nucleus and the symmetrical mode. Nuclear temperature, defined by $1/T=d\ln\omega(E)/dE$, can be estimated from the expression for nuclear level densities; $\omega(E)=C\exp[2(E/a)^{\frac{1}{2}}]$. For uranium $C=10$ and $a=0.244$ gives the observed 5 electron volt spacing of levels at excitation energy of about 6 Mev. Under the assumption that the energy level density in the distorted nucleus varies in the same way, b becomes $(0.244)^{-\frac{1}{2}}$ and ΔE is 4.7 Mev. Figure 1 then indicates that the symmetrical mode of fission requires the order of 4.7 Mev more energy than the asymmetrical mode.[4]

The authors wish to express their thanks to Professor John A. Wheeler of Princeton University for his very valuable discussion and suggestions concerning the theoretical interpretation.

[4] *Note added in proof.*—Professor V. F. Weisskopf (private communication) uses $C=0.005$ and $a=0.08$ for the level densities in uranium. With these constants, ΔE becomes 2.7 Mev.

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Non-Adiabatic Treatment of the Relativistic Two-Body Problem

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The method of Tamm and Dancoff, for the non-adiabatic treatment of the relativistic interaction of two nucleons, is generalized in order to include nucleon pair creation and higher order effects in the exchange of mesons. This generalized form of the Tamm-Dancoff method is shown to give results which are equivalent to those obtained from the relativistic equation of Bethe and Salpeter. A detailed study is made of two limiting cases: (a) no pair of nucleons is created in the intermediate states, but an arbitrary number of mesons can be present at the same time; (b) the maximum number of mesons present at a given time is one, but the number of pairs is unrestricted.

The two methods are applied to the calculation of the lowest order correction to the scalar meson interaction of two nucleons. It is shown that the exact correction, which is of the second order in the nucleon velocities, can only be obtained through the inclusion of the fourth- and sixth-order interaction processes involving, in the corresponding Feynman diagrams, the crossing of the meson lines.

1. INTRODUCTION

THE study of a system of two bound particles interacting through a quantized field has usually been done by means of the so-called "adiabatic" approximation, that is: an effective potential between the interacting particles is calculated, neglecting their motion during the exchange of field quanta; this potential is then introduced into a wave equation, and the motion of the interacting particles is worked out as a second step. This method yields good results for nonrelativistic potentials but fails in the case of intrinsically relativistic interactions, such as that yielded by the pseudoscalar meson theory. It has been shown, in this case,[1] that the adiabatic approximation does not allow the system to bind, even if a relativistic calculation is made to the second order in the coupling constant. Furthermore, the notion of effective potential becomes ambiguous when higher order effects are considered.[2]

Since the physical evidence has focused the attention on the pseudoscalar meson theory, it is important to calculate the energy levels of a system of two bound nucleons by means of a method which does not separate the calculation of the interaction to a given order in the coupling constant, from the derivation of the equations of motion, valid to the same order. Such an approach has first been made independently by Tamm[3] and Dancoff,[4] who derived an approximate second-order equation for two particles interacting through a scalar meson field. Another way to treat non-adiabatically a bound system of two particles is provided by the covariant equation which has been proposed by Bethe and Salpeter[5] and subsequently derived from field theory by Gell-Mann and Low.[6]

It is the purpose of the present paper:

(a) to generalize the method of Tamm and Dancoff (abbreviated as T.D. in the following) as to include pair creation and higher order effects in the exchange of mesons, in order to bring it into a suitable form for the study of the pseudoscalar meson interaction between nucleons;

(b) to compare this extended form of the T.D. formalism, where the physical meaning of all quantities is clear at all stages, with the results obtained from the equation of Bethe and Salpeter (abbreviated as B.S. in the following); it will be shown that both descriptions are equivalent, although the way in which the interaction is expanded in powers of the coupling constant is quite different;

[1] M. M. Lévy, Phys. Rev. **84**, 441 (1951).
[2] Y. Nambu, Prog. Theor. Phys. **5**, 614 (1950).
[3] I. Tamm, J. Phys. (U.S.S.R.) **9**, 449 (1945).
[4] S. M. Dancoff, Phys. Rev. **78**, 382 (1950).
[5] E. Salpeter and H. A. Bethe, Phys. Rev. **84**, 1232 (1951).
[6] M. Gell-Mann and F. Low, Phys. Rev. **84**, 350 (1951).

(c) to illustrate this equivalence and to compare the practical value of the two methods, by calculating the lowest order non-adiabatic corrections to the scalar meson interaction between two nucleons; it will be shown that the "ladder" approximation, in the B.S. equation, as well as the second-order approximation in the T.D. formalism, give rise to corrections of the first order in nucleon velocities; however, when the fourth-order effects in the exchange of mesons, especially those which correspond to interaction diagrams involving meson lines crossings, are taken into account, the non-adiabatic corrections of the first order in v/c are cancelled exactly, and the remaining corrections are of the order of v^2/c^2.

From the preceding study it will be concluded that it is more convenient to use the T.D. method when only convergent processes are taken into account; on the other hand, when renormalizations of mass and charge have to be carried out explicitly, it appears easier to use the B.S. equation. The results of the present paper provide a correspondence between the two methods, and therefore make it possible to use them concurrently, depending on the effects which have to be calculated.

The application of the general theory to the analysis of the pseudoscalar meson interaction of two nucleons and the calculation of the low energy properties of the neutron-proton system will be considered in a subsequent paper.

2. EXTENSION OF THE TAMM-DANCOFF FORMALISM

2.1. General Equations

Let us consider two nucleons (1) and (2), with wave fields $\psi(1)$ and $\psi(2)$ interacting through a neutral meson[7] field ϕ. We write the Schrödinger equation for the wave-functional Ψ of the system:[8]

$$-i\delta\Psi/\delta t=(H_0+H')\Psi, \tag{1}$$

in which H_0 is the free Hamiltonian

$$H_0=\int\{\bar{\psi}(1)H_1\psi(1)+\bar{\psi}(2)H_2\psi(2)+[\pi^2+(\nabla\phi)^2+\mu^2\phi^2]\}d^3x_1d^3x_2, \tag{2}$$

where H_1 and H_2 are the free Hamiltonian operators of particles (1) and (2), and π the field canonically conjugate to ϕ. H' is an interaction Hamiltonian, the matrix element of which describe the simultaneous creation (or annihilation) of one meson and zero or one pair of nucleons. Looking for stationary expressions of Ψ, as a function of the (common) time t, we expand it as a series of the complete and orthonormal set of eigenfunctions of H_0:

$$\Psi(t)=\sum_{\lambda,m,n}a_\lambda^{(m,n)}\psi_\lambda^{(m,n)}e^{iWt}. \tag{3}$$

In this equation, W is the total energy of the system, m the number of mesons present in the free state which is considered, n the corresponding number of pairs of nucleons; λ is a variable which specifies the momenta, spins, etc., $\cdots$ of the particles of the systems and $a_\lambda^{(m,n)}$ the quantized probability amplitude[9] of the state (λ, m, n). The solution of Eq. (1) leads to the following set of coupled integral equations:

$$[W-E_\lambda^{(m,n)}]a_\lambda^{(m,n)}=\sum_{q=n-1}^{n+1}\sum_{p=m\pm1}\sum_{\mu}(\lambda, m, n|H'|\mu, p, q)a_\mu^{(p,q)}, \tag{4}$$

where we have put:

$$(\lambda, m, n|H'|\mu, p, q)=(\psi_\mu^{(p,q)}, H'\psi_\lambda^{(m,n)}). \tag{5}$$

One of the ways of solving the system of Eqs. (4) [other ways will be considered later] consists in eliminating all the amplitudes except $a_\lambda^{(0,0)}$, by means of successive substitutions. The resulting equation is of the following form:

$$[W-E_\lambda^{(0,0)}]a_\lambda^{(0,0)}=\sum_{p,q}\sum_{\lambda'}\Delta^{(p,q)}(\lambda,\lambda')a_{\lambda'}^{(0,0)}, \tag{6}$$

where $\Delta^{(p,q)}(\lambda,\lambda')$ is an interaction term which is proportional to the $2p$th power of the coupling constant G, and which results from a series of virtual processes such that the maximum numbers of mesons and nucleon pairs in the intermediate states are, respectively, p and q.

The series on the right-hand side of (6) includes all kinds of divergent processes, such as self-energy, vacuum polarization, etc. In the framework of the present formalism, it is not possible, even if the meson interaction belongs to a renormalizable type,[10] to separate covariantly these divergences and to reinterpret them as renormalization effects of mass and charge.[11] It is possible, however, to treat them separately, by using the correspondence between the present formalism and the covariant equation of Bethe and Salpeter, which will be discussed in Sec. 3.

