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Diffusion of Neutrons*

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It is the object of the present investigation to include the thermal motion of the medium in the treatment of neutron diffusion, under the simplifying assumption that the collisions are such as between hard elastic spheres. The methods used are those previously established by the author in radiative and gas-kinetic transfer. They only have to be adapted to the present purpose. This is done in Sec. II. Boltzmann's original equation which is quadratic in the distribution functions is linearized by the assumption that the velocity distribution of the molecules is Maxwellian. The treatment of the resultant transport equation is different for "thermal" and for "fast" neutrons. Section III deals with thermal neutrons by the method of "moments." The distribution function is assumed to be of the form: Maxwellian factor times arbitrary function of direction, the latter written as a series of general spherical harmonics. Thereby, and by the use of a theorem due to Maxwell, the fundamental equation may be reduced to an infinite system of linear differential equations which have been previously treated. The case of arbitrary geometry is treated to second-order moments (inclusive). For the case of cylindrical symmetry a recurrence relation for the moments, to any order, is derived. In Sec. IV the case of fast neutrons is treated by the method of iterated integrations. In order to make them straightforward the kernel of the integral equation is transformed into the standard form. This is achieved by characterizing the collisions in a way which is different from that usually applied. A simple example for the working of the method is given and the results are discussed.

I. INTRODUCTION

DURING the last decade important progress has been made in the theory of neutron diffusion by a variety of methods.[1] As far as the author is aware, however, all previous treatments are subject to an important restriction. They do not take into account the thermal motion of the ambient medium, the "moderator," since they invariably assume that the molecules of this medium are at rest when they are hit by the neutrons. As long as this assumption is made, it is evidently impossible to investigate the influence of temperature on the diffusion process, or even to derive in a rational way the simple fact that the neutrons are not slowed down indefinitely but arrive ultimately at "thermal velocities." It is the object of the present investigation to close this gap of the theory.

Methods for dealing with the problem have been available for a good many years. As early as 1930 the author[2] treated the fundamental Boltzmann equation in a way which makes it directly applicable to the present purpose. By considering what happens in a molecular beam directed in a given direction, rather than dealing with a certain neighborhood in the field, the author transformed the Boltzmann equation to the form which is now frequently called a transport equation. Furthermore, he treated this equation by two methods, the method of "moments" and the method of "iteration," which he had developed still earlier in his theory of the anisotropic radiation field.[3]

Neither the author's work on gas-kinetic transfer, nor the earlier work on radiative transfer, seem to have been noticed by the later workers in the field, and some work has been duplicated. Naturally, important results which have been derived since then are not to be found in those early papers. However, in some regards they go considerably beyond what has been achieved since that time, such as by not being restricted to specific geometric conditions,[4] by including time-dependent processes (A, p. 624; B, p. 458; J, p. 249), and even the existence of a given external force field (J, p. 248), which case corresponds to an inhomogeneous medium in radiation theory (B, p. 501; C, p. 470).

It is the object of the author in the present paper to establish methods for dealing with the problem, not to arrive at results of practical applicability.

II. THE FUNDAMENTAL EQUATION

If the thermal motion of the medium is to be included in the theory, it becomes imperative to go back to the original form of the Boltzmann equation which is quadratic in the distribution functions and not to start from a linear transport equation as occurs in the theory of radiative transfer. We shall write the fundamental equation in the form which was obtained in J, but for the sake of simplicity we shall restrict ourselves to the simple case of elastic collisions between spherically shaped particles. Furthermore, we shall assume that there is no external force field.

We refer the index 1 to the neutrons of mass m_1 and the index 2 to the molecules of mass m_2 of the medium. Let $f(\mathbf{r}_1, \mathbf{c}_1, t)$, abbreviated by f_1, be the distribution function for the former, and $F(\mathbf{r}_2, \mathbf{c}_2, t)$, abbreviated by F_2, the corresponding distribution function for the

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[1] For a summary of the work done see R. E. Marshak, Revs. Modern Phys. **19**, 185–238 (1947).

[2] G. Jaffé, Ann. Physik **6**, 195–252 (1930). We refer to this paper in the text with J.

[3] G. Jaffé, (A) Ann. Physik **68**, 583 (1922); (B) Physik. Z. **23**, 500 (1922); (C) Ann. Physik **70**, 457 (1923). We refer to these papers in the text with the symbols A, B, and C.

[4] The linear problem was treated merely as an instance for various aspects of the general theory (A, p. 608, 619; C, p. 465), and the case of spherical symmetry was developed by a pupil of the author in Leipzig: H. W. Becker, Ann. Physik **81**, 1 (1926).

molecules of the medium.[5] Furthermore, let σ be the radius of the "sphere of action," i.e., the sum of the radii of m_1 and m_2, $\mathbf{g}$ the relative velocity of m_1 *versus* m_2, and ϑ, $(0\leq\vartheta\leq\pi/2)$, the angle between $\mathbf{g}$ and the line of centers $\mathbf{z}$, drawn from m_1 to m_2, at the moment of collision. This line is to be contained within an infinitesimal solid angle $d\Omega_z$ in order to make the collision well defined. Finally, $d\omega_1$ and $d\omega_2$ are the "cells in velocity space" containing the velocities $\mathbf{c}_1$ and $\mathbf{c}_2$ before collision. Primes, wherever they appear, refer to variables after collision.

With all these conventions the rate of change of f_1 along any straight line s of given direction may be written in the form (see J, pp. 207–208, with the changes appropriate to elastic collisions)

$$(1/c_1)\partial f_1/\partial t+\partial f_1/\partial s=J(f_1, F_2), \tag{1}$$

where the integral operator

$$J(f_1, F_2)=(1/c_1)\int(f_1'F_2'-f_1F_2)\sigma^2 g\cos\vartheta d\Omega_z d\omega_2 \tag{2}$$

represents in the well-known way the excess of gains over losses by collisions between neutrons and molecules.

In general, Eq. (1) should contain a second term referring to collisions between neutrons themselves [Bo, Eq. (25)]. However, at this point the first simplification of the neutron problem as compared with the general gas-kinetic problem, may be introduced. Under observable conditions, the collisions between neutrons are such rare events that the term referring to them may be omitted.

Of equal importance is the second simplifying assumption which we introduce, namely, that the distribution function of the molecules may be considered to be given independently of their collisions with the neutrons. Again, the collisions of neutrons with molecules occur at such relatively long intervals of time that after each collision the molecule can restore its thermal equilibrium before it is hit again. This means that we may assume F_2 to be a given function, and no second equation for F_2 besides (1) is required.

