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Diffraction in Time*

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In a previous note a dynamical description of resonance scattering was given, and transient terms appeared in the wave function describing the process. To understand the physical significance of these terms, the transient effects that appear when a shutter is opened are discussed in this paper. For a nonrelativistic beam of particles, the transient current has a close mathematical resemblance with the intensity of light in the Fresnel diffraction by a straight edge. This is the reason for calling the transient phenomena by the name of diffraction in time.

The shutter problem is discussed for particles whose wave functions satisfy the Schrödinger equation, the ordinary wave equation, and the Klein-Gordon equation. Only for the Schrödinger time-dependent equation do the transient wave functions resemble the solutions that appear in Sommerfeld's theory of diffraction. The connection of transient phenomena with the time-energy uncertainty relation, and the interpretation of the transient current in a scattering process, are briefly discussed. The relativistic wave functions for the shutter problem may play an important role in the dynamical description of a relativistic scattering process.

I. INTRODUCTION

IN a previous paper[1] we analyzed the dynamical behavior of a resonance scattering process, when the scatterer (represented by an appropriate boundary condition) was introduced into the beam of incident particles at a definite time. As a result of the introduction of the scatterer, transient terms appeared in the wave function representing the process, and with their help we could make the transition from the initial state (plane wave) to the final state (plane plus scattered waves).

Transient terms are to be expected in a dynamical description of resonance scattering, from the analogy that this description has with the theory of resonant electric circuits.[2] As is well known, in circuit theory the appearance of resonances in the stationary current is closely related with the transients of the circuit, as the same parameters (resonant frequencies and damping factors) appear in both.[3]

The transient terms in a scattering process contain, besides those that could be considered the analogous of electric circuit theory, terms that are related to the time-energy uncertainty relation,[4] as quantum mechanics is used in the description of the scattering.

To understand the physical meaning of the transient terms in the resonance scattering process, it seemed of interest to analyze first the transient effects that appear when the propagation of a beam of particles is interrupted. The more complicated phenomenon, where an actual scatterer (represented by a potential or by boundary conditions[5]) is introduced into the beam of incident particles, will be discussed briefly at the conclusion of this paper.

We deal here with the transient terms in the wave function that appear when a shutter is opened. It will be shown in the next section, that when the state of the beam of particles is represented by a wave function satisfying the time-dependent Schrödinger equation, the transient current has a remarkable mathematical similarity with the intensity of light in the Fresnel diffraction by a straight edge.[6] The transient phenomena have therefore been given the name of diffraction in time.

The form of the transient terms of ψ that appear when the shutter is opened, is strongly dependent on the type of wave equation satisfied by ψ. In the present paper we analyze the transient terms that appear when the ψ's satisfy the Schrödinger equation, the ordinary wave equation, and the Klein-Gordon equation. Only for the Schrödinger equation is there an analogy with the phenomena of optical diffraction, which has to do with the resemblance that the solutions have with those that appear in Sommerfeld's[7] theory of diffraction.

II. THE SHUTTER PROBLEM

The problem we shall discuss in this section is the following: a monochromatic beam of particles of mass m, energy $(\hbar^2k^2/2m)$, moving parallel to the x-axis, is interrupted at $x=0$ by a shutter perpendicular to the beam, as illustrated in Fig. 1. If at $t=0$ the shutter is opened, what will be the transient particle current observed at a distance x from the shutter?

To set the mathematical problem, we must first give the behavior of the shutter, i.e., if it acts as a perfect absorber (no reflected wave), or a perfect reflector (an infinite potential barrier), or something between the two. For simplicity we will assume that the shutter

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1 M. Moshinsky, Phys. Rev. **84**, 525 (1951) to be referred to as (A).

2 E. P. Wigner, Am. J. Phys. **17**, 99 (1949).

3 J. C. Jaeger, *An Introduction to the Laplace Transformation* (Methuen and Company, London, 1949), p. 31.

4 M. Moshinsky, Rev. Mex. Fis. **1**, 28 (1952).

5 E. P. Wigner, Phys. Rev. **70**, 15 (1946); Feshbach, Peaslee, and Weisskopf, Phys. Rev. **71**, 145 (1947).

6 M. Born, *Optik* (Julius Springer, Berlin, 1933), pp. 192–5.

7 A. Sommerfeld, *Theorie der Beugung*, Chap. XX of the Frank-v. Mises, *Differential- und Integralgleichungen der Mechanik und Physik* (Fried. Vieweg and Sohn, Braunschweig, 1935), Vol. II, pp. 808–871.

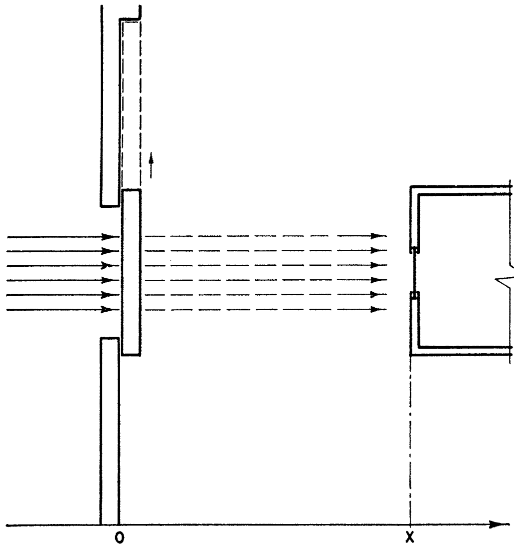

FIG. 1. The shutter problem.

acts as a perfect absorber, though it can be easily shown that for $x \gg \lambda$ (where λ is the wavelength $\lambda=(2\pi/k)$), the transient current obtained below holds for any type of shutter, so long as it acts as a device that, when closed, keeps the beam of particles on only one side of it.