In the remaining paragraphs of this Section, we shall only consider virtual processes which, in quantum electrodynamics and spinless meson theories, are essentially convergent. We shall further limit ourselves to two special cases:

[7] The words "nucleon" and "meson" are used, in this section, as a matter of convenience. The formalism applies generally to two Fermi-Dirac particles interacting through a Boson field.

[8] A system of units in which $\hbar=c=1$ is used throughout this paper.

[9] These amplitudes have, of course, to be properly symmetrized. The two originally interacting nucleons are treated separately from the virtual pairs, since they cannot annihilate each other.

[10] P. T. Matthews, Phil. Mag. **41**, 185 (1950); A. Salam, Phys. **82**, 217 (1951); **84**, 426 (1951).

[11] A covariant form of the configuration space treatment of the two-body problem has recently been proposed by S. Schweber and A. Wightman (to be published). Renormalization becomes possible in their formalism, but meets with considerable difficulties, mainly because of the vacuum fluctuations of which some finite parts, depending on the total energy of the system, have to be separated and re-interpreted. The simplest way to treat radiative effects seems to be to use the equivalence between the T.D. formalism and the B.S. equation (see Sec. 3).

(a) No pair of nucleons can be produced, but an arbitrary number of mesons can simultaneously be present in the system.

(b) The maximum number of mesons at any time is one, but an arbitrary number of nucleon pairs can be present at the same time.

2.2. Higher Order Effects in the Exchange of Mesons (No Pairs Present)

For this problem we have to put $n=q=0$ in (5). The two special cases $m\leqslant 1$ and $m\leqslant 2$ will first be discussed in detail, as the results will be needed in Sec. 4, for the calculation of the relativistic corrections to the scalar interaction.

2.21. $m\leqslant 1$

This simple case has been treated by Tamm[3] and Dancoff.[4] The system (4) contains only two equations connecting $a_\lambda^{(0,0)}$ and $a_\mu^{(1,0)}$. Eliminating the latter, one obtains an equation like (6), with only one interaction term:

$$\left.\begin{aligned}\Delta_2^{(1,0)}=&\frac{(\mathbf{p}_1,\mathbf{p}_2,s_1,s_2|H'|\mathbf{p}_1-\boldsymbol{\kappa},\mathbf{p}_2,s_1',s_2)(\mathbf{p}_1-\boldsymbol{\kappa},\mathbf{p}_2,s_1',s_2|H'|\mathbf{p}_1-\boldsymbol{\kappa},\mathbf{p}_2+\boldsymbol{\kappa},s_1',s_2')}{W-E_{p_1-\kappa}-E_{p_2}-\omega_\kappa}\\&+\text{a symmetrical term with respect to }\mathbf{p}_1\text{ and }\mathbf{p}_2,\end{aligned}\right\}\quad(7)$$

which can be considered, in the momentum representation, as an "equivalent potential," valid to the second order in G. In this equation, $\mathbf{p}_1$, $\mathbf{p}_2$ are the momenta of the nucleons, M their mass, $\boldsymbol{\kappa}$ the momentum of the exchanged meson; s_1, s_2, s_1', s_2' represent the spin states of the nucleons. We have set

$$E_p=(p^2+M^2)^{\frac{1}{2}},\quad \omega_\kappa=(\kappa^2+\mu^2)^{\frac{1}{2}}.$$

2.22. $m\leqslant 2$

The elimination of $a_\mu^{(1,0)}$ and $a_\nu^{(2,0)}$ would already lead, at this stage, to an Eq. (6) with an infinite series of interaction terms. It is, however, possible to replace this series by an interaction kernel obeying an integral equation. By eliminating $a_\nu^{(2,0)}$, one first gets an inhomogeneous integral equation for $a_\mu^{(1,0)}$:

$$[W-E_\mu^{(1,0)}]a_\mu^{(1,0)}=\sum_{\lambda'}(\mu,1,0|H'|\lambda',0,0)a_{\lambda'}^{(0,0)}+\sum_{\nu,\mu'}\frac{(\mu,1,0|H'|\nu,2,0)(\nu,2,0|H'|\mu',1,0)}{W-E_\nu^{(2,0)}}a_{\mu'}^{(1,0)}.\quad(8)$$

Calculating the Neuman-Liouville series corresponding to this equation, one finds that $a_\mu^{(1,0)}$ can be written in the form:

$$[W-E_\mu^{(1,0)}]a_\mu^{(1,0)}=\sum_{\lambda'}K_{\mu\lambda'}^{(1)}a_{\lambda'}^{(0,0)},\quad(9)$$

where $K_{\mu\lambda'}^{(1)}$ is a solution of the following equation:[12]

$$K_{\mu\lambda'}^{(1)}=(\mu,1,0|H'|\lambda',0,0)+\sum_{\nu,\mu'}\frac{(\mu,1,0|H'|\nu,2,0)(\nu,2,0|H'|\mu',1,0)}{[W-E_\nu^{(2,0)}][W-E_{\mu'}^{(1,0)}]}K_{\mu'\lambda'}^{(1)}.\quad(10)$$

By substituting (9) in the first equation of (5), one finally obtains the equation for $a_\lambda^{(0,0)}$:

$$[W-E_\lambda^{(0,0)}]a_\lambda^{(0,0)}=\sum_{\mu,\lambda'}\frac{(\lambda,0,0|H'|\mu,1,0)}{W-E_\mu^{(1,0)}}K_{\mu\lambda'}^{(1)}a_{\lambda'}^{(0,0)}.\quad(11)$$

The other amplitudes can then simply be expressed as functions of $a_\lambda^{(0,0)}$.

It should be noted that, to this order (where only terms up to the fourth order in the coupling constant are correctly taken into account), it is entirely consistent to replace $K_{\mu\lambda'}^{(1)}$ by its first Born approximation. This leads again to an equation like (6), with two interaction terms: the first one is identical with Δ_2, defined by (7); the second can be split into two parts $\Delta_4^{(a)}$ and $\Delta_4^{(b)}$, which can be written explicitly:

$$\Delta_4^{(a)}=2\frac{(\mathbf{p}_1\,\mathbf{p}_2,s_1,s_2|H'|\mathbf{p}_1-\boldsymbol{\kappa},\mathbf{p}_2,s_1',s_2)(\mathbf{p}_1-\boldsymbol{\kappa},\mathbf{p}_2,s_1',s_2|H'|\mathbf{p}_1-\boldsymbol{\kappa}-\boldsymbol{\kappa}',\mathbf{p}_2,s_1'',s_2)}{(W-E_{p_1-\kappa}-E_{p_2}-\omega_\kappa)(W-E_{p_1-\kappa-\kappa'}-E_{p_2}-\omega_\kappa-\omega_{\kappa'})}\times\frac{(\mathbf{p}_1-\boldsymbol{\kappa}-\boldsymbol{\kappa}',\mathbf{p}_2,s_1'',s_2|H'|\mathbf{p}_1-\boldsymbol{\kappa}-\boldsymbol{\kappa}',\mathbf{p}_2+\boldsymbol{\kappa},s_1'',s_2')\times(\mathbf{p}_1-\boldsymbol{\kappa}-\boldsymbol{\kappa}',\mathbf{p}_2+\boldsymbol{\kappa},s_1'',s_2'|H'|\mathbf{p}-\boldsymbol{\kappa}-\boldsymbol{\kappa}',\mathbf{p}_2+\boldsymbol{\kappa}+\boldsymbol{\kappa}',s_1'',s_2'')}{(W-E_{p_1-\kappa-\kappa'}-E_{p_2+\kappa}-\omega_{\kappa'})},\quad(12a)$$

$$\Delta_4^{(b)}=2\frac{(\mathbf{p}_1,\mathbf{p}_2,s_1,s_2|H'|\mathbf{p}_1-\boldsymbol{\kappa}',\mathbf{p}_2,s_1',s_2)(\mathbf{p}_1-\boldsymbol{\kappa}',\mathbf{p}_2,s_1',s_2|H'|\mathbf{p}_1-\boldsymbol{\kappa}-\boldsymbol{\kappa}',\mathbf{p}_2,s_1'',s_2)}{(W-E_{p_1-\kappa'}-E_{p_2}-\omega_{\kappa'})(W-E_{p_1-\kappa-\kappa'}-E_{p_2}-\omega_\kappa-\omega_{\kappa'})}\times\frac{(\mathbf{p}_1-\boldsymbol{\kappa}-\boldsymbol{\kappa}',\mathbf{p}_2,s_1'',s_2|H'|\mathbf{p}_1-\boldsymbol{\kappa}-\boldsymbol{\kappa}',\mathbf{p}_2+\boldsymbol{\kappa},s_1'',s_2')\times(\mathbf{p}_1-\boldsymbol{\kappa}-\boldsymbol{\kappa}',\mathbf{p}_2+\boldsymbol{\kappa},s_1'',s_2'|H'|\mathbf{p}_1-\boldsymbol{\kappa}-\boldsymbol{\kappa}',\mathbf{p}_2+\boldsymbol{\kappa}+\boldsymbol{\kappa}',s_1'',s_2'')}{(W-E_{p_1-\kappa-\kappa'}-E_{p_2+\kappa}-\omega_{\kappa'})}\quad(12b)$$

[12] It is to be noted that, when the Neuman-Liouville expansion does not converge, Eq. (10) continues to have a meaning.

plus the symmetrical terms with respect to $\mathbf{p}_1$ and $\mathbf{p}_2$. These interaction terms correspond to virtual processes which are illustrated[13] in Fig. 1: $\Delta_4^{(a)}$ corresponds to graphs (a) and (a'); $\Delta_4^{(b)}$ to (b) and (b').