It will depend on the physical circumstances which form for F_2 is adequate. We shall assume that the temperature is high enough to eliminate quantum effects and that the influence of the binding of the molecules in the lattice may be disregarded. The latter assumption, implicit already in Eq. (2), is rather far reaching;[6] however, it certainly represents a first approximation in dealing with the thermal motion of the medium.

With these assumptions the velocity distribution of the molecules may be considered to be Maxwellian, i.e., of the form

$$F_2=A_2\exp(-h_2c_2^2), \tag{3}$$

with

$$A_2=n_2(h_2/\pi)^{\frac{3}{2}},\quad h_2=m_2/(2kT), \tag{4}$$

and, thereby, our fundamental Eq. (1) becomes linear with regard to the unknown distribution function f_1.

For some purposes it is desirable to generalize Eq. (1) so as to include true absorption, characterized by a coefficient of absorption α_{abs}, and formation of new neutrons, characterized by a source function $S_1(\mathbf{r}_1, \mathbf{c}_1, t)$, which is not necessarily isotropic. Then the fundamental equation takes the form

$$(1/c_1)\partial f_1/\partial t+\partial f_1/\partial s=J(f_1, F_2)-\alpha_{\text{abs}}f_1+S_1. \tag{5}$$

It was shown by the author that, in the theory of anisotropic radiation, the treatment has to be different according to whether absorption is "strong" or "weak," and in the treatment of Boltzmann's original equation the cases of "short" and "long" free path have to be distinguished.[7] Correspondingly, in the present case the treatment of the fundamental equations (1) or (5) becomes different for "thermal" and for "fast" neutrons. This distinction introduces in a most natural way the two methods of moments and of iterated integrations.

III. THERMAL NEUTRONS

If the neutrons are supposed to be of thermal velocities, but not by any means isotropically distributed, it is reasonable to represent the distribution function as the product of a Maxwellian distribution into a series progressing with general spherical harmonics. In order to simplify the writing we shall use the notation previously introduced (J, pp. 211–212).[8]

Thus, we assume for f_1 a representation of the form

$$f_1(\mathbf{r}_1, \mathbf{c}_1)=\exp(-h_1c_1^2)f_1^*(\mathbf{r}_1, \lambda_1, \mu_1), \tag{6}$$

with

$$f_1^*(\mathbf{r}_1, \lambda_1, \mu_1)=\sum_{m=0}^{\infty}\sum_{\mu=0}^{2m}A_m{}^{\mu}\Pi_m{}^{\mu}(\lambda_1, \mu_1), \tag{7}$$

and

$$h_1=m_1/(2kT),\quad \int f_1 d\omega_1=n_1. \tag{8}$$

This expresses that we have represented the vector $\mathbf{c}_1$ by polar coordinates, c_1, λ_1, μ_1, with regard to any fixed system of coordinates in velocity space.

Limiting ourselves to the stationary case we assume that the coefficients $A_m{}^{\mu}$ depend on the position vector $\mathbf{r}_1$ only, and neither on time nor on the quantity c_1. This latter assumption is an important restriction as compared with the more general treatment in J and will be discussed later.

[5] Our notation follows as closely as feasible the one used in J and in L. Boltzmann's classical *Vorlesungen über Gastheorie* (I. Teil, Joh. Ambr. Barth, Leipzig, 1896). We refer to the latter with the symbol Bo.

[6] See reference 1, p. 186.

[7] Naturally it has to be defined accurately what is meant by strong and weak absorption (A, pp. 601, 614) and by short and long mean free path (J, p. 226).

[8] The $\Pi_m{}^{\mu}$ introduced by us are, except for the normalization, essentially the same with the Y_{lm} frequently used in wave mechanics (e.g., see: Leonard I. Schiff, *Quantum Mechanics* (McGraw-Hill Book Company, Inc., New York, 1949), pp. 72, 73. We were forced to use the old notation and normalization in order to retain agreement with the older papers.

When the expressions (3) and (6) are substituted into the integral operator (2) the sum of the kinetic energies after collisions can be replaced by the corresponding sum before collision, and the factor $\exp(-h_1c_1^2)$ can be drawn before the sign of integration. Thereby Eq. (5), with $\partial/\partial t=0$, reduces to

$$df_1^*/ds=J_1-(\alpha(c_1)+\alpha_{\mathrm{abs}})f_1^*+S^*. \tag{9}$$

Here we have introduced the following abbreviations:

$$J_1=(A_2\sigma^2/c_1)\int \exp(-h_2c_2^2)\times(\sum_{m,\mu} A_m{}^\mu\Pi_m{}^\mu(\lambda_1', \mu_1'))g\cos\vartheta d\Omega_z d\omega_2, \tag{10}$$

$$\alpha(c_1)=(A_2\sigma^2/c_1)\int \exp(-h_2c_2^2)g\cos\vartheta d\Omega_z d\omega_2, \tag{11}$$

and

$$S^*=S_1(\mathbf{r}_1, \mathbf{c}_1)/(\exp(-h_1c_1^2)). \tag{12}$$

It is our object to transform Eq. (9) into an infinite system of partial differential equations for the coefficients $A_m{}^\mu$. In order to do that we have to develop the left-hand side into a series of general spherical harmonics and to perform the integrations in (10) and (11) in such a way that the corresponding terms in (9) also become series of this type. Finally the coefficients of equal harmonics on both sides of (9) have to be equated.

The integral in (11) is well known (Bo. p. 62), namely

$$\alpha(c_1)=1/\lambda(c_1)=(n_2\sigma^2\pi^{\frac{1}{2}}/\xi^2)\times\left\{\xi\exp(-\xi^2)+(2\xi^2+1)\int_0^\xi \exp(-y^2)dy\right\}, \tag{13}$$

with

$$\xi^2=h_2c_1^2. \tag{14}$$

Its reciprocal, $\lambda(c_1)$, represents the mean free path of a molecule m_1 of speed c_1 traveling through a gas of molecules m_2, n_2 in number per cc. The mean free path so defined appears here without any ambiguity, and its dependence on c_1 will prove to be of importance in the later work. For $(m_2/m_1)\rightarrow\infty$, or for $T\rightarrow 0$, $\alpha(c_1)$ reduces to

$$\alpha_\infty=1/\lambda_\infty=n_2\sigma^2\pi, \tag{15}$$

which is larger than the Maxwellian value by the factor $2^{\frac{1}{2}}$.[9]

The integral in (10) is not as simple, since it contains the angles λ_1', μ_1' after collision. Fortunately, these can be eliminated by the use of a theorem on spherical harmonics due to Maxwell.