For nonrelativistic particles, the wave function $\psi(x, t)$ that represents the state of the beam of particles for $t>0$, satisfies the time dependent Schrödinger equation:

$$-i(\partial\psi/\partial t)=(\hbar/2m)(\partial^2\psi/\partial x^2), \qquad (1)$$

and the initial conditions:

$$\psi(x,0)=\begin{cases}\exp(ikx) & \text{if } x<0\\ 0 & \text{if } x>0.\end{cases} \qquad (2)$$

The solution of (1, 2) can be immediately given with the help of the functions $\chi(x, k, t)$, introduced in (A), and defined by:

$$\chi(x,k,t)=\exp(imx^2/2\hbar t)\exp(y^2)\,\mathrm{erfc}(y), \qquad (3a)$$

where:

$$\mathrm{erfc}(y)=2\pi^{-\frac{1}{2}}\int_y^\infty \exp(-y^2)dy, \qquad (3b)$$

$$y=\exp(-i\pi/4)(2\hbar t/m)^{-\frac{1}{2}}(x-vt),\quad v=(\hbar k/m). \qquad (3c)$$

The function $\chi(x, k, t)$ satisfies (1) as:

$$(\hbar/2m)(\partial^2\chi/\partial x^2)+i(\partial\chi/\partial t) \equiv (4it)^{-1}\exp(imx^2/2\hbar t)[d^2/dy^2 - 2y d/dy - 2]\exp(y^2)\,\mathrm{erfc}(y), \qquad (4)$$

and from the definition (3b) of the error integral function, we see that the right-hand side of (4) vanishes. Furthermore, when $t\to 0$, $|y|\to\infty$ and $\arg y$ is either $-(\pi/4)$ if $x>0$ or $(3\pi/4)$ if $x<0$. From the asymptotic properties of the error integral function when $|y|\to\infty$, summarized in appendix 2 of (A), we have:

$$\exp(y^2)\,\mathrm{erfc}(y)\to\begin{cases}0 & \text{if } -(\pi/2)<\arg y<(\pi/2)\\ 2\exp(y^2) & \text{if } (\pi/2)<\arg y<(3\pi/2).\end{cases} \qquad (5)$$

We conclude from the above that:

$$\psi(x,t)=\tfrac{1}{2}\chi(x,k,t), \qquad (6)$$

satisfies Eq. (1) and the initial condition (2).

When $t\to\infty$, $|y|\to\infty$ and $\arg y=(3\pi/4)$, so using (5) again, we obtain:

$$\psi(x,t)\to\exp i[kx-(\hbar k^2/2m)t], \qquad (7)$$

which would be the expected stationary form of the wave function.

The transient current $J(x, t)$ takes from (36) the form:

$$J(x,t)=(\hbar/2im)[\psi^*(\partial\psi/\partial x)-\psi(\partial\psi^*/\partial x)] = (v/4)\left[|\chi(x,k,t)|^2+(\tfrac{1}{2}\pi kvt)^{-\frac{1}{2}}\,\mathrm{Im}\{\exp i[(\pi/4) - (mx^2/2\hbar t)]\chi(x,k,t)\}\right], \qquad (8)$$

where Im stands for the imaginary part of the curly bracket. The expression (8) simplifies considerably if we assume that $x\gg\lambda$. As we shall show below, $|\chi(x, k, t)|$ differs appreciably from zero only when t is of the order or larger than the time of flight $T=(x/v)$. For $t\geqslant T$ we have that $(kvt)^{-\frac{1}{2}}\leqslant(kx)^{-\frac{1}{2}}$ is very small. Therefore, the second term in (8) is $\simeq 0$ for all $t>0$ if $x\gg\lambda$, and in this case the current becomes:

$$J(x,t)=(v/4)|\mathrm{erfc}[-(\pi/2i)^{\frac{1}{2}}u]|^2, \qquad (9)$$

where:

$$u=(kx)^{\frac{1}{2}}(\pi t/T)^{-\frac{1}{2}}[(t/T)-1]. \qquad (10)$$

From the asymptotic behavior (7) of ψ when $t\to\infty$, we see that the stationary current is $J_0=v$. Using a simple relation[8] between the error integral function and

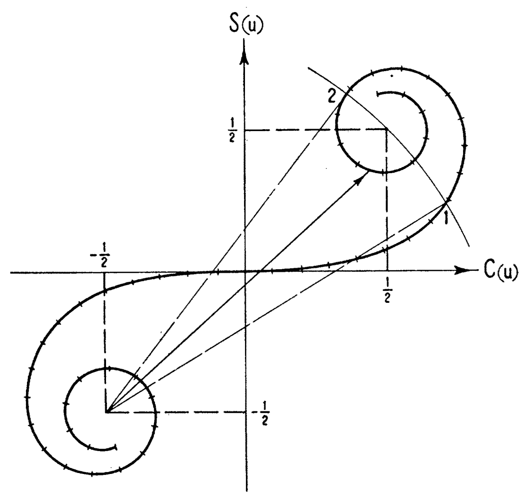

FIG. 2. Cornu spiral.