The process where meson κ' is absorbed by particle (1) before κ has been emitted has already been taken care of in the first term of the right-hand side of (6). This can be seen by iterating Eq. (6) an arbitrary number of times. An important feature of the present formalism is, consequently, that an infinite number of virtual processes is already included in a single interaction term. It is, therefore, suited to the treatment of bound states problems, where the interaction between the nucleons occurs during a long time.

2.23. *m Arbitrary*

We first suppose that the probability amplitudes of the states where the number of mesons is higher than $N+1$ (N being an arbitrary positive integer) are sufficiently small and can be neglected. The system (5) consists, in this case, in a set of $N+2$ coupled equations, the mth of which connects $a_\lambda^{(m-1,0)}$, $a_\mu^{(m,0)}$ and $a_\nu^{(m+1,0)}$. By eliminating all the amplitudes except $a_\lambda^{(0,0)}$, one gets for the latter amplitude an equation identical with (9); but the kernel $K_{\mu\lambda'}^{(1)}$, does not satisfy any longer Eq. (10). Instead, one introduces a series of kernels

$$K_{\lambda_1\mu_1}^{(1)}, \cdots, K_{\lambda_{N+1},\mu_{N+1}}^{(N+1)},$$

defined by the following equations:

$$K_{\lambda\mu}^{(m)} = (\lambda, m, 0|H'|\mu, m-1, 0) + \sum_{\mu',\lambda'} \frac{(\lambda, m, 0|H'|\mu', m+1, 0)}{[W-E_{\mu'}^{(m+1,0)}][W-E_{\lambda'}^{(m,0)}]} K_{\mu'\lambda'}^{(m+1)} K_{\lambda'\mu}^{(m)}, \tag{13}$$

with $m=1, 2\cdots N+1$, and $K_{\lambda\mu}^{(N+2)}=0$. All the amplitudes $a_\lambda^{(m,0)}$ are then expressed as functions of $a_{\lambda'}^{(0,0)}$, through the kernels $K^{(1)}, \cdots, K^{(m)}$, by means of the equation:

$$[W-E_\lambda^{(m,0)}]a_\lambda^{(m,0)} = \sum_{\mu_1,\cdots\mu_{m-1},\lambda'} \frac{K_{\lambda\mu_1}^{(m)} K_{\mu_1\mu_2}^{(m-1)} \cdots K_{\mu_{m-1},\lambda'}^{(1)}}{[W-E_{\mu_1}^{(m-1,0)}]\cdots[W-E_{\mu_{m-1}}^{(1,0)}]} a_{\lambda'}^{(0,0)}. \tag{14}$$

These equations are valid for any value of the coupling constant G, if there exist a finite number of intermediate states having higher probability amplitudes than all the others. On the other hand, they are more convenient than the system (5), because one has now to solve a set of uncoupled equations, the kernel of each one being defined by the solution of the preceding one. The first N equations are inhomogeneous; the eigenvalue problem occurs only in the solution of the last one [Eq. (9)], which is homogeneous. Furthermore, the problem is now all set for an application of perturbation theory: having solved the system of equations corresponding to $m\leq N+1$, it is possible, by doing only some integrations, to calculate the correction to the energy levels which is provided by the system corresponding to $m\leq N+2$.

When it is possible to expand the interaction in powers of G^2, one can replace Eqs. (9) and (13) by Eq. (6), with $q=0$ and

$$\Delta^{(p,0)}(\lambda, \lambda') = \sum_{\mu_i} \sum_{(r_i)} \frac{(\lambda, 0, 0|H'|\mu_1, 1, 0)(\mu_1, 1, 0|H'|\mu_2, 2, 0)}{[W-E_{\mu_1}^{(1,0)}][W-E_{\mu_2}^{(2,0)}]} \times \left\{\prod \frac{(\mu_{i-1}, r_{i-1}, 0|H'|\mu_i, r_i, 0)}{[W-E_{\mu_i}^{(r_i,0)}]}\right\} \times \frac{(\mu_{2p-2}, 2, 0|H'|\mu_{2p-1}, 1, 0)(\mu_{2p-1}, 1, 0|H'|\lambda', 0, 0)}{[W-E_{\mu_{2p-1}}^{(1,0)}]}. \tag{15}$$

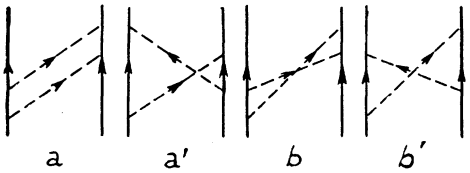

FIG. 1. Virtual processes involving two mesons (and no pairs) in the intermediate states.

The summation of the right-hand side is extended on all possible sets of $2p-4$ numbers (r_i), with $i=3, \cdots, 2p-2$, which satisfy the following conditions: $r_i\leq p$, $r_{i+1}=r_i\pm 1$, $r_2=r_{2p-2}=2$. The formal expression (15) might allow, in certain cases, the summation of the interaction terms which appear on the right-hand side of (6).

2.3. Lowest Order Pair Creation Effects

We consider the case where the maximum number of mesons which can be present at a given time in the intermediate states is one, the number of nucleon pairs simultaneously present being unrestricted. This case includes the lowest order exchange processes in which virtual pair creation can affect the interaction of the two particles of the system, and not merely their self-energy. After a brief study of the general equations, we shall derive in an explicit fashion the effective interaction resulting from the creation of two virtual pairs.

2.31. *General Equations for $m\leq 1$, n Arbitrary*

For each value of n, the system (4) includes two equations, defining the amplitudes $a_\lambda^{(0,n)}$ and $a_\lambda^{(1,n)}$. The

[13] In the graphical description of the virtual exchange effects, a conventional time ordering is used by which the emission and absorption of mesons, the creation or annihilation of pairs, occur in the same order as the corresponding matrix elements in $\Delta^{(p,q)}(\lambda, \lambda')$, counted from the right to the left.

latter can be eliminated so that the transformed system contains only the probability amplitudes of states where no meson is present:

$$[W-E_\lambda^{(0,n)}]a_\lambda^{(0,n)} = \sum_{0\leqslant n'=n-2}^{n+2} \sum_{\lambda'} Q^{(n,n')}(\lambda,\lambda')a_{\lambda'}^{(0,n')}, \quad (16)$$

where we have set:

$$Q^{(n,n')}(\lambda,\lambda') = \sum_{p=n-1}^{n+1} \sum_{\mu} \frac{(\lambda,0,n|H'|\mu,1,p)(\mu,1,p|H'|\lambda',0,n')}{[W-E_\mu^{(1,p)}]}, \quad (17)$$

in which p and n' are restricted by the inequalities: $n-1\leq p\leq n+1$, $p-1\leq n'\leq p+1$. $Q^{(n,n')}(\lambda,\lambda')$ is a "reduced" matrix element for the transitions between two states where n' and n nucleon pairs are present respectively.

The form of Eqs. (16) is such that it is possible to apply to them the method of integral equations and successive kernels developed in Sec. 2.23. The situation

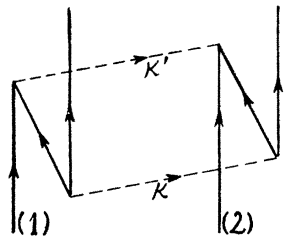

FIG. 2. A virtual process involving two pairs, in the intermediate states, but not more than one meson at a given time.

is, however, a little more complicated, since the summation of the right-hand side extends over five amplitudes, instead of just two as in the case where no pairs are present. Two series of kernels have to be defined, by means of the equations:

$$[W-E_\lambda^{(0,n)}]a_\lambda^{(0,n)} = \sum_{\lambda'} L^{(n)}(\lambda,\lambda')a_{\lambda'}^{(0,n-1)} + M^{(n)}(\lambda,\lambda')a_{\lambda'}^{(0,n-2)}, \quad (18)$$

and these kernels are solutions of integral equations, the kernels of which are functions of other kernels corresponding to higher values of n.