Since $\mathbf{c}_1$ is to be kept constant for the integrations in (10) we may fix $\mathbf{c}_2$ in polar velocity coordinates, c_2, ϕ, χ, *versus* $\mathbf{c}_1$ as polar axis. Thereby $d\omega_2$ becomes

$$d\omega_2=c_2^2\sin\phi d\phi d\chi dc_2. \tag{16}$$

The direction of the line of centers, $\mathbf{z}$, may be fixed in the usual way by two angles, ϑ and ϵ, relatively to $\mathbf{g}$, with the result

$$d\Omega_z=\sin\vartheta d\vartheta d\epsilon. \tag{17}$$

Now we perform the integrations in the order $d\chi$, $d\epsilon$, $d\vartheta$, $d\phi$, dc_2. The first of these integrations can be achieved by the use of Maxwell's theorem already mentioned. In our notation it states (Bo. p. 171)

$$\int \Pi_m{}^\mu(\lambda_1', \mu_1')d\chi=2\pi\Pi_m{}^\mu(\lambda_1, \mu_1)P_m(\cos\phi), \tag{18}$$

where P_m is the Legendre polynomial of order m. When the integration with regard to $d\chi$ is performed, those with regard to $d\epsilon$ and $d\vartheta$ become immediate and we obtain

$$J_1=(2\pi A_2\sigma^2/c_1)\sum_{m,\mu} A_m{}^\mu\Pi(\lambda_1, \mu_1)\times\int_0^\infty \exp(-h_2c_2^2)c_2^2dc_2\int_{-1}^{+1} P_m(x)gdx. \tag{19}$$

In order to integrate with regard to $dx=\sin\phi d\phi$ we develop the relative velocity

$$g=(c_1^2+c_2^2-2c_1c_2\cos\phi)^{\frac{1}{2}} \tag{20}$$

into a series progressing with Legendre polynomials

$$g=\sum_{n=0}^\infty b_n(c_1, c_2)P_n(\cos\phi),\quad b_n(c_1, c_2)=[(2n+1)/2]\int_{-1}^{+1} g(x)P_n(x)dx. \tag{21}$$

If this development is substituted into (19) only the term with $n=m$ remains on account of the orthogonality of the Legendre polynomials, and we obtain

$$J_1=\sum_{m,\mu} \alpha_m A_m{}^\mu\Pi_m{}^\mu(\lambda_1, \mu_1), \tag{22}$$

where we have set

$$\alpha_m=[4\pi^2\sigma^2A_2/((2m+1)c_1)]\times\int_0^\infty \exp(-h_2c_2^2)b_mc_2^2dc_2. \tag{23}$$

Thereby we have obtained our object of representing the integral J_1 in form of a series like (7). It remains to evaluate the coefficients b_m and α_m. This can be done by quite straightforward procedures based on the definitions (21) and (23).

The first coefficient, α_0, is identical with $\alpha(c_1)$ as defined by (13). The subsequent ones become increasingly more involved. Therefore, we shall be satisfied to state the results for ξ [see Eq. (14)] larger than about 3.

[9] The function in (13) is tabulated in: J. Jeans, *An Introduction to the Kinetic Theory of Gases* (The Macmillan Company, New York, 1940), Appendix V.

Neglecting terms of the order $\exp(-\xi^2)$ we find

$$\alpha_0=\alpha_\infty(1+1/\xi^2), \tag{24a}$$

$$\alpha_1=[\alpha_\infty\{4/(3\pi^{\frac{1}{2}})\}/\xi][-1+1/(5\xi^2)], \tag{24b}$$

$$\alpha_2=[\alpha_\infty/(10\xi^2)][-1+15/(14\xi^2)]. \tag{24c}$$

These results indicate that, with the exception of α_0, all α's are negative and become rapidly smaller as ξ increases. Now ξ increases with m_2 and, in the limiting case of $(m_2/m_1)\gg 1$, α_0 becomes α_∞ and all other α's vanish. In that case our results become identical with those of our old radiation theory where the "emission," i.e., the term corresponding to the gain J_1, is treated as isotropic.

This leads to an important remark regarding a generalization of our theory. The factor g in (10) expresses, by its dependence on ϕ, how the gain in the beam $\mathbf{c}_1$ depends on the angle between the velocities $\mathbf{c}_1$ and $\mathbf{c}_2$ of the two colliding particles. This dependence is, of course, limited to the specific nature of the collisions, i.e., elastic collisions in our case. If the nature of the collision is changed the angular dependence may vary and may be expressed by any other function of c_1, c_2, and ϕ, say one suggested by experimental results. This would not invalidate our method as long as the function in question shows cylindrical symmetry and, therefore, may be written as a series like (21). This would change only the values of the constants b_m and α_m. Thus any kind of anisotropic scattering, in gas-kinetic as in radiative transfer, may be treated by the same method, if only the scattering is cylindrically symmetrical.

Following the program given above, we have now to develop the left-hand side of Eq. (9) into a series of harmonics. This has been carried through completely in the author's radiation theory [A, Eqs. (24) to (26)], and there is no need for reproducing the lengthy formulas. Nor is it necessary to write down the differential equations which are obtained by the comparison of the two sides of (9). They are identical with those given in the old paper [A, Eqs. (27), (42), (43)] with only one exception. In the case of isotropic scattering to which the older theory applies the absorption coefficient (there called α_ν) is the same in all orders. In the present case the effective absorption coefficient, α_m^*, which replaces α_ν, will be different for each group of $(2m+1)$ equations of order m, namely,

$$\alpha_m^*=\alpha_{\text{abs}}+\alpha_0-\alpha_m. \tag{25}$$

Therefore also the treatment of the linear differential equations remains exactly the same as exposed in A, and we shall only quote the generalized results. In first approximation the series (7) is broken off after the four first terms. This means restricting the representation to the local density, n_c, and the (vector) flux, $\mathbf{F}_c$, both referring to neutrons of specified velocity c_1. These physical quantities are related to the coefficients $A_m{}^\mu$ in the following manner:

$$n_c=4\pi\exp(-h_1c_1^2)c_1^2A_0{}^0, \tag{26}$$

and

$$(F_c)_i=(4\pi/3)\exp(-h_1c_1^2)c_1^3A_1{}^i,\quad i=0,1,2. \tag{27}$$

The differential equations of zero and first order lead to the equation

$$\nabla^2 n_c=3\alpha_1^*(\alpha_{\text{abs}}n_c-4\pi c_1^2S_1) \tag{28}$$

for the density n_c. The flux is given by

$$\mathbf{F}_c=-(c_1/3\alpha_1^*)\,\text{grad}n_c. \tag{29}$$

The treatment can be carried through in a similar way if the five moments of second order are added,[10] which means taking into consideration the pressure tensor (J, p. 221).