[8] E. Jahnke and F. Emde, *Tables of Functions* (Dover Publications, New York, 1945), fourth edition, p. 36.

the Fresnel integrals, the ratio of the transient to the stationary current becomes:

$$[J(x,t)/J_0]=\tfrac{1}{2}\{[\tfrac{1}{2}+C(u)]^2+[\tfrac{1}{2}+S(u)]^2\}, \quad (11)$$

where:

$$C(u)=\int_0^u \cos(\tfrac{1}{2}\pi u^2)du, \quad S(u)=\int_0^u \sin(\tfrac{1}{2}\pi u^2)du. \quad (12)$$

The right-hand side in (11) is identical to the expression[6] for the intensity of light in the Fresnel diffraction by a straight edge. The variable u has though a different meaning from the optical problem, as it is now the function of time given by (10).

In the Cornu spiral diagram of Fig. 2, the right-hand side of (11) is one-half of the square of the radius vector from the point $(-\frac{1}{2}, -\frac{1}{2})$ to the point on the spiral whose distance from the origin, along the curve, is u. From (10) we see that when $0<t\leqslant 0$, then $-\infty<u\leq 0$, while when $T\leqslant t<\infty$ then $0\leqslant u<\infty$. With the help of the Cornu spiral we see that when t goes from 0 to T, the ratio (J/J_0) increases monotonically from 0 to $\frac{1}{4}$, while when t is larger than T, then (J/J_0) behaves as a damped oscillation around the value $(J/J_0)=1$, tending to this value when $t\to\infty$. This behavior is illustrated in Fig. 3.

From the standpoint of classical mechanics, it is clear that $(J/J_0)=0$ if t is less that the time of flight T, while $(J/J_0)=1$ if $t\geqslant T$. This behavior is also illustrated in Fig. 3 by the straight line.

A good measure of the "width" of this diffraction effect in time, can be obtained from the difference τ between the first two times at which (J/J_0) takes the classical value, i.e., $\tau=t_2-t_1$, as shown in Fig. 3. The times at which the curve of Fig. 3 intersects with the straight line $(J/J_0)=1$, correspond to the values of u obtained from the intersection of the Cornu spiral with the circle of radius $\sqrt{2}$ and center $(-\frac{1}{2}, -\frac{1}{2})$. The values u_1, u_2 that correspond to t_1, t_2 are the lengths along the Cornu spiral from the origin to the points 1, 2 in Fig. 2, so that we have: $u_2-u_1=0.85$. As $x\gg\lambda$ we see from (10) that t_1, t_2 are very close to T, so we may write:

$$\tau\simeq(u_2-u_1)(\pi/kx)^{\frac{1}{2}}T=0.85(\pi x\hbar/mv^3)^{\frac{1}{2}}. \quad (13)$$

As an example (of possible interest in the operation of neutron velocity selectors), we have that for 300°K neutrons at a distance $x=1$ m, the diffraction width would be:

$$\tau=0.27\times 10^{-8}\text{ sec.}$$

Borrowing from the terminology of optics, we could say that in classical mechanics a shutter casts a sharp shadow in time, i.e., the beam current jumps suddenly from zero to the stationary value at $t=T$, while in quantum mechanics there is a diffraction effect in time as illustrated in Fig. 3. When we make Planck's constant h tend to zero, we see from (13) that the diffraction width τ tends also to zero, and the quantum-mechanical current reduces to the classical current, as one should expect.

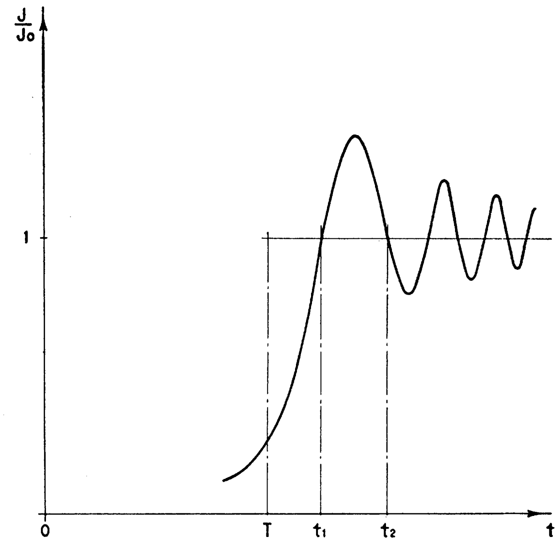

FIG. 3. Diffraction in time.