When it is possible to expand the interaction in powers of G^2, one can replace Eqs. (16) by Eq. (6) with $p=1$, where the interaction terms $\Delta^{(1,q)}$ can be written explicitly:

$$\Delta^{(1,q)}(\lambda,\lambda') = \sum_{\alpha=1,2} \sum_{\beta=1,2} \sum_{(r_i)} \sum_{\mu} \frac{Q^{(0,\alpha)}(\lambda,\mu_1)}{W-E_{\mu_1}^{(0,\alpha)}} \times \left\{ \prod_{i=1}^{q-2} \frac{Q^{(r_i,r_{i+1})}(\mu_i,\mu_{i+1})}{W-E_{\mu_{i+1}}^{(0,r_{i+1})}} \right\} Q^{(\beta,0)}(\mu_{q-1},\lambda'). \quad (19)$$

The summation of the right-hand side extends over all possible sets of $q-1$ positive integers (r_i), with $i=1,2,\cdots,q-1$, which satisfy the conditions: $0<r_i\leq q$, $r_i-2\leq r_{i+1}\leq r_i+2$, $r_1=\alpha$, $r_{q-1}=\beta$.

The generalization of Eqs. (16) and (19) to the case where more than one meson is present at one time is straightforward, although the equations become complicated. The only modification which has to be made is in the form of $Q^{(n,n')}(\lambda,\lambda')$, which becomes analogous to $\Delta^{(p,0)}(\lambda,\lambda')$ of Eq. (15).

2.32. *Effective Interaction Resulting from the Creation of Two Virtual Pairs*

This calculation, which illustrates the formal equations of the preceding paragraph, will also be used in the next section for comparison with the results obtained from the B.S. equation: it will then provide an example of the analytical differences which occur between the two treatments when virtual nucleon pairs are present in the intermediate states.

After inserting in the system (4) the conditions $m\leq 1$, $n=0$ or 2, we obtain for $a_\lambda^{(0,0)}$, an equation like (6), with two interaction terms on the right-hand side. The first is identical with $\Delta_2^{(1,0)}$ of (7); the second, which results from the virtual process illustrated in Fig. 2 (plus those which are symmetrical with respect to particles 1 and 2), has the following expression:

$$\Delta_4^{(1,2)} = \frac{1}{2V^2} \frac{\Gamma_{12}^{(1)}(\mathbf{p}_1,-\mathbf{p}_1-\boldsymbol{\kappa}')\Gamma_{12}^{(2)}(\mathbf{p}_2,-\mathbf{p}_2+\boldsymbol{\kappa}')\Gamma_{21}^{(1)}(-\mathbf{p}_1-\boldsymbol{\kappa}',\mathbf{p}_1+\boldsymbol{\kappa}+\boldsymbol{\kappa}')\Gamma_{21}^{(2)}(-\mathbf{p}_2+\boldsymbol{\kappa}',\mathbf{p}_2-\boldsymbol{\kappa}-\boldsymbol{\kappa}')}{\omega_\kappa\omega_{\kappa'}[W-E_{p_1}-E_{p_2}-E_{p_2-\kappa'}-E_{p_2-\kappa-\kappa'}-\omega_\kappa]} \quad (20)$$
$$\times[W-E_{p_1}-E_{p_2}-E_{p_1+\kappa'}-E_{p_2-\kappa'}-E_{p_1+\kappa+\kappa'}-E_{p_2-\kappa-\kappa'}]$$
$$\times[W-E_{p_1}-E_{p_1+\kappa'}-E_{p_1+\kappa+\kappa'}-E_{p_2-\kappa-\kappa'}-\omega_{\kappa'}]$$

where we have used the notation

$$(\lambda,0,0|H'|\mu,1,1)=[2V\omega_\kappa]^{-\frac{1}{2}}\Gamma_{12}^{(r)}(\mathbf{p}_r,-\mathbf{p}_r-\kappa), \quad (21)$$

for the matrix element describing the creation of one meson and one nucleon pair; $\mathbf{p}_r$ is the initial momentum of nucleon (r), $\boldsymbol{\kappa}$ the momentum of the emitted meson and V the volume in which the nucleon wave functions are normalized. We have set $\omega_\kappa=(\kappa^2+\mu^2)^{\frac{1}{2}}$. Another slightly more general definition of $\Gamma_{ij}^{(r)}(\mathbf{p},\mathbf{p}')$ will be given in 3.1.

3. NON-ADIABATIC CORRECTIONS IN THE BETHE AND SALPETER EQUATION

We write B.S. equation, in coordinate space, as follows:[14]

$$\chi(x_1, x_2) = -\int S_F^{(1)}(x_1, x_1')S_F^{(2)}(x_2, x_2') \times G(x_1', x_2'; x_1'', x_2'')\chi(x_1'', x_2'')dx_1'dx_2'dx_1''dx_2'', \quad (22)$$

where the indices (1) and (2) refer to the nucleons. For the interaction kernel G, which is a covariant function of $x_1'-x_2'$, $x_1''-x_2''$, Bethe and Salpeter have proposed an expansion in powers of the coupling constant:

$$G=G_2+G_4+\cdots \quad (23)$$

where G_2 is the Feynman expression corresponding to the second-order diagram of the S-matrix:

$$G_2(x_1', x_2'; x_1'', x_2'') = G^2\Delta_F(x_1'-x_2')\delta(x_1', x_1'')\delta(x_2', x_2''). \quad (24)$$

The functions S_F and Δ_F, which appear in (22) and (24) are defined according to Dyson.[15] The replacement of G by G_2 corresponds to the so-called "ladder approximation," to which we shall restrict ourselves in the following. The extension of our calculations to the other terms of the interaction can be carried out in a straightforward way.

3.1. Expression of Eq. (22) in the Momentum Representation

The "wave function" $\chi(x_1, x_2)$ is expanded by means of its Fourier coefficients, in the following way:

$$\chi(x_1, x_2)=\sum_{i,j}\sum_{p_1,p_2}\frac{1}{V}A_{ij}(p_1, p_2) \times u_i^{(1)}(\mathbf{p}_1)u_j^{(2)}(\mathbf{p}_2)e^{i(p_1x_1+p_2x_2)}, \quad (25)$$

where the function $u_i^{(r)}(p_r)$ is the amplitude of the Dirac spinor, obeying the following equation:

$$[i\boldsymbol{\gamma}^{(r)}\cdot\mathbf{p}_r+\gamma_4^{(r)}E_i+M]u_i^{(r)}(\mathbf{p}_r)=0. \quad (26)$$

The indices i, j can take the values 1, 2. We have set $E_1(\mathbf{p}_r)=-E_2(\mathbf{p}_r)=E_{pr}$. Substituting (25) into (22), using the momentum representation of the S_F and Δ_F functions and multiplying both sides on the right with $\bar{u}_i^{(1)}(\mathbf{p}_1)\bar{u}_j^{(2)}(\mathbf{p}_2)(i\gamma_\mu^{(1)}p_\mu^{(1)}+M)(i\gamma_\nu^{(2)}p_\nu^{(2)}+M)$, the following equation is obtained:

$$A_{ij}(p_1, p_2) = \frac{-iG^2}{2V[E^{(1)}-E_i(\mathbf{p}_1)+i\epsilon_i\eta][E^{(2)}-E_j(\mathbf{p}_2)+i\epsilon_j\eta]} \times\sum_{k,l}\sum_{\kappa}\frac{\Gamma_{ik}^{(1)}\Gamma_{jl}^{(2)}A_{kl}(p_1-\kappa, p_2+\kappa)}{\kappa^2+\mu^2-i\eta}, \quad (27)$$

+symmetrical term with respect to particles (1) and (2), where the following notations have been used:

$$E^{(r)}=-ip_4^{(r)}$$

and

$$\Gamma_{ik}^{(r)}=\bar{u}_i^{(r)}(\mathbf{p}_r)\gamma_4^{(r)}\mathcal{O}^{(r)}u_k^{(r)}(\mathbf{p}_r+\epsilon_r\boldsymbol{\kappa}),$$

$\mathcal{O}^{(r)}$ being an operator connected with the properties of the meson field which is considered: $\mathcal{O}^{(r)}=1$ for scalar, $i\gamma_5^{(r)}$ for pseudoscalar, $\gamma_\mu^{(r)}$ for vector fields, etc. We have also $\epsilon_1=+1$, $\epsilon_2=-1$, η is a small positive quantity which tends to zero after the summation over κ on the right hand side of (27) has been carried out.

