents the total central force, defined by (67) and curve II the tensor force, defined by (68). The dotted curve (Ia) represents the central force when the eighth-order corrections are taken into account. Curves III, IV, V represent the respective contributions of the second-, fourth-, and eighth-order central parts of the potential.

C. Determination of $G^2/4\pi$ and x_c

The two constants which define the interaction have been calculated by fitting exactly the binding energy of the deuteron and the singlet scattering length, in the following way:

(a) For the ${}^1S$ state of zero energy, the radial function obeys the equations

$$\begin{aligned} &v(x_c)=0,\\ &d^2v/dx^2=(M/\mu)V_c(x)v(x) \text{ for } x\geqslant x_c. \end{aligned} \tag{69}$$

These equations have been integrated numerically[28] for several values of $G^2/4\pi$, starting from the asymptotic form $v(x)\sim 1+x(\mu a_s)^{-1}$ and determining in each case the value of x at which the wave function goes to zero. The initial integrating point was 4.5 and the difference interval $\Delta x=0.1$. This procedure permits to draw point by point the curve $x_c=f_1(G^2/4\pi)$ which results from an exact fitting of the zero energy scattering length.

(b) For the $({}^3S+{}^3D)$ state, the wave function may be written[29]

$$\phi^{(0,0)}(\mathbf{r})=(1/r)[u(x)+(1/8^{\frac{1}{2}})S_{12}w(x)]\chi_m, \tag{70}$$

where χ_m is the spin function with magnetic quantum number m, $u(x)$ and $w(x)$ being solutions of the following system of equations:

$$\begin{aligned} \frac{d^2u}{dx^2}&=\left[\eta^2+\frac{M}{\mu}V_c(x)\right]u(x)+2^{\frac{1}{2}}\frac{M}{\mu}V_t(x)w(x),\\ \frac{d^2w}{dx^2}&=\left[\eta^2+\frac{M}{\mu}V_c(x)-\frac{2M}{\mu}V_t(x)+\frac{6}{x^2}\right]w(x)\\ &\qquad+2^{\frac{1}{2}}\frac{M}{\mu}V_t(x)u(x), \end{aligned} \tag{71}$$

with the secondary condition

$$u(x_s)=w(x_c)=0, \tag{72}$$

η^2 being equal to $-M\epsilon/\mu$. The system of equations (71) has similarly been integrated numerically for several values of $G^2/4\pi$, starting from the asymptotic form

$$u(x)\sim e^{-\eta x},\quad w(x)\sim \rho e^{-\eta x}\left[1+\frac{3}{\eta x}+\frac{3}{(\eta x)^2}\right], \tag{73}$$

where ρ is adjusted by repeated integrations so that both $u(x)$ and $w(x)$ go to zero at the same point. This procedure yields a curve $x_c=f_2(G^2/4\pi)$, resulting from an exact fitting of the deuteron binding energy.

(c) The required values of x_c and $G^2/4\pi$ are determined by the relations $x_c=f_1(G^2/4\pi)=f_2(G^2/4\pi)$, which gives[30]

$$r_c=x_c/\mu=(0.38\pm 0.03)(1/\mu),\quad G^2/4\pi=9.7\pm 1.3. \tag{74}$$

The corresponding potential functions are drawn in Fig. 5, together with the eighth-order correction to the central force, which is defined by Eq. (26). When this correction is incorporated into $V_c(x)$, it has been found that the value of x_c which results from an exact fitting of the zero energy scattering length is modified by about 1 percent.

D. Calculation of the Derived Quantities

Using the numerically integrated function $v(x)$ corresponding to the constants (74), the singlet effective range is calculated as follows:

$${}^1r_0=\frac{2}{\mu}\int_0^\infty\left[\left(1+\frac{x}{\mu a_s}\right)^2-v^2(x)\right]dx=1.78(1/\mu). \tag{75}$$

Similarly, the numerically integrated functions $u(x)$ and $w(x)$ corresponding to the same constants yield a

[28] The integration method which has been used allows the calculation of the solution at one point from its value at the two preceding ones. The error at a point x is equal to $(1/120)(\Delta x)^6 \times u^{(6)}(x)$. We are indebted to Dr. R. Christian for valuable advice in the numerical calculations.