The analogy between the transient current (11) and the diffraction phenomena in optics, raises the question of whether the transient currents for other types of wave equations show this analogy. We shall see in the following sections that this is not the case, and that only when ψ obeys the Schrödinger equation, does it resemble the wave functions that appear in Sommerfeld's theory of diffraction.[7]

The transient current (11) increases monotonically from the very moment in which we open the shutter, and therefore, in principle, an observer at a distance x from the shutter could detect particles before a time (x/c), where c is the velocity of light. This would imply that some of the particles travel with velocities larger than c, and the error is due of course, to employing the nonrelativistic Schrödinger equation in the analysis.

We shall discuss in the following sections the transient wave functions associated with two relativistic equations, the ordinary wave equation and the Klein-Gordon equation. It will be shown in this case, that at a distance x from the shutter, the wave function $\psi(x, t)$ is zero for $t<(x/c)$, thus safeguarding the relativity principle. When in the solution for the Klein-Gordon equation we make $c\to\infty$, it will reduce essentially to (6), thus giving the transient current (11) in the nonrelativistic approximation.

III. THE ORDINARY WAVE EQUATION

We shall now deal with the same problem of the last section, but assume that ψ satisfies the ordinary wave equation

$$c^2(\partial^2\psi/\partial x^2)=(\partial^2\psi/\partial t^2). \quad (14)$$

If initially ψ and its time derivative are given by:

$$\psi(x,0)=F(x),\quad (\partial\psi/\partial t)_{t=0}=G(x), \tag{15}$$

then as is well known,[9] $\psi(x,t)$ has the form:

$$\psi(x,t)=\tfrac{1}{2}\left[F(x+ct)+F(x-ct)+c^{-1}\int_{x-ct}^{x+ct}G(x')dx'\right]. \tag{16}$$

For $t<0$ when the shutter was closed, we had on the left side of the shutter:

$$\psi(x,t)=\exp[ik(x-ct)],\quad \text{for}\quad x<0.$$

This implies that at $t=0$ we have:

$$F(x)=\exp(ikx),\quad G(x)=-ikc\exp(ikx),\quad \text{if}\quad x<0, \tag{17}$$

while $F(x)=G(x)=0$ if $x>0$.

Introducing these initial conditions in (16) we obtain:

$$\psi(x,t)=\begin{cases} 0 & \text{if}\quad t<(x/c) \\ \exp[ik(x-ct)]-\tfrac{1}{2} & \text{if}\quad t>(x/c). \end{cases} \tag{18}$$

The absolute value of the wave function jumps suddenly to the value $\frac{1}{2}$ at $t=(x/c)$, and it oscillates periodically thereafter. This behavior has certainly no resemblance to the diffraction in time effect obtained in the previous section.

IV. THE RELATIVISTIC SHUTTER PROBLEM

If we wish to extend the discussion of the transient effects of Sec. II, to a beam of electrons at relativistic energies, we would have, of course, to replace Eq. (1) by Dirac's equation. A fundamental change creeps in though by the fact that the wave function will involve the negative, as well as the positive energy eigenfunctions. A complete description of the phenomenon would necessitate then the introduction of hole theory, and the single particle picture would have to be abandoned.

As our main interest concerns the transients associated with the different wave equations, we shall not attempt to discuss here the problem when hole theory is involved, but rather will restrict ourselves to the single particle picture in which states of both positive and negative energy are available. Furthermore, the features of spin and statistics are irrelevant to our analysis, so that for simplicity, instead of Dirac's equation we shall use the Klein-Gordon equation. We shall briefly indicate though, at the end of this section, the relation of the solution for the Dirac equation with the solution to be obtained for the Klein-Gordon equation.

The wave function $\psi(x,t)$ satisfies now the equation:

$$(\partial^2\psi/\partial x^2)=c^{-2}(\partial^2\psi/\partial t^2)+\mu^2\psi, \tag{19}$$

where $\mu=(mc/\hbar)$.

If initially ψ and its time derivative are given by:

$$\psi(x,0)=F(x),\quad (\partial\psi/\partial t)_{t=0}=G(x), \tag{20}$$

then using the Fourier integral theorem, we obtain:

$$\psi(x,t)=(2\pi)^{-\frac{1}{2}}\int_{-\infty}^{\infty}[f(\kappa)\cos(\epsilon ct) +(\epsilon c)^{-1}g(\kappa)\sin(\epsilon ct)]\exp(i\kappa x)d\kappa, \tag{21}$$

where $f(\kappa)$, $g(\kappa)$ are the ordinary Fourier transforms of $F(x)$, $G(x)$, and $\epsilon=(\kappa^2+\mu^2)^{\frac{1}{2}}$.

For $t<0$, when the shutter was closed, we assume as in the previous section, that:

$$\psi(x,t)=\exp[i(kx-Ect)],\quad \text{for}\quad x<0,$$

where E is the energy in units of reciprocal length, i.e., $E=(k^2+\mu^2)^{\frac{1}{2}}$. This implies that at $t=0$ we have:

$$F(x)=\exp(ikx),\quad G(x)=-iEc\exp(ikx),\quad \text{for}\quad x<0, \tag{22}$$

while $F(x)=G(x)=0$ for $x>0$.