We use in the following the center of gravity system, defined as: $p^{(1)}+p^{(2)}=P(0,0,0,W)$, $p^{(1)}-p^{(2)}=p(\mathbf{p}, ip_0)$. By going to the limit of a continuous distribution of the momenta, we obtain the equation:

$$A_{ij}(\mathbf{p}, p_0)=\frac{1}{[\Lambda_i(\mathbf{p})-p_0][\Lambda_j(\mathbf{p})+p_0]}\sum_{k,l}\frac{-iG^2}{(2\pi)^4} \times\int\frac{\Gamma_{ik}^{(1)}\Gamma_{jl}^{(2)}A_{kl}(\mathbf{p}+\boldsymbol{\kappa}, p_0+\kappa_0)d^3\kappa d\kappa_0}{\omega_\kappa^2-\kappa_0^2-i\eta}, \quad (28)$$

where we have put:

$$\Lambda_i(\mathbf{p})=\tfrac{1}{2}W-E_i(\mathbf{p})+i\epsilon_i\eta. \quad (29)$$

When the times of the two particles are equalled in $\chi(x_1, x_2)$ the corresponding amplitude is given by:

$$a_{ij}(\mathbf{p})=\frac{1}{2\pi i}\int_{-\infty}^{+\infty}A_{ij}(\mathbf{p}, p_0)dp_0. \quad (30)$$

3.2. Adiabatic Approximation in the B.S. Equation

In order to obtain the adiabatic approximations to Eq. (28), we neglect the motion of the nucleons while the mesons are in flight. Mathematically, this corresponds to replacing Δ_F, in Eq. (22), by its time integral, multiplied by a δ-function operating on the time coordinate.

$$\Delta_F(x)\rightarrow\delta(x_4)\int_{-\infty}^{+\infty}\Delta_F(\mathbf{x}, x_4)dx_4. \quad (31)$$

As is well known, the integral of the right-hand side of (31) is equal to the Yukawa potential. With the help of

[14] From Eq. (22) to Eq. (31) inclusive, light type is used for 4-dimensional quantities, heavy type for three-dimensional ones. In the rest of the paper, light type is used for one-dimensional quantities, unless otherwise stated.

[15] F. J. Dyson, Phys. Rev. **75**, 486, 1736 (1949).

(30), Eq. (28) becomes:

$$A_{ij}^{(0)}(\mathbf{p}, p_0)=\frac{F_{ij}^{(0)}(\mathbf{p})}{[\Lambda_i(\mathbf{p})-p_0][\Lambda_j(\mathbf{p})+p_0]}, \quad (32)$$

where we have put:

$$F_{ij}^{(0)}(\mathbf{p})=\lambda\sum_{k,l}\int\frac{\Gamma_{ik}^{(1)}\Gamma_{jl}^{(2)}a_{kl}^{(0)}(\mathbf{p}+\boldsymbol{\kappa})d^3\kappa}{\omega_\kappa^2}, \quad (33)$$

with $\lambda=G^2(2\pi)^{-3}$. In this approximation, the "odd" components of a_{ij} (corresponding to $i\neq j$) vanish. The "even" components obey the equation:

$$[W-2E_i(\mathbf{p})]a_{ii}^{(0)}(\mathbf{p})=-\lambda\epsilon_i\sum_k\int\frac{\Gamma_{ik}^{(1)}\Gamma_{ik}^{(2)}a_{kk}^{(0)}(\mathbf{p}+\boldsymbol{\kappa})d^3\kappa}{\omega_\kappa^2}. \quad (34)$$

This equation does not exactly correspond to the non-relativistic limit of (28), because it still involves the exact spin matrix elements and one small component of the amplitude. Its form is that of a two-particle Dirac equation, where the "odd" components[16] of the wave-function are equalled to zero.

3.3. Non-Adiabatic Corrections (Pairs Excluded)

The non-adiabatic corrections to Eq. (34) are calculated by inserting on the right-hand side the "adiabatic" approximation and iterating the equation an arbitrary number of times. When no pairs are created during the interaction of the two nucleons, Eq. (28) is restricted to the "large" components of the amplitude $A_{ij}(i=j=1)$. In this subsection, all the indices will be dropped; we shall write, for example, $A_{11}=A$, $a_{11}=a$, $\Lambda_1(\mathbf{p})=\Lambda_p$, $\Gamma_{11}^{(r)}(\mathbf{p}, \mathbf{p}')=\Gamma^{(r)}(\mathbf{p}, \mathbf{p}')$ etc.

In order to calculate the first non-adiabatic correction, we insert in the right-hand side of (28) the trial-function $A(\mathbf{p}, p_0)=-2\Lambda_p a(\mathbf{p})[\Lambda_p^2-p_0^2]^{-1}$, which has the same dependence on p_0 as the adiabatic solution, and satisfies Eq. (30). Integrating over κ_0 yields the equation:

$$A(\mathbf{p}, p_0)=\frac{i\lambda}{\Lambda_p^2-p_0^2}\int\frac{\Gamma^{(1)}(\mathbf{p}, \mathbf{p}+\boldsymbol{\kappa})\Gamma^{(2)}(-\mathbf{p}, -\mathbf{p}-\boldsymbol{\kappa})[\Lambda_{p+\kappa}-\omega_\kappa]a(\mathbf{p}+\boldsymbol{\kappa})d^3\kappa}{\omega_\kappa[(\Lambda_{p+\kappa}-\omega_\kappa)^2-p_0^2]}. \quad (35)$$

Inserting this expression into the right-hand side of (30) leads to an equation for $a(\mathbf{p})$ which can be written, with the help of (29), as follows:

$$(W-2E_p)a(\mathbf{p})=\lambda\int\frac{\Gamma^{(1)}(\mathbf{p}, \mathbf{p}+\boldsymbol{\kappa})\Gamma^{(2)}(-\mathbf{p}, -\mathbf{p}-\boldsymbol{\kappa})a(\mathbf{p}+\boldsymbol{\kappa})d^3\kappa}{\omega_\kappa(W-E_p-E_{p+\kappa}-\omega_\kappa)}. \quad (36)$$

The interaction term is readily found to be identical with (7) if the latter is written in the center-of-mass system ($\mathbf{p}_1=+\mathbf{p}$, $\mathbf{p}_2=-\mathbf{p}$). The second approximation to Eq. (34) is obtained by inserting in the right-hand side of (28) the expression (35) which has just been calculated for $A(\mathbf{p}, p_0)$. By making use of (30), one obtains the equation:

$$\begin{aligned}(W-2E_p)a(\mathbf{p})=&\lambda\int\frac{\Gamma^{(1)}(\mathbf{p}, \mathbf{p}+\boldsymbol{\kappa})\Gamma^{(2)}(-\mathbf{p}, -\mathbf{p}-\boldsymbol{\kappa})a(\mathbf{p}+\boldsymbol{\kappa})d^3\kappa}{\omega_\kappa(W-E_p-E_{p+\kappa}-\omega_\kappa)}\\&+\frac{\lambda^2}{2}\int\frac{\Gamma^{(1)}(\mathbf{p}, \mathbf{p}+\boldsymbol{\kappa})\Gamma^{(2)}(-\mathbf{p}, -\mathbf{p}-\boldsymbol{\kappa})\Gamma^{(1)}(\mathbf{p}+\boldsymbol{\kappa}, \mathbf{p}+\boldsymbol{\kappa}+\boldsymbol{\kappa}')\Gamma^{(2)}(-\mathbf{p}-\boldsymbol{\kappa}, -\mathbf{p}-\boldsymbol{\kappa}-\boldsymbol{\kappa}')a(\mathbf{p}+\boldsymbol{\kappa}+\boldsymbol{\kappa}')d^3\kappa d^3\kappa'}{\omega_\kappa\omega_{\kappa'}(W-E_{p+\kappa'}-E_{p+\kappa+\kappa'}-\omega_\kappa)(W-E_p-E_{p+\kappa+\kappa'}-\omega_\kappa-\omega_{\kappa'})(W-E_p-E_{p+\kappa'}-\omega_{\kappa'})}.\end{aligned} \quad (37)$$

On the right-hand side, the kernel of the first term is again identical with (7); the kernel of the second term is equal to $\frac{1}{2}\Delta_4^{(a)}$, defined by (12a), and corresponds to the virtual process where two mesons can be present at the same time, without crossing of meson lines illustrated in Fig. 1(a).