[29] H. Feshbach and J. Schwinger, Phys. Rev. 84, 194 (1951).

[30] The value of r_c is in agreement with our estimate of the distance at which the interaction (30) becomes important.

triplet effective range

$$^3r_0=\frac{2}{\mu}\int_0^\infty\left[e^{-2\eta x}-\frac{u^2(x)+w^2(x)}{1+\rho^2}\right]dx=1.185(1/\mu),\quad (76)$$

an electric quadrupole moment

$$Q=\frac{2^{\frac{1}{2}}}{10\mu^2}\int_0^\infty x^2[uw-8^{-\frac{1}{2}}w^2]dx\bigg/\int_0^\infty (u^2+w^2)dx$$
$$=2.08\times 10^{-27}\ \mathrm{cm}^2,\quad (77)$$

and a proportion of D state:

$$p_D=\int_0^\infty w^2(x)dx\bigg/\int_0^\infty (u^2+w^2)dx=0.051.\quad (78)$$

A quantity of physical interest is the mean value of the square of the momentum in the ground state

$$\left\langle\frac{p^2}{\mu^2}\right\rangle=-\frac{\int_0^\infty\left[u\left(\frac{d^2}{dx^2}\right)u+w\left(\frac{d^2}{dx^2}-\frac{6}{x^2}\right)w\right]dx}{\int_0^\infty (u^2+w^2)dx}=1.76,\quad (79)$$

which corresponds to a mean value of the kinetic energy of 18.5 Mev per nucleon. This result indicates that the momentum distribution is mainly concentrated around μ, and is therefore not too sensitive to the shape of the interaction at distances of order $(1/M)$.

The preceding results are in good agreement with experiment, except the quadruple moment, which is too small by about 20 percent. This quantity is, however, sensitive to an increase of the tensor force, and can probably be improved by including the remaining portion of the fourth-order interaction, $V_4''(r)$ defined by (20), and the sixth-order potential [Eq. (23)].

E. Neutron-Proton Scattering at 40 Mev

The neutron-proton scattering cross section for a neutron energy of 40 Mev in the laboratory system has been calculated, using the potential $V(r)=V_2(r)+V_4(r)$ defined by (13) and (14) for $r\geqslant r_c$, a repulsive core of radius r_c, and the constants (74). For the (^1S) and ($^3S+{}^3D$) scattering states, the wave equation has been integrated numerically, starting at the point $x=x_c$ (in the case of the $^3S+{}^3D$ state, two numerical integrations were necessary, for the S and D dominant modes, respectively). The phase shifts corresponding to P and D singlet and triplet states were calculated by means of the Born approximation, the coupling (due to the tensor force) with states corresponding to $L>2$ being neglected. Due to the presence of the repulsive core, the phase shift corresponding to a state (J, L, s) in the Born approximation is given by the equation

$$\tan{}^s\delta_{J,L}=-\frac{M}{\mu k}\int_{x_c}^\infty [g_{L+\frac{1}{2}}(\xi)-\rho(x_c)g_{-(L+\frac{1}{2})}(\xi)]^2$$
$$\times{}^sV_{J,L}(\xi)d\xi-(-1)^l\rho(x_c),\quad (80)$$

with $k^2=\frac{1}{2}ME$ (E is the neutron energy in the laboratory system),

$$g_\nu(x)=(\tfrac{1}{2}\pi kx)^{\frac{1}{2}}J_\nu(x),\ \text{and}\ \rho(x_c)=g_{L+\frac{1}{2}}(x_c)/g_{-(L+\frac{1}{2})}(x_c).$$

The calculated value of the total cross section is equal to $\sigma_t=216$ millibarns. The corresponding mean value of the two experimentally measured cross sections[31] is $\sigma_t=194\pm 20$ millibarns. The agreement is therefore satisfactory.

