conduction band. This naive model neglects, of course, such possibly important influences as electronic work functions and other surface properties at the interfaces. Nevertheless, it is useful in correlating some of the main features of the film behavior.

The linear variation of photocurrent with voltage below the forming potential is of interest, particularly since the leakage current is very nonlinear. The situation appears somewhat similar to that in thin semiconductor films in vacuum.[4]

We wish to thank J. K. Bragg, M. H. Hebb, J. P. Howe, D. Turnbull, and D. A. Vermilyea for valuable conversations.

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Multiple Scattering of Fast Protons in Photographic Emulsions*†

MARTIN J. BERGER‡
Department of Physics and Institute for Nuclear Studies, University of Chicago, Chicago, Illinois
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Measurements have been made of the multiple Coulomb scattering of protons with energies of 337±1 Mev and 218±2 Mev in Ilford G-5 emulsions. A total of 261 tracks with a combined length of 130.4 cm were analyzed by the determination of the lateral multiple scattering deflections according to the "coordinate method." The results are summarized in terms of the customary scattering factors which, for a given cell length, indicate the proportionality of the average multiple scattering deflection to the quantity "charge/momentum×velocity" of the scattered particle. Scattering factors are given for cell lengths of 250, 500, 750, and 1000 microns. These results are compared with the predictions of various theories of multiple scattering, including those of Moliere, Snyder, and Scott, as well as a proposed extension of the theory of Goudsmit and Saunderson, which has the advantage of being directly and conveniently applicable to the analysis of lateral multiple scattering deflections. The predicted theoretical scattering factors are in fair agreement with the experimental values, but slightly larger; the discrepancies are either within, or just outside of, the limits of the experimental error (3–6 percent).

I. INTRODUCTION

THE energy of charged particles can be determined by the measurement of the deflections of their tracks in photographic plates caused by multiple scattering, i.e., frequent and predominantly small-angle Coulomb scattering. This method has been developed by various investigators in the course of cosmic ray research[1–5] and has more recently been applied to fast electrons from nuclear reactions and from accelerators.[6,7] It was the purpose of this investigation to determine the multiple scattering of artificially accelerated high energy protons. A preliminary account of some of the results of the experiment has been presented in a previous communication.[8] Measurements of the multiple scattering of fast protons under controlled conditions have also been reported by Gottstein *et al.*[9]

The evidence presented here is based on the analysis of the lateral multiple scattering deflections that occurred in 130.4 cm of track of protons with energies of 337±1 Mev and 218±2 Mev in Ilford G-5 emulsions that had been exposed to the external beam of the 184-in. Berkeley cyclotron.[10] Since the energy of this beam is known with an accuracy of 0.3 percent, the multiple scattering calibration was thus freed of the error introduced by the uncertainty of the particle energy that occurs with cosmic-ray particles whose energies must be determined by grain counts or range measurements, or with fast electrons which may suffer considerable radiative energy loss.

II. ANALYSIS OF THE TRACKS

The method of determining the multiple scattering from the deflections of the tracks is illustrated by Fig. 1. The heavy winding curve represents a track of a particle passing successively through points P_1, P_2, and P_3, as projected onto a plane parallel to the surface of the emulsion. Point P_2 divides the portion of track shown

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† Based on a doctoral dissertation submitted to the University of Chicago.
‡ Now Postdoctoral Fellow in statistics and physics, University of Chicago, Chicago, Illinois.
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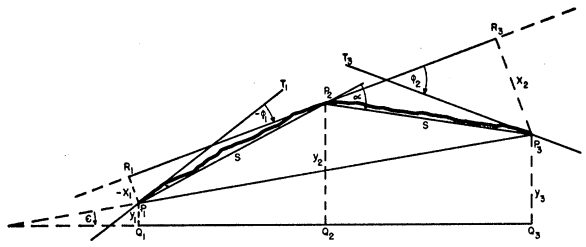

FIG. 1. Geometrical analysis of a track.

in Fig. 1 into two equal cells of length s. Lines P_1T_1, R_1R_3, and T_3P_3 are tangent to the track at P_1, P_2, and P_3, respectively. Line Q_1Q_3 is a reference line, approximately parallel to the track with respect to which the coordinates of the points P_1, P_2, and P_3 are, respectively, y_1, y_2, and y_3.[11] In the absence of scattering the track would be rectilinear. Hence in Fig. 1, $-x_1$ and x_2 are the lateral spatial multiple scattering deflections in cells (P_1, P_2) and (P_2, P_2), as projected on planes containing the direction of emergence, respectively, incidence, of the particle.[12] The probability distributions of $-x_1$ and x_2 are the same, provided one averages over all possible directions of incidence in cell (P_1, P_2) and emergence in cell (P_2, P_3) (see Appendix, Sec. C).