The previous procedure can be continued to all orders in the coupling constant. The amplitude $a(\mathbf{p})$ obeys the following general equation, for which a recurrence proof can be given through lengthy but straightforward calculations:

$$(W-2E_p)a(\mathbf{p})=2\sum_{n=1}^{\infty}\left(\frac{\lambda}{2}\right)^n\int\frac{D^{(n)}(\mathbf{p}, \boldsymbol{\kappa}_1, \cdots, \boldsymbol{\kappa}_n)}{\omega_{\kappa_1}\cdots\omega_{\kappa_n}}a(\mathbf{p}+\boldsymbol{\kappa}_1+\cdots+\boldsymbol{\kappa}_n)d^3\kappa_1\cdots d^3\kappa_n \quad (38)$$

with

$$D^{(n)}=\frac{\prod_{q=1}^{n-1}[\Gamma^{(1)}(\mathbf{p}+\boldsymbol{\kappa}_1+\cdots+\boldsymbol{\kappa}_{q-1}, \mathbf{p}+\boldsymbol{\kappa}_1+\cdots+\boldsymbol{\kappa}_q)\Gamma^{(2)}(-\mathbf{p}-\boldsymbol{\kappa}_1-\cdots-\boldsymbol{\kappa}_{q-1}, -\mathbf{p}-\boldsymbol{\kappa}_1-\cdots-\boldsymbol{\kappa}_q)}{\prod_{q=1}^{n}[W-E(\mathbf{p}+\boldsymbol{\kappa}_q+\cdots+\boldsymbol{\kappa}_n)-E(\mathbf{p}+\boldsymbol{\kappa}_1+\cdots+\boldsymbol{\kappa}_n)-\omega_{\kappa_1}-\cdots-\omega_{\kappa_q}]\times\prod_{q=1}^{n-1}[W-E_p-E(\mathbf{p}+\boldsymbol{\kappa}_{q+1}+\cdots+\boldsymbol{\kappa}_n)-\omega_{q+1}-\cdots-\omega_{\kappa_n}]}. \quad (39)$$

[16] Namely, those which correspond to $i\neq j$.

Equation (38) is identical with Eqs. (6) and (15) for $a_\lambda^{(0,0)}$ if, in all the processes formally included in the latter, only the exchange effects with no meson lines crossing are taken into account. In the ladder approximation, when no pairs are present, one has therefore the result:

$$a^{(0,0)}(\mathbf{p}, -\mathbf{p}) = a(\mathbf{p}); \tag{40}$$

the Fourier coefficient of the wave function corresponding to equal times of the particles is identical with the probability amplitude of the state where only two nucleons, and no meson, are present.

3.4. Calculation of the First Non-Adiabatic Approximation, Pair Creation Being Included

In this case, the whole set of components of A_{ij} must be considered. One inserts on the right-hand side of (28) the adiabatic solution defined by (32). Making use of (30) and (33), the equation for the amplitude corresponding to equal times of the particles is obtained:

$$[W - E_i(\mathbf{p}) - E_j(\mathbf{p})]a_{ij}(\mathbf{p}) = -\lambda \sum_{k,l} \int \Gamma_{ik}^{(1)} \Gamma_{jl}^{(2)} K_{ij}^{kl}(\mathbf{p}, \mathbf{p}+\boldsymbol{\kappa}) a_{kl}(\mathbf{p}+\boldsymbol{\kappa}) d^3\kappa, \tag{41}$$

where the kernel has the following expression:

$$K_{ij}^{kl} = \frac{1 - \frac{1}{4}(\epsilon_i - \epsilon_j)(\epsilon_k - \epsilon_l)}{4\omega_\kappa} \times \sum_{\substack{\mu=i,j \\ \nu=k,l}} \frac{\epsilon_\mu \epsilon_\nu}{\omega_\kappa - \epsilon_\mu \Lambda_\mu(\mathbf{p}) - \epsilon_\nu \Lambda_\nu(\mathbf{p}+\boldsymbol{\kappa})}. \tag{42}$$

By eliminating all the amplitudes except a_{11} by means of successive substitution, one obtains an equation of the general form:

$$(W - 2E_p) a_{11}(\mathbf{p}) = \sum_n \lambda^n \int \Delta_n(\mathbf{p}, \mathbf{p}') a_{11}(\mathbf{p}') d^3p', \tag{43}$$

where all the interaction kernels corresponding to virtual processes which do not involve pair creation are identical with those defined by Eq. (15). When nucleon pairs are created in the intermediate states, the corresponding kernels of Eq. (43) coincide with those defined by Eq. (19) only in the nonrelativistic limit. For example, the process illustrated by Fig. 2 leads to the interaction term:

$$\Delta_4^{(1,2)}(\mathbf{p}, \mathbf{p}') = \int \frac{\Gamma_{12}^{(1)}(\mathbf{p}, \mathbf{p}'') \Gamma_{12}^{(2)}(-\mathbf{p}, -\mathbf{p}'') \Gamma_{21}^{(1)}(\mathbf{p}'', \mathbf{p}') \Gamma_{21}^{(2)}(-\mathbf{p}'', -\mathbf{p}') d^3p''}{\omega(\mathbf{p}, \mathbf{p}'') \omega(\mathbf{p}'', \mathbf{p}') (W + 2E_{p''})(\omega_{p,p''} + E_p + E_{p''})(\omega_{p'',p'} + E_{p'} + E_{p''})} \tag{44}$$

which differs from the corresponding expression (20) in the T.D. formalism by corrections proportional to $(2E_p - W)$ or $(2E_{p'} - W)$. These differences tend to zero in the nonrelativistic limit, since, in that case, the amplitudes $a_{11}(\mathbf{p})$ and $a_{11}(\mathbf{p}')$ are mainly concentrated in the region of low momenta.[17]

4. RELATIVISTIC CORRECTIONS TO THE SCALAR MESON INTERACTION OF TWO NUCLEONS

In this section we calculate the lowest order relativistic corrections to the scalar neutral meson interaction of two nucleons. The purpose of this application is to compare in practical calculations the two nonadiabatic methods which have been discussed, in general terms, in Secs. 2 and 3.

Before relativistic "corrections" can be calculated, it is necessary to define briefly the nonrelativistic equation, wave function and coupling constant which have to be corrected.

4.1. Nonrelativistic Schrödinger Equation

In the scalar (neutral) meson theory, the nonrelativistic amplitude $a_0(\mathbf{p})$ is constituted by four "large" components, all of them obeying the same equation (there is no spin-separation between the stationary states):

$$[p^2 + M|w|]a_0(\mathbf{p}) = \lambda_0 M \int \frac{a_0(\mathbf{p}+\boldsymbol{\kappa}) d^3\kappa}{\omega_\kappa^2}, \tag{45}$$

where w is the binding energy: $w = W - 2M$. The simplest approximation to the solution of this equation[18] is provided by the "trial"-function of Wilson:[19] $a_0(\mathbf{p}) \sim (p^2 + \gamma^2)^{-2}$. The eigenvalue corresponding to the ground state is obtained by means of the variational principle:[20] $\delta\lambda_0 = 0$, where λ_0 is defined by:

$$\frac{1}{\lambda_0} = \frac{\int a_0^*(\mathbf{p}) \omega_\kappa^{-2} a_0(\mathbf{p}+\boldsymbol{\kappa}) d^3p d^3\kappa}{\int a_0^*(\mathbf{p})[p^2 + M|w|]a_0(\mathbf{p}) d^3p}. \tag{46}$$

The corresponding values of the constants are as follows: $\mu = 140.25$ Mev, $(\mu/M = 0.150)$, $|w| = 2.185$ Mev,

[17] These differences result from the fact that the "one-meson part" of the "ladder" approximation includes a small fraction of the contributions of the exchange effects where, in the intermediate states, more than two pairs are present. It is to be noted that the difference $2E_{p''} - W$, where $\mathbf{p}''$ is the intermediate momentum which appears in Eq. (44), cannot be considered as small, even in the non-relativistic limit. Terms involving $\mathbf{p}''$ are in fact, identical in both expressions (20) and (44).

[18] For a general study of the solution of this equation in the momentum representation, see: M. M. Lévy, Proc. Roy. Soc. (London) **A204**, 145 (1950), and J. phys. et radium (to be published).

[19] A. H. Wilson, Proc. Cambridge Phil. Soc. **34**, 365 (1938).

[20] It is simpler, and more in accordance with the practical situation, where the binding energy is known, and the coupling constant unknown, to vary λ_0 instead of $|w|$.