The angular distribution of the cross section in the center-of-mass system, represented in Fig. 6, shows very little asymmetry around 90 degrees. This is due to the fact that, for odd states, the weak long range second order potential (13) becomes repulsive, whereas the strong short-range fourth-order potential (14) remains attractive. The phase shifts corresponding to odd states

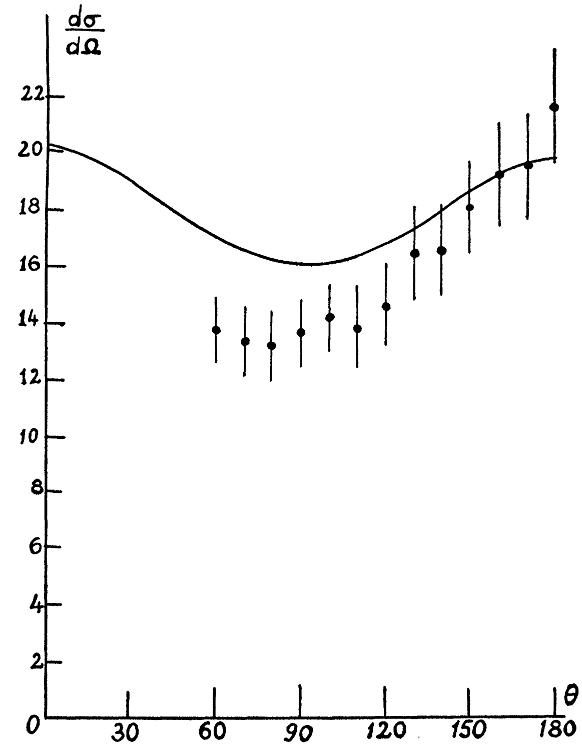

FIG. 6. Angular distribution of the differential cross section (in millibarns per steradian) in the center-of-mass system for scattering of 40-Mev neutrons by protons. The experimental points are those of Hadley *et al.* (reference 31), normalized in order to make the total cross section agree with the mean experimental value. An arbitrary experimental error of 10 percent has been assumed at each point.

[31] Hadley, Kelly, Leith, Segrè, Wiegand, and York, Phys. Rev. **75**, 351 (1949); R. H. Hildebrand and C. E. Leith, Phys. Rev. **80**, 842 (1950).

are therefore greatly reduced. In the particular case of the 40-Mev n–p scattering, there is the additional helpful fact that the singlet and triplet P-phase shifts turn out to have opposite signs. The main defect of the theoretical angular distribution curve is that the D phase shifts are somewhat too small, the curve being consequently too flat by about 20 percent. This can be greatly improved, however, by taking into account the energy dependence of the potential which, as was shown at the end of Sec. III, increases the range of the interaction. For a neutron energy of 40 Mev, the effect would be of the order of 15 percent.

VI. CONCLUDING REMARKS

The fact that the pseudoscalar interaction calculated in this paper leads to a good agreement with the low energy experimental data on the neutron-proton system, might perhaps appear not too surprising, if one remarks that this interaction contains a certain number of features which were actually qualitatively predicted on the basis of other investigations:

(a) A phenomenological treatment of the nuclear forces[29] has led to the conclusion that the central force should be more singular and have a shorter range than the tensor force.

(b) A detailed analysis of nucleon-nucleon scattering at intermediate energies[32] shows that, in the case of a phenomenological Yukawa potential, a better agreement with experiment is obtained by choosing a range of the central force corresponding to a higher mass than that of the π-meson.