So far the treatment is entirely general concerning the geometry of the field. Whenever a solution of Eq. (28) can be found, Eq. (29) yields the flux. However, the boundary conditions can be given only in a rather restricted form, by prescribing either the density, or the normal component of the flux at the boundaries.

A more general, and more rational way of giving the boundary conditions is to prescribe the distribution function itself at the boundaries in all directions leading into the field (A, p. 592). In order to treat this more comprehensive problem higher moments have to be introduced. The more anisotropic the incoming radiation is, the more terms have to be considered.

From here on we shall restrict ourselves to problems of cylindrical symmetry, i.e., to cases where the distribution function reduces to a simple series of Legendre polynomials. We shall prove a theorem which permits of calculating all moments by a recurrence relation.

Since we are assuming cylindrical symmetry now there will be a distinguished direction. This we choose as x axis and measure the angles, λ_1, μ_1, with respect to the x axis as polar axis. Then Eq. (9) can be written in the form

$$\cos\lambda_1\, df_1^*/dx=\int\varphi(c_2)f_1^*(\lambda_1',\mu_1')g\times\cos\vartheta d\Omega_z dc_2 d\Omega_1'-\alpha f_1^*+S^*, \tag{30}$$

where we have introduced the abbreviations

$$\varphi(c_2)=(A_2\sigma^2/c_1)\exp(-h_2c_2^2)c_2^2, \tag{31}$$

and

$$\alpha=\alpha(c_1)+\alpha_{\text{abs}}, \tag{32}$$

and where $d\Omega_1'$ signifies an infinitesimal solid angle which contains the direction λ_1', μ_1'.

Now we multiply Eq. (30) by $P_m(\cos\lambda_1)$ and integrate over

$$d\Omega_1=\sin\lambda_1 d\lambda_1 d\mu_1. \tag{33}$$

This yields

$$\frac{d}{dx}\int P_m(\cos\lambda_1)f_1^*d\Omega_1=C_m-\alpha K_m+S_m, \tag{34}$$

[10] See Appendix I.

if the mth moment, K_m, is defined by

$$K_m=\int f_1{}^*P_m d\Omega_1=(4\pi/(2m+1))A_m{}^0, \qquad (35)$$

and if the abbreviations

$$C_m=\int \varphi(c_2)P_m \cos\vartheta d\Omega_z dc_2 d\Omega_1 \int g f_1{}^* d\Omega_1{}', \qquad (36)$$

and

$$S_m=\int S^* P_m d\Omega_1 \qquad (37)$$

are introduced. If S^* is isotropic all S_m's vanish with the exception of S_0.

Transforming the left-hand side of (34) by the recurrence relation of the Legendre polynomials we obtain

$$((m+1)/(2m+1))dK_{m+1}/dx$$
$$=-(m/(2m+1))dK_{m-1}/dx-\alpha K_m+C_m+S_m. \qquad (38)$$

Now we treat the integral C_m in exactly the same way as we treated J_1 above introducing a series of form (7) for $f_1{}^*$, making use of the Maxwell theorem (18), and adding in the end one integration with regard to $d\Omega_1$ (Eq. (33)). This yields

$$C_m=\alpha_m K_m. \qquad (39)$$

If this is introduced into Eq. (38) and the integration with regard to x performed we obtain the recurrence relation

$$((m+1)/(2m+1))K_{m+1}=-(m/(2m+1))K_{m-1}$$
$$-\alpha_m{}^*\int_0^x K_m dx+\int_0^x S_m dx+B_{m+1}, \qquad (40)$$

where $\alpha_m{}^*$ is defined by Eq. (25), and where B_{m+1} is a constant of integration.

It should be remarked again that the recurrence relation remains valid if $g(\phi)$ is replaced by a different function of ϕ, provided this function has cylindrical symmetry and can be developed into a series like (21). Only the values of the constants b_m and $\alpha_m{}^*$ are changed.

The recurrence relation (40) can serve to calculate successively any number of moments, and thereby the distribution function, provided the first two, K_0 and K_1 are known. These two can be obtained from the first-order solution given above [Eqs. (28) and (29)].[11] Since each step introduces a new constant of integration, sufficient constants may be obtained to satisfy the boundary conditions to any desired degree of accuracy.

[11] It has been shown by the author (A, pp. 609–610) that in the plane problem which we are dealing with now $A_0{}^0=-\alpha_\nu C_1 x+C_2$, $A_1{}^0=C_1$, (with C_1 and C_2 constants), is a rigorous solution of the infinite set of differential equations in the case $S_1=0$. It is easy to generalize this solution for a nonvanishing source function, and it may serve as a starting point for the application of (40). Also the higher approximation for the first four moments (A, pp. 610–613) may be used for this purpose.

The procedure has been shown at length in the older paper (A, pp. 607–612).

It is easy to extend the method of this section to the more general case where the temperature of the medium is a prescribed function of space. It becomes necessary only to combine the coefficients $A_m{}^\mu$ of Eq. (7) with the exponential factor of Eq. (6) into new coefficients. All transformations remain the same, and the system of differential equations for the new coefficients is unaltered.

The contents of this entire section are, however, limited to neutrons which are already slowed down to thermal velocities, though they may still exhibit anisotropy to any amount.[12] This restriction is due to our assumption that the $A_m{}^\mu$'s are independent of c_1. If this assumption were dropped the development (6), (7) could represent distribution functions which depart much more strongly from the Maxwellian distribution (as in J, p. 213). However, Maxwell's theorem, which proved to be essential for the treatment in this Section, would not be applicable any more, since $c_1{}'$ would be dependent on χ, and the first integration in J_1 or C_m could not be performed by the aid of that theorem. Therefore, another approach is indicated for "fast" neutrons.

IV. FAST NEUTRONS

The method which lends itself to the treatment of this case, and even to the treatment of all velocities, is that of iterated integrations (A, Sec. 2(b), p. 615 and J, Sec. 4, p. 238). However, the Boltzmann equation in its original form (1) makes the actual performance of the necessary integrations very inconvenient, since in the "gain term" the integration is over $d\omega_2$, i.e., over the velocities of m_2 *before* collision, whereas the distribution functions depend on the velocities *after* collision. Therefore, we shall first transform the gain term in such a way that it corresponds to the standard form in the theory of integral equations. This transformation will prove to be the crucial step in this section.