$\gamma=0.843\mu, G^2/4\pi=2\pi^2\lambda_0=2.453(\mu/M)$. The mean value of the momentum is given by: $\langle p^2/M^2\rangle=\gamma^2/M^2$. Consequently, $G^2/4\pi$, $\langle p/M\rangle=\langle v/c\rangle$ and μ/M are all of the same order of magnitude.

4.2. Variational Principle in a Perturbation Expansion

We need a formula which defines, through a variational principle, the corrections to λ_0 introduced by a relativistic correction to the potential. When only one large amplitude $a(\mathbf{p})$ is involved, it obeys an integral equation which can be written in general:

$$T(\mathbf{p})a(\mathbf{p})=\lambda\int K(\mathbf{p},\,\mathbf{p}+\boldsymbol{\kappa})a(\mathbf{p}+\boldsymbol{\kappa})d^3\kappa. \tag{47}$$

We suppose that the kernel of this equation can be split into two parts:

$$K=K_0(\mathbf{p},\,\mathbf{p}+\boldsymbol{\kappa})+K_1(\mathbf{p},\,\mathbf{p}+\boldsymbol{\kappa}), \tag{48}$$

where $K_0=\omega_\kappa^{-2}$, and K_1 is a relativistic correction which is supposed to be small. We write: $a(\mathbf{p})=a_0(\mathbf{p})+a_1(\mathbf{p})$, $\lambda=\lambda_0+\lambda_1$. Multiplying both sides of (47) with $a_0^*(\mathbf{p})$, and integrating over $\mathbf{p}$, we obtain by taking into account the fact that $a_0(\mathbf{p})$ is solution of the variational principle (46), the following expression:

$$\frac{\lambda_1}{\lambda_0^2}=-\frac{\int a_0^*(\mathbf{p})K_1(\mathbf{p},\,\mathbf{p}+\boldsymbol{\kappa})a_0(\mathbf{p}+\boldsymbol{\kappa})d^3pd^3\kappa}{\int a_0^*(\mathbf{p})K_0(\mathbf{p},\,\mathbf{p}+\boldsymbol{\kappa})a_0(\mathbf{p}+\boldsymbol{\kappa})d^3pd^3\kappa}. \tag{49}$$

In the general case where the amplitude $a(\mathbf{p})$ has "large" and "small" components $a_{ij}(\mathbf{p})$, the generalizations of Eqs. (47) and (49) are as follows:

$$T_{ij}a_{ij}(\mathbf{p})=\lambda\sum_{k,l}\int\Gamma_{ik}^{(1)}\Gamma_{jl}^{(2)}K_{ij}^{kl}(\mathbf{p},\,\mathbf{p}+\boldsymbol{\kappa})a_{kl}(\mathbf{p}+\boldsymbol{\kappa})d^3\kappa, \tag{47g}$$

$$\frac{1}{\lambda_0}-\frac{\lambda_1}{\lambda_0^2}=\frac{\sum_{i,j,k,l}\int a_{ij}^{(0)*}(\mathbf{p})\Gamma_{ik}^{(1)}\Gamma_{jl}^{(2)}K_{ij}^{kl}(\mathbf{p},\,\mathbf{p}+\boldsymbol{\kappa})a_{kl}^{(0)}(\mathbf{p}+\boldsymbol{\kappa})d^3pd^3\kappa}{\sum_{i,j}\int a_{ij}^{(0)}(\mathbf{p})T_{ij}(\mathbf{p})a_{ij}^{(0)}(\mathbf{p})d^3p}. \tag{49g}$$

In this last formula, $a_{ij}^{(0)}$ is the solution of the equation:

$$T_{ij}a_{ij}^{(0)}(\mathbf{p})=\lambda_0\sum_{kl}\int\Gamma_{ik}^{(1)}\Gamma_{jl}^{(2)}\omega_\kappa^{-2}a_{kl}^{(0)}(\mathbf{p}+\boldsymbol{\kappa})d^3\kappa, \tag{50}$$

and $\Gamma_{ik}^{(r)}$ are the spin matrix elements defined by Eq. (28).

4.3. Relativistic Corrections (Pair Formation Neglected)

In this sub-section, the corrections which come from the retardation of the meson field, and those contributed by the relativistic part of the spin-matrix elements connecting positive energy states, are estimated.

The spin matrix elements are easily calculated in the scalar case. One gets, for example, for particle (1):

$$\Gamma_{ik}^{(1)}=\left\{\frac{[M+E_i(\mathbf{p})][M+E_k(\mathbf{p}+\boldsymbol{\kappa})]}{4E_i(\mathbf{p})E_k(\mathbf{p}+\boldsymbol{\kappa})}\right\}^{\frac{1}{2}}\times\{1-\mathbf{R}_i(\mathbf{p})\cdot\mathbf{R}_k(\mathbf{p}+\boldsymbol{\kappa})+i\boldsymbol{\sigma}^{(1)}\cdot[\mathbf{R}_i(\mathbf{p})\times\mathbf{R}_k(\mathbf{p}+\boldsymbol{\kappa})]\}, \tag{51}$$

with $\mathbf{R}_i(\mathbf{p})=[M+E_i(\mathbf{p})]^{-1}\cdot\mathbf{p}$. The relativistic part of the matrix elements which connect positive energy states only $(i=j=k=l=1)$ contributes corrections which are clearly of the order of v^2/c^2, or higher. The separation singlet-triplet is also of the same order.

4.31. Lowest Order Correction in the B.S. *Equation*

It is convenient, in order to calculate the relativistic corrections included in the B.S. equation, to use Eq. (38), which involves only the amplitude $a(\mathbf{p})$. The two first terms of the expansion of the right-hand side are written more explicitly in Eq. (37). On the right-hand side of this equation, the first term includes the Yukawa potential, plus an interaction term which is of the order of v/c, since:

$$\langle(E_p+E_{p+\kappa}-W)/\omega_\kappa\rangle\sim\langle p^2/M\mu\rangle\sim(\mu/M).$$

The contribution of the second term is also a correction of the order of v/c, since λ_0 is itself of the order of μ/M. In this last term, all the differences between W and the sum of two free energies of the interacting particles can be dropped in the denominator of the kernel. The higher order terms, in the expansion of the right of (55), are proportional to $(\lambda/2)^n$, with $n\geqslant 3$. They are, therefore, of the order v^2/c^2, or higher. The equation which, in the "ladder" approximation of the B.S. equation, includes all the corrections of the first order in v/c is, consequently, of the form (47) with:

$$K_1(\mathbf{p},\,\mathbf{p}+\boldsymbol{\kappa})=\frac{W-E_p-E_{p+\kappa}}{\omega_\kappa^3}+\frac{\lambda_0}{2}\int\frac{d^3\kappa'}{\omega_{\kappa'}^2\omega_{\kappa-\kappa'}^2(\omega_\kappa+\omega_{\kappa-\kappa'})}. \tag{52}$$

If this expression is introduced into (49), the resulting value of λ_1/λ_0 is obviously of the order of μ/M. Incidentally, the equation which was derived by Tamm[3] and Dancoff[4] for the scalar meson theory included, as a non-adiabatic correction, only the first term of the right-hand side of (52). The second one, which is proper to the "ladder" approximation of the B.S. equation, corresponds to the exchange process illustrated in graph (1a). Both terms are of the same order of magnitude.[21]

4.32. *Calculation of the Exact Correction (In the Absence of Pair Formation)*

In the calculation of the preceding sub-section, three contributions to the relativistic corrections of the order of v/c have been neglected: these come from the exchange processes illustrated by diagrams (1a'), (1b), (1b'), which involve meson lines crossings. They can be taken into account if one considers instead of Eq. (38) the general Eq. (6) obtained, in the T.D. formalism, for $a_\lambda^{(0,0)}$. The interaction terms of that equation are written up to the fourth order in Eqs. (7), (12a), and (12b). Adding these terms, and keeping only the parts which are of the order of μ/M, the following equation is obtained:

$$\frac{p^2+M|w|}{M}a(\mathbf{p}) = \lambda\int\frac{1}{\omega_\kappa{}^2}\left(1-\frac{E_p+E_{p+\kappa}-W}{\omega_\kappa}\right)a(\mathbf{p}+\boldsymbol{\kappa})d^3\kappa + \lambda^2\int\frac{a(\mathbf{p}+\boldsymbol{\kappa}+\boldsymbol{\kappa}')d^3\kappa d^3\kappa'}{\omega_\kappa{}^3\omega_{\kappa'}{}^2}. \quad (53)$$

The second term of the right-hand side includes, in the limit where all the differences between W and the sum of two free energies are equalled to zero, the contributions of the kernels $\Delta_4^{(a)}$, $\Delta_4^{(b)}$ of Eq. (12). Since we are using perturbation theory, we can replace $a(\mathbf{p}+\boldsymbol{\kappa}+\boldsymbol{\kappa}')$ by $a_0(\mathbf{p}+\boldsymbol{\kappa}+\boldsymbol{\kappa}')$. After integrating over $\boldsymbol{\kappa}'$, with the help of (45), Eq. (53) becomes of the form (47), with: $K_1(\mathbf{p},\ \mathbf{p}+\boldsymbol{\kappa})=[\mathbf{p}+\boldsymbol{\kappa})^2-\mathbf{p}^2]/2M\omega_\kappa{}^3$. Introducing this expression into (49), we get, because of the symmetry of the integrations over $\mathbf{p}$ and $\mathbf{p}+\boldsymbol{\kappa}$: $\lambda_1=0$. We see therefore that the contributions of the order of v/c coming from all the exchange processes which do not involve pair formation cancel each other, leaving a correction of the order of v^2/c^2 only.