(c) Discussions of the nucleons' magnetic moments[33] and of multiple meson production in high energy nucleon-nucleon collisions[34] suggest that the probability of the presence of two mesons in the field around the nucleons is much higher than that of one meson. A more severe test for the validity of the theory will probably be provided by the analysis of high energy nucleon-nucleon and meson-nucleon scattering. In the case of neutron-proton scattering, the potential which has been obtained here contains an added feature which will probably be helpful in interpreting the experimental data: For odd states, the weak long range second order potential becomes repulsive, whereas the strong short-range fourth-order force remains attractive; as was shown at the end of Sec. V, this reduces the P-phase-shifts and leads to angular distributions in the center-of-mass system which are nearly symmetrical around 90 degrees. However, it is the opinion of the writer that it would not be very consistent to use the same potential at very high energies,[35] because the shape of the repulsive interaction would then become a significant factor. Moreover, the dependence of the potential on the total energy W should then carefully be taken into account.

In the case of meson-nucleon scattering, a weak coupling treatment of the interaction Hamiltonian (22) would probably lead to a serious disagreement (see reference 17) with recent experimental data.[36] It is, however, not unreasonable to think that the role of the radiative corrections is much more important in this case than in the nucleon-nucleon interaction.

The author would like to express his gratitude to Professor Oppenheimer for his kind hospitality at the Institute for Advanced Study. This work could not have been concluded without his continued encouragement and his stimulating criticisms and suggestions. The author is also very indebted to Professors Pais and Peierls, and to Dr. F. Low, for numerous and valuable discussions. His thanks are finally due Professors Bethe and Weisskopf, for two illuminating conversations.

APPENDIX

1. The functions $I_n(r)$. We have

$$I_n(r)=(2\pi)^{-\frac{3}{2}}\int\frac{e^{i\mathbf{kr}}}{\omega_k^{2n+1}}d\mathbf{k}=\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\frac{1}{r}\int_0^\infty\frac{\sin kr k dk}{(k^2+\mu^2)^{n+\frac{1}{2}}}. \tag{1a}$$

By using the integral representation of the Hankel functions of imaginary argument (see reference 15, p. 172):

$$K_n(\mu r)=\pi^{-\frac{1}{2}}\Gamma(n+\tfrac{1}{2})\left(\frac{2\mu}{r}\right)^n\int_0^\infty\frac{\cos kr dk}{(k^2+\mu^2)^{n+\frac{1}{2}}}, \tag{2a}$$

$I_n(r)$ can easily be put into the form

$$I_n(r)=-\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\frac{1}{r}\frac{d}{dr}\left[\frac{\pi^{\frac{1}{2}}}{\Gamma(n+\frac{1}{2})}\left(\frac{r}{2\mu}\right)^n K_n(\mu r)\right]. \tag{3a}$$

Taking into account the relations (reference 15, p. 79)

$$\begin{aligned}-2K_n'(x)&=K_{n-1}(x)+K_{n+1}(x),\\ -2(n/x)K_n(x)&=K_{n-1}(x)-K_{n+1}(x),\end{aligned} \tag{4a}$$

one finally obtains

$$I_n(r)=\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\frac{2^n n!}{(2n)!}\left(\frac{r}{\mu}\right)^{n-1}K_{n-1}(\mu r). \tag{5a}$$

2. The functions $J_n(r)$. These functions are defined by

$$J_n(r)=(2\pi)^{-3}\int\frac{e^{-i(\mathbf{k}_1+\mathbf{k}_2)\mathbf{r}}}{\omega_1\omega_2(\omega_1+\omega_2)^n}d\mathbf{k}_1d\mathbf{k}_2$$
$$=\frac{2}{\pi r^2}\int_0^\infty\int_0^\infty\frac{\sin k_1r\sin k_2r}{\omega_1\omega_2(\omega_1+\omega_2)^n}k_1k_2dk_1dk_2. \tag{6a}$$

[32] G. Breit, Phys. Rev. **84**, 1053 (1951).
[33] R. G. Sachs, Phys. Rev. **87**, 1100 (1952).
[34] H. W. Lewis (private communication).
[35] An estimate of the order of energy at which the potential is still valid is provided by the mean value of the kinetic energy in the deuteron ground state, calculated in Sec. V. In the laboratory system, it corresponds to about 37 Mev.
[36] H. L. Anderson *et al.*, Phys. Rev. **85**, 936 (1952); **86**, 793 (1952).