We limit ourselves again to stationary problems, we retain the assumption (3) for F_2 (with much better justification now) and add a term for true absorption and a source function, as in (5). Then the fundamental equation takes the form

$$df_1/ds=J_1-\alpha f_1+S_1, \qquad (41)$$

where α is given by (32), and J_1 is now defined by

$$J_1=(A_2\sigma^2/c_1)\int \exp(-h_2 c_2{}'^2) f_1{}' g \cos\vartheta d\Omega_z d\omega_2, \qquad (42)$$

f_1 being not expressed in any specific form.

Since the integral J_1 "collects" the gains out of all directions and velocities we wish to represent it as an integral with regard to $d\omega_1{}'$. This, however, cannot be

[12] A beam of thermal neutrons emerging from a reactor would present such a behavior.

achieved by transforming $d\omega_2$ into $d\omega_1'$ by the use of the collision equations: the Jacobian becomes zero.

This is readily understood by examination of the well-known Maxwell vector diagram for an elastic collision (Bo. Fig. 2). It is then seen that the definition of the collision gets lost by the indicated transformation. A collision can be specified in a variety of ways. The only one used until now is that underlying the integral operator (2), i.e., the collision is characterized by $\mathbf{c}_1$, $\mathbf{c}_2$, and the direction of the line of centers, $\mathbf{z}$, at the moment of collision. This represents eight independent parameters altogether. If, now, it is attempted to replace $\mathbf{c}_2$ by $\mathbf{c}_1'$ the collision cannot be specified any more by the direction of $\mathbf{z}$, since this direction is given already by that of $\mathbf{c}_1'-\mathbf{c}_1$. Therefore it becomes necessary to specify the collision in a new way. This can be done by giving, besides $\mathbf{c}_1$ and $\mathbf{c}_1'$, the direction of the relative velocity $\mathbf{g}$ before collision. Then Maxwell's diagram can again be constructed in a unique way.

It follows from this consideration that $d\Omega_z$, i.e., the infinitesimal solid angle containing $\mathbf{z}$, has to be replaced by $d\Omega_g$, i.e., an infinitesimal solid angle containing $\mathbf{g}$. Then the collision is again specified by eight independent parameters. Consequently we have to calculate the Jacobian in the equation

$$d\Omega_z d\omega_2 = \Delta d\Omega d_g \omega_1'. \tag{43}$$

This calculation is somewhat lengthy but straightforward and yields

$$\Delta = (\mu/\cos\vartheta)^3, \tag{44}$$

where

$$\mu = (m_1+m_2)/(2m_2), \tag{45}$$

and ϑ is the angle between the relative velocity, $\mathbf{g}$, and the line of centers, $\mathbf{z}$, as heretofore.[13]

Furthermore, it follows from Maxwell's diagram that the relative velocity can be expressed in the new variables by

$$g = \mu c_{11}/\cos\vartheta, \tag{46}$$

where

$$c_{11} = |\mathbf{c}_1'-\mathbf{c}_1|. \tag{47}$$

In consequence of (44) and (46) the integral (42) can be written in the form

$$J_1 = (\sigma^2\mu^4/c_1)\int f_1' K(\mathbf{c}_1, \mathbf{c}_1')d\omega_1', \tag{48}$$

where the kernel, K, has the desired representation and is given by

$$K(\mathbf{c}_1, \mathbf{c}_1') = A_2\int \exp(-h_2 c_2'^2)c_{11}(\sin\vartheta/\cos^3\vartheta)d\vartheta d\eta. \tag{49}$$

Here ϑ and η are the polar angles for $\mathbf{g}$ with regard to $\mathbf{c}_1-\mathbf{c}_1'$ as polar axis.

The kernel can be simplified by performance of the two integrations. The integration with regard to η leads to the result[14]

$$K(\mathbf{c}_1, \mathbf{c}_1^*) = 2\pi A_2 c_{11}\exp(-h_2X_0) \times\int_0^\infty J_0(i\gamma\zeta)\exp(-\beta^*\zeta^2)\zeta d\zeta, \tag{50}$$

with the abbreviations

$$X_0 = \mu^2c_1^2-(1-\mu)^2c_1'^2+2\mu(1-\mu)c_1c_1'\cos\phi, \tag{51}$$

and

$$\beta^* = h_2\mu^2c_{11}^2, \quad \gamma = 2h_2\mu c_{11}c_1'\sin\vartheta_1'. \tag{52}$$

The angle ϑ_1' is the one between $\mathbf{c}_1'$ and $(\mathbf{c}_1'-\mathbf{c}_1)$, ϕ is the angle between $\mathbf{c}_1$ and $\mathbf{c}_1'$ as previously, and $J_0(iy)$ is the Bessel function of order zero and purely imaginary argument.

The integration in (50) cannot be performed in closed form, but two important limiting cases can be obtained. If the velocity of the neutrons, $\mathbf{c}_1$, is of the thermal order the factor γ will be of the order 1 in the domain where the integrand contributes essentially. Therefore $J_0(i\gamma\zeta)$ may then be replaced by unity and the remaining integration is immediate.

We are not interested in thermal neutrons at present. On the other hand, if $\mathbf{c}_1$, and therefore also $\mathbf{c}_1'$ are large compared to the thermal velocity $\approx(1/h_2)^{\frac{1}{2}}$, $\gamma\zeta$ will be large in the important region[15] and $J_0(i\gamma\zeta)$ may be replaced by its asymptotic value. Then the integration can be performed again[16] and leads to the final form of the kernel for *fast neutrons*, namely

$$K(\mathbf{c}_1, \mathbf{c}_1') = n_2(h_2/\pi)^{\frac{1}{2}}\exp(-h_2X)/c_{11}, \tag{53}$$

where

$$X = X_0 - c_1'^2\sin^2\vartheta_1' = [\mu c_1^2-(1-\mu)c_1'^2+(1-2\mu)c_1c_1'\cos\phi]^2/c_{11}^2. \tag{54}$$

Combining (53) with (48) and (41) we obtain the fundamental equation in the form

$$df_1/ds = \beta\int \mathbf{K}(\mathbf{c}_1, \mathbf{c}_1')f_1'd\omega_1' - \alpha f_1 + S_1, \tag{55}$$

with

$$\mathbf{K}(\mathbf{c}_1, \mathbf{c}_1') = \exp(-h_2X)/c_{11}, \quad \beta = \sigma^2\mu^2n_2(h_2/\pi)^{\frac{1}{2}}/c_1. \tag{56}$$

Since the kernel has the standard form now, Eq. (55) can be solved by a Neumann series[17]

$$f_1 = \sum_{i=0}^{\infty}\beta^i f_1^{(i)}. \tag{57}$$

[13] See Appendix II.