This result is valid for neutral and symmetrically charged scalar meson theories.[22] It is not valid for a purely charged theory, since the diagrams involving meson lines crossings are, in this case, excluded by charge conservation. In the present state of the physical evidence, however, it is not easy to justify a purely charged meson theory.

4.4. Corrections Arising from Pair Formation

We finally want to prove that, for the scalar meson interaction, pair formation contributes only corrections of the order of $(v/c)^4$ to the corrective ratio λ_1/λ_0. The most convenient way is, in this case, to use Eq. (41), which belongs to the general type (47g), and to calculate λ_1/λ_0 from Eq. (49g).

The introduction of the small components of $a_{ij}(\mathbf{p})$ produces a singlet-triplet separation. It is, however, easy to see that the corrections to the coupling constant are of the same order for both kinds of states. It is, therefore, sufficient and simpler to consider singlet states only. In this case, the nonvanishing components of the zero-order amplitude defined by (45) are: $a_{11}^{(2)}=-a_{11}^{(3)}=a_0(\mathbf{p})$. The amplitude $a_{ij}^{(0)}(\mathbf{p})$ however, which satisfies Eq. (50), possesses, in addition, the components:[23]

$$a_{12}^{(1)}=-a_{21}^{(1)}=u_1(\mathbf{p}),\quad a_{12}^{(2)}=-a_{21}^{(3)}=u_2(\mathbf{p}),$$
$$a_{12}^{(3)}=-a_{21}^{(2)}=u_3(\mathbf{p}),\quad a_{12}^{(4)}=-a_{21}^{(4)}=u_4(\mathbf{p}),$$

with the following definition:

$$u_1=u_4{}^*=\frac{i\lambda_0}{W}\int\frac{|\mathbf{p}+\boldsymbol{\kappa}|}{2M}[S_\kappa^{(1)}-iS_\kappa^{(2)}]\frac{a_0(\mathbf{p}+\boldsymbol{\kappa})}{\omega_\kappa{}^2}d^3\kappa,$$

$$u_2=-u_3{}^*=-\frac{p}{2M}\frac{[p^2+M|w|]}{MW}a_0(\mathbf{p})$$
$$-\frac{\lambda_0}{W}\int\frac{|\mathbf{p}+\boldsymbol{\kappa}|}{2M}\left[\frac{\mathbf{p}\cdot(\mathbf{p}+\boldsymbol{\kappa})}{|\mathbf{p}|\cdot|\mathbf{p}+\boldsymbol{\kappa}|}+iS_\kappa^{(3)}\right]\frac{a_0(\mathbf{p}+\boldsymbol{\kappa})}{\omega_\kappa{}^2}d^3\kappa, \quad (54)$$

where $\mathbf{S}_\kappa=[\mathbf{p}\times(\mathbf{p}+\boldsymbol{\kappa})]\cdot[|\mathbf{p}|\cdot|\mathbf{p}+\boldsymbol{\kappa}|]^{-1}$. Using Eq.

[21] Another way to obtain the correction λ_1/λ_0 from the B.S. equation would be to solve directly Eq. (28), by means of a variational principle. It is clear, however, that as far as the scalar interaction is concerned, the method would give results identical with those obtained above: firstly because, when pair formation is not considered, Eqs. (28) and (38) are equivalent; secondly, because, to obtain the expression (52) of $K_1(\mathbf{p},\ \mathbf{p}+\kappa)$, only terms of the order of v^2/c^2 or higher have been neglected. Furthermore, a direct solution of Eq. (28) by means of a variational principle is not only unnecessary in the present case, but also hazardous, since any "trial" function is likely to become a bad approximation in the immediate neighborhood of the poles of $A(\mathbf{p},\ \mathbf{p}_0)$. A large number of variation-iterations are necessary, in order to test the accuracy of the value of λ which is obtained. Otherwise, the errors introduced by the variational principle might well be of the same order as (if not bigger than) the "correction" which one wants to calculate.

[22] It holds for quantum electrodynamics and neutral vector meson theory, because the spin matrix elements give also, in these cases, corrections of the order of v^2/c^2.

[23] The amplitude a_{ij} possesses, for every value of i and j, 4 spin-components $a_{ij}^{(r)}$, $r=1,\ 2,\ 3,\ 4$. The components a_{22} have been dropped everywhere, since they are of a still higher order in v/c. See reference 1.

(49g), the following value of λ_1/λ_0 is obtained:

$$\frac{\lambda_1}{\lambda_0}=\frac{\lambda_0^2/2MW}{\int a_0^*(\mathbf{p})[p^2+M|w|]a_0(\mathbf{p})d^3p}$$

$$\times\int\frac{a_0^*(\mathbf{p})a_0(\mathbf{p}+\boldsymbol{\kappa}+\boldsymbol{\kappa}')}{\omega_\kappa^2\omega_{\kappa'}^2}\{|\mathbf{p}|\cdot|\mathbf{p}+\boldsymbol{\kappa}+\boldsymbol{\kappa}'|\mathbf{S}_\kappa\cdot\mathbf{S}_{\kappa'}$$

$$-|\mathbf{p}+\boldsymbol{\kappa}|^{-2}[|\mathbf{p}+\boldsymbol{\kappa}|^2+\mathbf{p}\cdot(\mathbf{p}+\boldsymbol{\kappa})][|\mathbf{p}+\boldsymbol{\kappa}|^2$$

$$+(\mathbf{p}+\boldsymbol{\kappa})\cdot(\mathbf{p}+\boldsymbol{\kappa}+\boldsymbol{\kappa}')]d^3pd^3\kappa d^3\kappa'\}. \quad (55)$$

The right-hand side of this equation is of the order of $(\mu/M)^4\sim(v/c)^4$. Since pair formation does not contribute to the corrections of the order of v^2/c^2, these can be obtained exactly by considering:

(a) the exchange processes, including those which involve meson lines crossings, represented by Δ_2, $\Delta_4^{(a,b)}$ and Δ_6, in Eqs. (7), (12a), (12b), (15).

(b) the relativistic part of the spin matrix elements connected positive energy states, in the expression of Δ_2 only. This contribution produces the lowest order singlet-triplet separation, which is, therefore, of the order of v^2/c^2.

5. SUMMARY AND CONCLUSION

The results of the investigation described in this paper can be summarized as follows:

(1) The T.D. formalism can be used for the non-adiabatic treatment of the relativistic two-body problem, even if it is physically necessary to take into account virtual pair formation and higher order exchange effects. It has been shown that the T.D. formalism provides, through the natural play of the equations of motion, an iteration of the exchange processes included in these equations; so that, to any order in the coupling constant, it describes an infinite series of exchange effects. This makes it suitable to the description of bound systems, in which the interaction operates during a long time.

(2) The B.S. equation has been found to give, within the limits of the "ladder" approximation, results which are equivalent to those which can be obtained, with the same approximation, from the T.D. formalism. While, in the absence of pair creation, these results are identical in both methods, there exist, when pair formation is taken into account, some small analytical differences, which, however, tend to zero in the nonrelativistic limit.

(3) A detailed study of the relativistic corrections to the scalar meson interaction of two nucleons shows that the "ladder" approximation of the B.S. equation leads to wrong results, because it neglects the processes involving meson lines crossings, which are thus physically important.

The general methods developed in this paper will be used subsequently for the treatment of the neutron-proton system in the case of the symmetrical pseudoscalar meson interaction. While most of the interaction processes will be investigated in the framework of the T.D. formalism, the equivalence between this formalism and the B.S. equation will be used in order to calculate the radiative corrections to the interaction, where an explicit renormalization of mass and charge has to be carried out.

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