The Fourier transforms $f(\kappa)$, $g(\kappa)$ become then:[10]

$$f(\kappa)=(2\pi)^{-\frac{1}{2}}\int_{-\infty}^{0}\exp[i(k-\kappa)x]dx=(2\pi)^{\frac{1}{2}}\delta_+(\kappa-k)$$
$$=(\pi/2)^{\frac{1}{2}}\{\delta(\kappa-k)+i[\pi(\kappa-k)]^{-1}\}, \tag{23}$$

$$g(\kappa)=-iEcf(\kappa). \tag{24}$$

Introducing (23), (24) in (21) we obtain:

$$\psi(x,t)=\tfrac{1}{2}\exp[i(kx-Ect)] -(4\pi i)^{-1}\mathrm{P}\int_{-\infty}^{\infty}(\kappa-k)^{-1}(\epsilon+E) \times\exp[i(\kappa x-\epsilon ct)]\epsilon^{-1}d\kappa -(4\pi i)^{-1}\mathrm{P}\int_{-\infty}^{\infty}(\kappa-k)^{-1}(\epsilon-E) \times\exp[i(\kappa x+\epsilon ct)]\epsilon^{-1}d\kappa. \tag{25}$$

As usual, when $\delta_\pm$ functions are employed, the integrals must be interpreted in the sense of Cauchy's principal value[11] as indicated by the P. The initial conditions at $t=0$, can be obtained from (25) by closing the contour in the κ-plane, from above if $x>0$ and from below if $x<0$. The first and second integrals in (25) give the contribution from the positive and negative energies, respectively.

To eliminate the branch points at $\kappa=\pm i\mu$ in (25), we introduce in the first integral the change of variable:

$$\zeta=\mu^{-1}(\kappa+\epsilon), \tag{26}$$

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[9] A. G. Webster, *Partial Differential Equations of Mathematical Physics* (G. E. Stechert and Company, New York, 1933), second edition, p. 78.

[10] W. Heisenberg, Z. Physik **120**, 519 (1943).

[11] E. T. Whittaker and G. N. Watson, *Modern Analysis* (Cambridge University Press, New York, 1943), American edition, p. 117.

from which we have: $\kappa=\frac{1}{2}\mu(\zeta-\zeta^{-1})$, $\epsilon=\frac{1}{2}\mu(\zeta+\zeta^{-1})$. The limits of integration $[-\infty, \infty]$ for the κ transform into $[0, \infty]$ for the ζ.

In the second integral we introduce a ζ corresponding to negative energies, i.e., $\zeta=\mu^{-1}(\kappa-\epsilon)$. It is clear that with this ζ, the integrand in the second integral becomes the same function of ζ that we have in the first integrand. Furthermore, for this ζ the limits of integration are $[-\infty, 0]$.

Combining the two integrals we obtain:

$$\psi(x,t)=\tfrac{1}{2}\exp[i(kx-Ect)] -(2\pi i)^{-1}\mathrm{P}\int_{-\infty}^{\infty}(\zeta+z)(\zeta-z)^{-1}(2\zeta)^{-1} \times\exp\{\tfrac{1}{2}i\mu[(x-ct)\zeta-(x+ct)\zeta^{-1}]\}d\zeta, \quad (27)$$

where $z=\mu^{-1}(k+E)$.

The integrand of (27) has a pole at $\zeta=z$ and an essential singularity at $\zeta=0$, but the integral is convergent, and in the appendix it is evaluated by a suitable deformation of the contour.

From the analysis given in the appendix, it follows that:

$$\psi(x,t)=0 \quad \text{if} \quad t<(x/c), \quad (28\text{a})$$

$$\psi(x,t)=\exp[i(kx-Ect)]+\tfrac{1}{2}J_0(\eta) -\sum_{n=0}^{\infty}(\xi/iz)^nJ_n(\eta), \quad \text{if} \quad t>(x/c), \quad (28\text{b})$$

where:

$$\xi=[(ct+x)/(ct-x)]^{\frac{1}{2}}, \quad \eta=\mu(c^2t^2-x^2)^{\frac{1}{2}}, \quad (29)$$

and the J_n are the Bessel functions of order n.

The wave function $\psi(x,t)$ of (28) satisfies the Klein-Gordon equation, as can be easily seen when we express this equation in terms of the variables ξ, η of (29), so that (19) takes the form:

$$\left(\frac{\partial^2}{\partial\eta^2}+\frac{1}{\eta}\frac{\partial}{\partial\eta}-\frac{\xi^2}{\eta^2}\frac{\partial^2}{\partial\xi^2}-\frac{\xi}{\eta^2}\frac{\partial}{\partial\xi}+1\right)\psi(\xi,\eta)=0. \quad (30)$$

As $\xi^\alpha J_\alpha(\eta)$ is a solution of (30) for any α, we see that each term of the series in (28b) satisfies the Klein-Gordon equation.

For $n=0$ the solution $J_0[\mu(c^2t^2-x^2)^{\frac{1}{2}}]$ of the Klein-Gordon equation is well known in quantum electrodynamics.[12] For $n\neq 0$ the solutions $\xi^nJ_n(\eta)$ do not seem to have had a wide use in quantum field theory, possibly because they are not in themselves Lorentz-invariant, but become so only after multiplication by z^{-n}.