(a) $n=1$: We multiply the numerator and the denominator of the integrand by $(\omega_1-\omega_2)$:

$$J_1(r)=\frac{2}{\pi r^2}\int_0^\infty\int_0^\infty\left(\frac{1}{\omega_2}-\frac{1}{\omega_1}\right)\frac{\sin k_1 r\,\sin k_2 r k_1 k_2}{k_1{}^2-k_2{}^2}dk_1dk_2. \tag{7a}$$

By exchanging the integrations over k_1 and k_2 in the second part of the right-hand side, we obtain

$$J_1(r)=\frac{4}{\pi r^2}\int_0^\infty\frac{\sin k_2 r}{\omega_2}F_1(k_2,r)k_2dk_2, \tag{8a}$$

where the function F_1 is defined by

$$F_1(k_2,r)=\int_0^\infty\frac{\sin k_1 r k_1}{k_1{}^2-k_2{}^2}dk_1=\frac{\pi}{2}\cos k_2 r \text{ for } r>0. \tag{9a}$$

It follows, therefore, that

$$J_1(r)=\frac{1}{r^2}\int_0^\infty\frac{\sin 2k_2 r}{\omega_2}k_2dk_2=\frac{\mu}{r^2}K_1(2\mu r), \tag{10a}$$

where use has been made of the definition of $I_0(r)$ in the previous paragraph.

(b) $n=2$: We differentiate $J_2(\mu r)$ with respect to μ:

$$\frac{\partial}{\partial\mu}J_2(\mu r)=-\frac{\mu}{(2\pi)^3}\int\frac{e^{-i(\mathbf{k}_1+\mathbf{k}_2)\mathbf{r}}}{\omega_1{}^3\omega_2{}^3}d\mathbf{k}_1d\mathbf{k}_2$$

$$=-\mu[I_1(r)]^2=-\frac{2\mu}{\pi}[K_0(\mu r)]^2. \tag{11a}$$

Since $J_2(\mu r)\rightarrow 0$ for $\mu\rightarrow\infty$, it follows therefore that

$$J_2(\mu r)=\frac{2}{\pi r^2}\int_{\mu r}^\infty[K_0(x)]^2xdx. \tag{12a}$$

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$C^{12}(p,pn)C^{11}$ Cross Section from Threshold to 340 Mev*

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The excitation function for the reaction $C^{12}(p,pn)C^{11}$ has been measured from threshold to 340 Mev using the Berkeley 40-ft linear accelerator and 184-in. cyclotron. Absolute cross-section measurements were made at various energies, using a Faraday cup and calibrated beta-counter. The threshold occurs at 18.5±0.3 Mev. The cross section has a broad maximum of 100 millibarns near 45 Mev and decreases to 43 millibarns at 340 Mev.

I. INTRODUCTION

THE formation of radioactive C^{11} from C^{12} by high energy particles (protons, neutrons, deuterons, and alpha-particles) has been widely used at this laboratory as a monitor and detector.[1–6] These reactions have thresholds near 20 Mev and therefore discriminate against low energy background. The positron activity of C^{11}(0.97 Mev, 20.5 min)[7] is convenient for short activation and counting periods. Carbon targets are readily available in the form of graphite or polystyrene.

Knowledge of the variation of cross section with energy and the absolute value of the cross section is important to the extensive use of such reactions. The $C^{12}(p,pn)C^{11}$ reaction is of particular interest because of the number of existing proton accelerators. A considerable amount of work, both experimental and theoretical, has already been done at the Radiation Laboratory on this reaction. Before the 184-in. cyclotron was converted from deuteron to proton acceleration, Chupp and McMillan[8] measured the relative excitation curve up to 140 Mev using protons "stripped" from 190-Mev deuterons inside the cyclotron vacuum tank. By using this proton source, McMillan and Miller[9] determined the absolute cross section at 62 Mev. More recently Panofsky and Phillips,[10] working with the Berkeley 32-Mev proton linear accelerator, established the excitation curve up to 27 Mev. In particular they studied the region just above the threshold in

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