[14] See Appendix III.

[15] Since, under our assumption, $\zeta\exp(-\beta^*\zeta^2)$ has its steep and narrow maximum at $\zeta_{\max}=(1/2\beta^*)^{\frac{1}{2}}$, $\gamma\zeta_{\max}$ will have the value $(2h_2)^{\frac{1}{2}}c_1'\sin\vartheta_1'$.

[16] We have neglected terms of the order $(1/(h_2\mu^2c_{11}^2))$.

[17] The singularity of the kernel, c_{11}^{-1}, does not affect the applicability of the theory of integral equations. Compare: D. Hilbert, *Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen* (B. G. Teubner, Leipzig und Berlin, 1912), pp. 267, 276.

This leads to the zero approximation

$$f_1^{(0)}=\bar{f}_1\exp(-\alpha s)+\exp(-\alpha s)\int_{s_0}^{s}\exp(\alpha y)S_1(y)dy, \quad (58)$$

along a straight line s of any direction in the field, if $\bar{f}_1$ is the given value of f_1 at the point s_0 where the line enters the field.

The subsequent approximations are given by

$$f_1^{(i)}=\exp(-\alpha s)\int_{s_0}^{s}\exp(\alpha y-h_2X)c_{11}^{-1}(f_1^{(i-1)})'d\omega_1'dy, \quad i=1,2,\cdots. \quad (59)$$

The method can be extended to time-dependent problems by taking the integrations in a "retarded" way (C, Sec. 5, p. 458 ff.), and also the case of an external force field can be included (J, p. 248). It should be pointed out, furthermore, that the solution (58), (59) remains valid if the temperature is a prescribed function of space. This generalization affects only the quadratures to be performed.

Before the method is applied to a simple example it may be of interest to point out two limiting cases to which (55) leads. If $m_2\gg m_1$, the parameter μ becomes $\frac{1}{2}$, and $h_2\rightarrow\infty$. The exponent h_2X [Eq. (54)] reduces to

$$h_2X=(h_2/4)(c_1'^2-c_1^2)^2/c_{11}^2, \quad (60)$$

indicating that the integral will contribute only for $c'=c_1$, as it should be. In performing the integration we may treat c_{11} as a constant for any fixed angle ϕ and obtain

$$df_1/ds=(\alpha_\infty/4\pi)\int(f_1')_{c_1'=c_1}d\Omega_1'-\alpha f_1+S_1. \quad (61)$$

Thus the gain, i.e., the scattering into the direction of s is isotropic and we have obtained one of the forms of the fundamental equation of our old radiation theory[18] with the absorption coefficient $\alpha_\nu=\alpha_\infty$.

We shall treat $m_2=1$, i.e., hydrogen, as the second limiting case, and we shall, furthermore, assume that the molecules are at rest when they collide. This means $\mu=1$ and $T\rightarrow 0$, or $h_2\rightarrow\infty$. Then the exponent becomes

$$h_2X=h_2c_1^2(c_1-c_1'\cos\phi)^2/c_{11}^2. \quad (62)$$

Consequently the integral contributes only for

$$c_1=c_1'\cos\phi, \quad 0\leq\phi\leq\pi/2,$$

as it should be. The integration yields

$$df_1'/ds=(1/4\pi)\int\sigma^*(f_1')_{c_1'=c_1/\cos\phi}d\Omega_1'-\alpha f_1+S_1, \quad (63)$$

with σ^*/n_2, the "differential cross section for gain," given by

$$\sigma^*/n_2=4\pi\alpha_\infty/(n_2\cos^3\phi). \quad (64)$$

[18] Compare A, p. 618 Eq. (IId) combined with the subsequent equation.

The cross section just introduced has to be distinguished from the cross section for scattering which is usually considered and which might be obtained by the same procedure from the expression (11). The difference is caused by the fact that, in the process of scattering, it need not be considered what change the speed undergoes, whereas for the gain the change in speed has to be just such that the neutron is brought into the velocity group $\mathbf{c}_1$. This is the essential circumstance which makes the gain term so much more difficult to handle than the loss term, and it results in a different angular dependence of the differential cross section for gain as stated above. The result (63) can be verified by direct derivation.

As a simple example for our method we treat the case of a parallel, monochromatic beam of neutrons impinging normally on the plane surface of a semi-infinite scattering medium. The beam is to have equal density over the entire surface.

We choose the normal to the surface, pointing inward, as x axis with its origin in the surface. Directions will be fixed by the polar angles, Θ and Φ, with the x axis as polar axis. Thus s in the fundamental equation (55) signifies a straight line forming an angle Φ with the x axis.

We describe the incident beam by a δ-function

$$\bar{f}_1(c_1,\Theta_1)=C\delta(c_1-c^*,\Theta_1), \quad (65)$$

with

$$\int\delta(c_1-c^*,\Theta_1)c_1^2dc_1d\Omega_1=1\ (\text{cm/sec})^3. \quad (66)$$

Hence $c_1=c^*$ is its speed and $\Theta_1=0$ its direction. We disregard true absorption and formation processes ($\alpha_{\text{abs}}=0$, $S_1=0$).

Then the zero solution becomes

$$f_1^{(0)}=C\exp(-\alpha(c^*)s)\delta(c_1-c^*,\Theta_1). \quad (67)$$

Substituting this into (59) (with $i=1$), we obtain for the first-order solution

$$f_1^{(1)}=K(\mathbf{c}_1,\mathbf{c}^*)\int\delta(c_1'-c^*,\Theta_1')d\omega_1'\Psi_1^{(1)}(x,\Theta_1), \quad (68)$$

where we have set

$$\Psi_1^{(1)}(x,\Theta_1)=\exp(-\alpha(c_1)s)\times\int_{s_0}^{s}\exp[(\alpha(c_1)-\alpha(c^*)\cos\Theta_1)y]dy. \quad (69)$$