When the rest mass of the particle vanishes, then $\eta\to 0$ and $z\to\infty$, so that all the terms in the series of (28b) vanish except $J_0(\eta)$ which becomes 1. The solution (28) reduces in this case, to the solution (18) of the ordinary wave equation, as we should expect.

We are now interested in the asymptotic values of $\psi(x,t)$ when $t\to(x/c)$ and when $t\to\infty$. In the first case, we see from (29) that $\eta\to 0$ and we can replace the Bessel functions by their asymptotic values:

$$J_n(\eta)\to(2^nn!)^{-1}\eta^n. \quad (31\text{a})$$

The series in (28b) becomes then:

$$\sum_{n=0}^{\infty}(\xi/iz)^nJ_n(\eta)\to\sum_{n=0}^{\infty}(n!)^{-1}(\xi\eta/2iz)^n =\exp(\xi\eta/2iz). \quad (31\text{b})$$

For $t=(x/c)$ the series takes the value $\exp[i(k-E)x]$ and the asymptotic form of $\psi(x,t)$ is:[13]

$$\lim_{t\to(x/c)}[\psi(x,t)]=\tfrac{1}{2}. \quad (32)$$

The wave function $\psi(x,t)$ jumps suddenly from zero to the value $\frac{1}{2}$ at $t=(x/c)$, just as in the case of the ordinary wave equation, and it does not build up continuously from $t=0$ as in the case of the Schrödinger equation.

To obtain the asymptotic form of $\psi(x,t)$ when $t\to\infty$, we notice first that in this case, $\xi\to 1$ and $\eta\to\infty$. Replacing in the series of (28b) the $J_n(\eta)$ by their asymptotic values:

$$J_n(\eta)\to(\tfrac{1}{2}\pi\eta)^{-\frac{1}{2}}\cos[\eta-\tfrac{1}{4}(2n+1)\pi], \quad (33\text{a})$$

we get:

$$\left|\sum_{n=0}^{\infty}(\xi/iz)^nJ_n(\eta)\right|\to\left|(\tfrac{1}{2}\pi\eta)^{-\frac{1}{2}}\sum_{n=0}^{\infty}(iz)^{-n} \times\cos[\eta-\tfrac{1}{4}(2n+1)\pi]\right|\leqslant(\tfrac{1}{2}\pi\eta)^{-\frac{1}{2}}\sum_{n=0}^{\infty}z^{-n}. \quad (33\text{b})$$

From its definition $z=\mu^{-1}(k+E)>1$ for any $k>0$, so that the series in (33b) converges, and therefore, when $\eta\to\infty$ the series in (28b) tends to zero. The asymptotic value of $\psi(x,t)$ when $t\to\infty$, becomes then:

$$\psi(x,t)\to\exp[i(kx-Ect)], \quad (34)$$

which is the stationary form of the wave function that we expect.

To pass from the relativistic wave function (28b) to the nonrelativistic wave function (6), we have to take $c\to\infty$. A simple way to make this transition is through the expression (25) for $\psi(x,t)$. When $c\to\infty$ we see that $\epsilon ct=(\kappa^2+\mu^2)^{\frac{1}{2}}ct$, tends to:

$$\epsilon ct\to\mu ct+(\hbar\kappa^2/2m)t+\cdots, \quad (35)$$

where all the other terms of the development vanish for $c\to\infty$. A similar result holds for Ect. Furthermore,

[12] P. A. M. Dirac, Proc. Cambridge Phil. Soc. **30**, 150 (1934); J. Schwinger, Phys. Rev. **75**, 678 (1949).

[13] The asymptotic behavior of $\psi(x,t)$ can also be derived, in a simple and rigorous fashion, from the integral representation of the wave function that is given in the appendix.

when $c\rightarrow\infty$ we see that $(E/\epsilon)\rightarrow 1$, and therefore, the negative energy part of (25) vanishes, while the rest takes the form:

$$\psi(x,t)\rightarrow\exp(-i\mu ct)\left\{\tfrac{1}{2}\exp i[kx-(\hbar k^2/2m)t]\right.$$
$$\left.-(2\pi i)^{-1}\mathrm{P}\int_{-\infty}^{\infty}(\kappa-k)^{-1}\exp i[\kappa x-(\hbar\kappa^2/2m)t]d\kappa\right\}$$
$$=\exp(-i\mu ct)\tfrac{1}{2}\chi(x,k,t). \quad (36)$$

The last result follows from the evaluation of the integral in (36), that was carried in appendix 1 of (A).

It follows from (36) that when $c\rightarrow\infty$, the transient current obtained with the relativistic wave function reduces precisely to (11).

Finally, we would like to remark that an analysis similar to the one carried above, can be given for the Dirac wave equation. Choosing a definite direction of spin, the Dirac wave function reduces to two components, each of which can be expressed, for $t>(x/c)$, in terms of linear combinations of the wave function (28b) and $J_0(\eta)$.

V. ANALOGIES WITH SOMMERFELD'S DIFFRACTION THEORY

In the previous sections, the shutter problem for different types of wave equations was analyzed, and we obtained the corresponding wave functions. Rather than to compare the related transient currents with the intensity of light in optical diffraction [as suggested by the form of the nonrelativistic current (11)], we shall look into the analogies between the wave functions themselves, and a family of solutions in Sommerfeld's theory of diffraction.