The integral in (69) is easily performed though the cases $\Theta_1\lesseqgtr\pi/2$ have to be distinguished. For $\Theta_1<\pi/2$ the integration begins in the surface ($s_0=0$), for $\Theta_1>\pi/2$ at infinity. This yields

$$\Psi_1^{(1)}(x,\cos\Theta_1)=[\exp(-\alpha(c^*)x)-\exp(-\alpha(c_1)x/\cos\Theta_1)]/N, \quad 0\leq\Theta_1\leq\pi/2, \quad (70a)$$

$$\Psi_1^{(1)}(x, \cos\Theta_1) = [\exp(-\alpha(c^*)x]/N, \quad \pi/2 \leq \Theta_1 \leq \pi, \tag{70b}$$

with the denominator

$$N = \alpha(c_1) - \alpha(c^*) \cos\Theta_1.^{19} \tag{70c}$$

Now the final expression for the first-order solution takes the form

$$f_1^{(1)} = C[\exp(-h_2X^*)/c_{1*}]\Psi_1^{(1)}(x, \Theta_1), \tag{71}$$

with

$$X^* = [\mu c_1^2 - (1-\mu)c^{*2} + (1-2\mu)c_1c^* \cos\Theta_1]^2/c_{1*}^2, \tag{72a}$$

and

$$c_{1*}^2 = c_1^2 + c^{*2} - 2c_1c^* \cos\Theta_1. \tag{72b}$$

It should be pointed out that the solution (71) has a singularity at $\mathbf{c}_1 = \mathbf{c}^*$ (i.e., at $c_1 = c^*$, $\Theta_1 = 0$), but the singularity is integrable if integrated over $d\omega_1$.

The velocity distribution is given by the factor $K(\mathbf{c}_1, \mathbf{c}_*)$; the distribution of intensity in the scattered beam, in its dependence on position, x, and direction, Θ_1, by $\Psi_1^{(1)}$. It is easy to show that the latter factor would reduce to $\exp(-\alpha(c^*)x)$ if an incident beam of negligible cross section would be considered.

The first order solution represents conditions as they result from the consideration of neutrons which have undergone one collision (J, p. 252). Therefore it would be valid if a slab of absorbent medium were to be considered whose thickness is appreciably smaller than the mean free path $\lambda^* = 1/\alpha(c^*)$.

We briefly discuss the two cases $m_2 \gg m_1$, and $m_2 = m_1$. In the former case $h_2 \rightarrow \infty$ and the exponent is given by (60). Hence the velocity spectrum in the scattered beam contracts to a narrow line about $c_1 = c^*$ as m_2/m_1 increases indefinitely. Integrating over $c_1^2dc_1$ we start from $\beta f_1^{(1)}$ (which is the contribution to $f_1^{(0)}$) and find

$$\beta \int f_1^{(1)} c_1^2 dc_1 = (C/4\pi)\alpha_\infty \Psi_1^{(1)}(x, \Theta_1). \tag{73}$$

Therefore the angular distribution is determined by $\Psi_1^{(1)}$ exclusively and becomes isotropic if a narrow incident beam is considered (see above). In the case $m_1 = m_2$ the exponent is given by (62) and the velocity distribution is somewhat involved. We treat the two cases that the velocity c_1 is either "thermal" (i.e., $c_1 \approx 1/(h_2)^{\frac{1}{2}}$) or "fast" (i.e., $c_1 \approx c^*$). For thermal neutrons[20] we have $h_2 = h_1$, $h_1c_1^2 \approx 1$. Consequently the kernel boils down to

$$K^* = \exp(-h_1c_1^2 \cos^2\Theta_1)/c^*. \tag{74}$$

Hence there will be a finite contribution of thermal neutrons, even after the first collision. By integrating over the velocity spectrum (for a given direction) we find

$$\beta \int f_1^{(1)} c_1^2 dc_1 = (C/4\pi)\alpha_\infty \times [1/(h_1c_1c^* \cos^3\Theta_1)]\Psi_1^{(1)}(x, \Theta_1). \tag{75}$$

In the case $c_1 \approx c^*$, i.e., $h_1c_1^2 \gg 1$, the kernel will be different from zero only for

$$c_1 \approx c_1' \cos\Theta_1, \quad \Theta_1 \leq \pi/2, \tag{76}$$

and the velocity spectrum contracts for every direction to a narrow line about the velocity (76). The total number of neutrons in this beam is found to be

$$\beta \int f_1^{(1)} c_1^2 dc_1 = (C/4\pi)4\alpha_\infty \Psi_1^{(1)}(x, \Theta_1) \tag{77}$$

relative to the contribution of $f_1^{(0)}$. Hence the number of fast neutrons is larger than that of the thermal ones by a factor of the order c^*/c_{therm}. There are no other groups of velocity in the scattered beam except the two discussed, since the exponent in K^* makes all other contributions negligible.

We shall not enter here into the calculation of higher approximations. They are all performed easily in an approximate way by treating the kernel in the preceding solution as a δ-function (on account of the denominator c_{11}); then this kernel reproduces itself. Otherwise lengthy calculations become necessary and it is difficult to obtain the asymptotic solution for $x \rightarrow \infty$ by the addition of the series (57). It can, however, be obtained by a more direct procedure by assuming it to be of the form

$$f_1 = \exp(-\alpha x)K(\mathbf{c}_1, \mathbf{c}^*)\Psi(\Theta_1), \tag{78}$$

where the kernel for thermal neutrons has to be used (see Eq. (49)).

The case of a point source in an infinitely extended medium can be treated in a way very similar to the linear problem.

Note added in proof:—The author is obliged to Dr. Richard K. Osborn for pointing out to him that the integral (50) is known in explicit form (see G. N. Watson, *Theory of Bessel Functions* (Cambridge University Press, Cambridge, 1944), Sec. 13.3, Eq. (1), p. 393). If the correct formula is used the result is the same as the one obtained by the author who used an asymptotic procedure. Hence the final form of the kernel, Eq. (56), is not limited to "fast neutrons," but holds for all values of the parameters.

APPENDIX I

If the series (7) is broken off after the second-order terms, the problem can still be treated in a general way if a potential function, Φ, (called in A the "radiation potential") is introduced. By similar calculations as in the old paper (A, pp. 605 to 607) it can be shown that the differential equation for Φ becomes

$$\nabla^2\Phi - a_1\Phi + a_2S_1 = 0, \tag{79}$$

[19] It is in the expression for N where the distinction between $\alpha(c_1)$ and $\alpha(c^*)$ becomes of importance. However, this fact makes itself felt only in the calculation of the higher approximations, i.e., for $i > 1$. For $i = 1$ the second term in (70a) prevents a singularity at $\Theta_1 = 0$ even if $\alpha(c_1) = \alpha(c^*)$.

[20] It might appear contradictory to apply our result to thermal neutrons since we used a kernel limited to fast neutrons (see p. 606). It should be pointed out, however, that for the calculation of the kernel in the first-order solution the value of c^* is relevant, as is shown by Eq. (68), and we still assume that c^* is much larger than $1/(h_1)^{\frac{1}{2}}$.

with the abbreviations

$$a_1 = 3\alpha_1{}^*\alpha_2{}^*\alpha_{\text{abs}}/a_3, \quad (80a)$$

$$a_2 = 4\pi c_1{}^3\alpha_2{}^*/a_3, \quad (80b)$$

and

$$a_3 = \alpha_2{}^* + (4/5)\alpha_{\text{abs}}. \quad (80c)$$

When Φ is determined from this differential equation the first four moments are obtained in the following way

$$n_c = (3\alpha_1{}^*\alpha_2{}^*/c_1a_3)\Phi + (16\pi c_1{}^2/5a_3)S_1, \quad (81)$$

and

$$\mathbf{F}_c = -\text{grad}\Phi. \quad (82)$$

The five second-order moments $A_2{}^i (i=0, 1, 2, 3, 4)$ can be expressed in terms of partial second differential coefficients of Φ. For $\alpha_{\text{abs}}=0$, $\alpha_1{}^*=\alpha_2{}^*=\alpha_\nu$, Eq. (81) reduces to the result of the old paper [A, Eq. (46)].

APPENDIX II

In the calculation of the Jacobian in Eq. (43) the following considerations should be taken into account. It is customary to fix the direction of the line of centers, $\mathbf{z}$, relatively to the relative velocity, $\mathbf{g}$. In a similar way in our choice of collision parameters the direction of $\mathbf{g}$ may be fixed relatively to $(\mathbf{c}_1'-\mathbf{c}_1)$. This is perfectly legitimate as long as integrations are to be performed which keep $\mathbf{c}_1$ and $\mathbf{c}_2$ in the old system, or $\mathbf{c}_1$ and $\mathbf{c}_1'$ in the new system, constant. However, if the five variables $\mathbf{c}_2$, $\mathbf{e}_z$ (with e_z a unit vector in the direction of z) are to be transformed into $\mathbf{c}_1'$, $\mathbf{e}_g$, the unit vectors $\mathbf{e}_z$ and $\mathbf{e}_g$ may not be fixed any longer relatively to $\mathbf{g}$ or $(\mathbf{c}_1'-\mathbf{c}_1)$, since these vectors are varied themselves in the procedure. Consequently all five variables of either system have to be defined in one and the same fixed coordinate system.

Let the components of $\mathbf{c}_1$ in this system be u_1, v_1, w_1, etc.; furthermore, let ϑ_z, η_z be the polar angles for $\mathbf{z}$, and ϑ_g, η_g the corresponding ones for $\mathbf{g}$. Then the five expressions relating the old set of variables with the new set become

$$u_2 = u_1 + \mu c_{11}{}^2 \cos\vartheta_g/V', \quad (83a)$$

$$v_2 = v_1 + \mu c_{11}{}^2 \sin\vartheta_g \cos\eta_g/V', \quad (83b)$$

$$w_2 = w_1 + \mu c_{11}{}^2 \sin\vartheta_g \sin\eta_g/V', \quad (83c)$$

$$\cos\vartheta_z = (u_1'-u_1)/c_{11}, \quad (83d)$$

$$\tan\eta_z = (w_1'-w_1)/(v_1'-v_1). \quad (83e)$$

Here the abbreviations

$$c_{11} = ((u_1'-u_1)^2 + (v_1'-v_1)^2 + (w_1'-w_1)^2)^{\frac{1}{2}}, \quad (84a)$$

$$V' = (u_1'-u_1)\cos\vartheta_g + (v_1'-v_1)\sin\vartheta_g\cos\eta_g + (w_1'-w_1)\sin\vartheta_g\sin\eta_g, \quad (84b)$$

have been introduced. The angle ϑ between $\mathbf{g}$ and $\mathbf{z}$ is given by

$$\cos\vartheta = V'/c_{11}. \quad (85)$$

Now the calculation of Δ is straightforward (some work may be shortened by introducing $x_z=\cos\vartheta_z$ and $x_g=\cos\vartheta_g$ as new variables).

A further remark may not be out of place. We have based our calculation of the gain term in (2) on Boltzmann's original expression, which makes use of the concept of "inverse collisions." One might think that this represents a detour and that the gain term could be calculated in a more direct manner by first setting up the usual expression for the number of direct collisions, by then expressing g in terms of c_{11} by Eq. (47), and by finally integrating over $d\Omega_g d\omega_1'$. The last step of this procedure would be erroneous because it would assume arbitrarily $\Delta=1$, in contradiction with the result in the text. In replacing g by c_{11} a change of variables is implied which necessitates a corresponding Jacobian. Therefore, the laborious calculation of the latter cannot be avoided by starting from direct collisions.

APPENDIX III

In order to perform the integration with regard to η in (50) we first have to obtain an expression for c_2' in our variables of integration. Now we have

$$\mathbf{c}_2' = \mathbf{c}_1' - \mathbf{g}', \quad (86)$$

or, by the aid of (46),

$$c_2'^2 = c_1'^2 + (c_{11}\mu/\cos\vartheta)^2 - 2c_1'(c_{11}\mu/\cos\vartheta)\cos(\mathbf{g}', \mathbf{c}_1'). \quad (87)$$

For the integration with regard to ϑ and η, $\mathbf{c}_1$ and $\mathbf{c}_1'$, hence also $\mathbf{c}_{11}=\mathbf{c}_1'-\mathbf{c}_1$ are being kept constant. Consequently we may fix the directions of $\mathbf{g}'$ and $\mathbf{c}_1'$ with regard to $\mathbf{c}_{11}$ as polar axis. Let the polar angles be ϑ', η', and ϑ_1', η_1', respectively. Now the angle between $\mathbf{g}'$ and $(\mathbf{c}_1'-\mathbf{c}_1)$ is the same as the angle between $\mathbf{g}$ and $(\mathbf{c}_1-\mathbf{c}_1')$, and the vectors $\mathbf{g}$, $\mathbf{g}'$ and $(\mathbf{c}_1-\mathbf{c}_1')$ are in the same plane; consequently ϑ', η' may be replaced by ϑ, η. This leads to the final expression for $(c_2')^2$, namely,

$$c_2'^2 = c_1'^2 + (c_{11}\mu/\cos\vartheta)^2 - 2\mu c_1'c_{11}[\cos\vartheta_1' + \tan\vartheta\sin\vartheta_1'\cos(\eta-\eta_1')]. \quad (88)$$

If now $\tan\vartheta=\zeta$ is introduced as a new variable the result (5) is obtained, since

$$\int_0^{2\pi} \exp(\gamma\zeta\cos(\eta-\eta_1'))d\eta = 2\pi J_0(i\gamma\zeta). \quad (89)$$