For electromagnetic diffraction problems in the plane, we need appropriate solutions of the two-dimensional Kirchhoff equation, which in polar coordinates has the form:

$$(\nabla^2+k^2)\phi\equiv(\partial^2\phi/\partial r^2)+r^{-1}(\partial\phi/\partial r)+r^{-2}(\partial^2\phi/\partial\theta^2)+k^2\phi=0, \quad (37)$$

where $k=(2\pi/\lambda)$ and λ is the wavelength of the radiation.

The well-known family of solutions of (37) that is obtained with the help of the Riemann surface[14] analysis of Sommerfeld, can be written in the form:

$$\chi'(r,\theta,\theta_0)=\exp(ikr)\exp(y'^2)\,\mathrm{erfc}(y'), \quad (38a)$$

where:

$$y'=-\exp(-i\pi/4)(2kr)^{\frac{1}{2}}\sin[\tfrac{1}{2}(\theta-\theta_0)], \quad (38b)$$

and θ_0 is a parameter.

[14] B. B. Baker and E. T. Copson, *The Mathematical Theory of Huygens' Principle* (Clarendon Press, Oxford, 1939), Chapter IV, pp. 130, 132, 142.

The χ' satisfy (37) as can be seen from the fact that:

$$[\nabla^2+k^2]\phi\equiv k^2(2ikr)^{-1}\exp(ikr)\times[d^2/dy'^2-2y'd/dy'-2]\exp(y'^2)\,\mathrm{erfc}(y'), \quad (39)$$

and as shown in (4), the right-hand side of (39) vanishes.

Despite the fact that the time dependent Schrödinger equation is of the parabolic type, while the Kirchhoff equation is of the elliptic type, they have solutions of the form (3) and (38) which are remarkably similar. No such similarity exists between the wave functions (18) and (28) of the hyperbolic relativistic wave equations, and the family of solutions (38), so that we do not expect in this case, a resemblance between the transient current and optical diffraction effects.

When a semi-infinite perfectly reflecting plane is introduced at $\theta=-(\pi/2)$, in the path of an electromagnetic wave polarized in the plane and propagating in the direction $\theta=0$, the wave function $\phi(r,\theta)$ that satisfies[14] the boundary conditions at $\theta=-(\pi/2)$, $(3\pi/2)$ and the asymptotic behavior at $r\rightarrow\infty$, becomes:

$$\phi(r,\theta)=\tfrac{1}{2}[\chi'(r,\theta,0)-\chi'(r,\theta,\pi)]. \quad (40)$$

A corresponding problem for the transient current appears when the shutter is represented as a perfect reflector, i.e., an infinite potential barrier. In this case, we must replace in the initial condition (2), $\exp(ikx)$ by $\exp(ikx)-\exp(-ikx)$, and with the help of (5) we immediately obtain that:

$$\psi(x,t)=\tfrac{1}{2}[\chi(x,k,t)-\chi(x,-k,t)]. \quad (41)$$

From (40), and assuming that $r\gg\lambda$, we obtain the characteristic Fresnel diffraction effect[6] for the intensity of the electromagnetic wave in the vicinity of $\theta=0$. From (41), and assuming $x\gg\lambda$, we obtain the transient current (11), which also shows a Fresnel diffraction effect in the vicinity of $t=T$, where T is the time of flight $T=(x/v)$.

IV. CONCLUSION

It is well known that ordinary diffraction phenomena for beams of particles are closely associated with the position-momentum uncertainty relations.[15] The appearance of diffraction in time effects for the transient current, when an obstacle is introduced or removed from the incident beam, suggests that these effects are closely connected with the time-energy uncertainty relation. This is shown to be the case by analyzing the transient wave function when an impenetrable spherical potential is introduced into the beam.[4] The ratio of the transient to the stationary scattered current has in this case, also the form (11). If the wave function is transformed to the energy-angular momentum representation,[4] it shows that when a scattered particle is detected at a time Δt after the introduction of the

[15] W. Heisenberg, *The Physical Principles of Quantum Theory* (Chicago University Press, Chicago, 1930), Chapter II.

scatterer, there is an indetermination ΔE in its energy, such that:

$$\Delta t \Delta E \simeq h. \tag{42}$$

The χ-functions that are used in the nonrelativistic shutter problem, appear also (in the form $\chi(r, \pm k'', t)$, where r is the distance from the scatterer and k'' the momentum of the incident particles) in the dynamical description of the scattering by an impenetrable spherical potential,[4] and in the resonance scattering processes.[1] These functions account then for the transient effects due to the time-energy uncertainty relation. When a scatterer has a structure, such as in the single level scattering process,[1] there are other terms in the transient wave functions that depend on the poles of the scattering matrix. The transient effects that these terms give rise to in the current, are then no longer general quantum-mechanical effects, but are related to the specific nature of the scatterer.

The important role that the χ-functions play in the dynamical description of nonrelativistic scattering processes, suggests that their relativistic generalization, given by the wave function $\psi(x, t)$ of (28), may play a similar role in any dynamical description of relativistic scattering processes.

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APPENDIX

In this appendix, we shall evaluate the integral in (27), so as to determine the explicit form of the relativistic wave function.

Instead of taking the principal value of the integral in (27), we shall make the integration along the contour C of Fig. 4, where the singular points $\zeta=0, z$ of the integrand, are by-passed from above by means of the semicircles C', C''. From Fig. 4 we obtain:

$$\mathrm{P}\int_{-\infty}^{\infty} - \int_C = \int_{-\delta}^{\delta} - \int_{C'} - \int_{C''}. \tag{43}$$

The integral around C'' must be interpreted as the limiting value when the radius of the semicircle tends to zero. As the integrand has a simple pole at $\zeta=z$, we obtain:

$$-\int_{C''} = \pi i \,\mathrm{Res}(\zeta=z) = \pi i \exp[i(kx-Ect)]. \tag{44}$$

From the form of the integrand in (27), we see that the first two integrals on the right-hand side of (43) vanish when $\delta \to 0$, assuming that $x, t > 0$. As the integrand has an essential singularity at $\zeta=0$, but is regular outside this point (except for the pole at $\zeta=z$), we conclude that the first two integrals of (43) add to zero for any $\delta < z$.

From (27) and (43) we obtain then:

$$\psi(x,t) = -(2\pi i)^{-1}\int_C (\zeta+z)(\zeta-z)^{-1}(2\zeta)^{-1} \times \exp\{i(\mu/2)[(x-ct)\zeta-(x+ct)\zeta^{-1}]\}d\zeta. \tag{45}$$

When $x > ct$ we can close the contour from above, and as the contribution from the dotted semicircle is clearly zero, and the integrand is analytic in that part of the ζ plane above C, we conclude that:

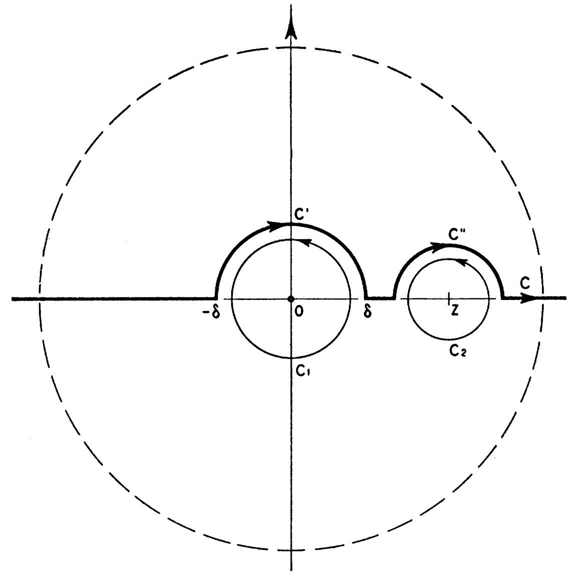

FIG. 4. Integration contour for the relativistic wave function.

$$\psi(x,t)=0, \quad \text{if} \quad x > ct. \tag{46}$$

When $x < ct$ we close the contour from below. Again the contribution of the dotted semicircle is zero, but now we have the singular points $\zeta=0, z$ inside the contour. As the singular points are not branch points, we obtain that:

$$\psi(x,t) = (2\pi i)^{-1}\left[\int_{C_1} + \int_{C_2}\right], \quad \text{for} \quad x < ct, \tag{47}$$

where the contours C_1, C_2 are now the closed circles surrounding the points 0 and z, respectively.

The second integral in (47) is given by twice the value of (44). For the first integral we introduce the variables ξ, η defined by (29), and changing to the variable of integration given by

$$\zeta' = -i\xi^{-1}\zeta, \tag{48a}$$

we obtain:

$$(2\pi i)^{-1}\int_{C_1} = (2\pi i)^{-1}\int_{C_1'} [(\zeta'-z')^{-1} - (2\zeta')^{-1}] \times \exp[\tfrac{1}{2}\eta(\zeta'-\zeta'^{-1})]d\zeta', \tag{48b}$$

where $z' = -i\xi^{-1}z$ and the radius of C_1' is less than $|z'| = \xi^{-1}z$.

Using the well-known formula:[16]

$$\exp[\tfrac{1}{2}\eta(\zeta'-\zeta'^{-1})] = \sum_{n=0}^{\infty} \zeta'^n J_n(\eta) + \sum_{n=1}^{\infty} (-1)^n \zeta'^{-n} J_n(\eta), \tag{49a}$$

as well as the fact that for $|\zeta'| < |z'|$ we have:

$$(z'-\zeta')^{-1} = z'^{-1}\sum_{n=0}^{\infty} (\zeta'/z')^n, \tag{49b}$$

we obtain straightforwardly that:

$$(2\pi i)^{-1}\int_{C_1} = \mathrm{Res}(\zeta'=0) = -\sum_{n=1}^{\infty} (\xi/iz)^n J_n(\eta) - \tfrac{1}{2}J_0(\eta). \tag{50}$$

Introducing this result into (47), we obtain finally the explicit form of $\psi(x, t)$ for $x < ct$ as given by (28b).

[16] See reference 11, p. 101.