It follows from simple geometric considerations that, if the multiple scattering deflections are small, the angle α between successive chords P_1P_2 and P_2P_3 is

$$\alpha=(x_2-x_1)/s, \tag{1}$$

and that

$$x_2-x_1=(2y_2-y_1-y_3)=-\Delta^2y_1.^{13} \tag{2}$$

Let y_t $(t=1,2,\cdots,k+1)$ be the set of coordinates of a track divided into k cells of lengths s. Then the lateral multiple scattering is specified by the set of second coordinate differences Δ^2y_t, or chord-angles α_t $(t=1,2,\cdots,k-1)$. It has become customary to express this information in terms of a scattering factor as follows:

$$K=\langle\Delta^2y\rangle\frac{1}{s}\left(\frac{100}{s}\right)^{\frac{1}{2}}\frac{pv}{Z}=\langle\alpha\rangle\left(\frac{100}{s}\right)^{\frac{1}{2}}\frac{pv}{Z}, \tag{3}$$

where Z, p, and v are the charge, momentum, and velocity of the scattered particle. The numerical values of the scattering factor K are given for the case that $\langle\alpha\rangle$ is expressed in degrees, s in microns and (pv) in Mev. The use of such a scattering factor is motivated by the fact that K is approximately constant, and thus expresses the universal scattering properties of the photographic emulsion.[14] The average values $\langle\Delta^2y\rangle$ and $\langle\alpha\rangle$ used in expression (3) may be defined in a number of different ways. We shall in this paper consider the following averages:

(a) Arithmetic mean

$$\langle|\Delta^2y|\rangle=\frac{1}{k-1}\sum_{t=1}^{k-1}|\Delta^2y_t|.$$

(b) Arithmetic mean with cutoff

$$\langle|\Delta^2y|\rangle_c=\frac{1}{k'}\sum_{t=1}^{k-1}W_t'|\Delta^2y_t|,$$

where

$$W_t'=1 \quad \text{if} \quad |\Delta^2y_t|\leq 4\langle|\Delta^2y|\rangle_c,$$
$$W_t'=0 \quad \text{otherwise};$$
$$k'=\text{number of nonzero } W_t'\text{s}.$$

(c) Root mean square value with cutoff

$$\langle(\Delta^2y)^2\rangle_c^{\frac{1}{2}}=\left\{\frac{1}{k''}\sum_{t=1}^{k-1}W_t''(\Delta^2y_t)^2\right\}^{\frac{1}{2}},$$

where

$$W_t''=1 \quad \text{if} \quad |\Delta^2y_t|\leq 4\langle(\Delta^2y)^2\rangle_c^{\frac{1}{2}},$$
$$W_t''=0 \quad \text{otherise};$$
$$k''=\text{number of nonzero } W_t''\text{s}.$$

The corresponding scattering factors for these three cases are K_1, K_1^c, and K_2^c, respectively. The averages of α are defined in an analogous fashion in the three cases. The purpose of the truncated averages (b) and (c) is to reduce the statistical fluctuations caused by the comparatively rare large single scattering deflections which may give rise to occasional very large second coordinate differences or chord-angles. Previous experimental multiple scattering determinations have been mainly concerned with the scattering factors K_1 and K_1^c, while the theory outlined in the next section is applicable to K_2^c.

III. THEORIES OF MULTIPLE SCATTERING

A. General Remarks

Multiple scattering deflections are the sum of numerous small and a few comparatively large single scattering deflections. Hence their probability distribution is approximately Gaussian with an added "single scattering tail." The Gaussian part of the distribution has been determined in the approximate theories of Williams and Fermi as presented by Rossi and Greisen.[15] The ac-

[11] In the present experiment, the angle ϵ between the chord to the track and the reference line was always smaller than 1°.

[12] The convention has been adopted of giving a negative (positive) algebraic sign to deflections projected on a plane containing the direction of emergence (incidence) when they are directed toward the reference line with respect to which the coordinates are determined, and the opposite sign when the deflections are directed away from the reference line.

Actually it is not strictly true that $-x_1$ and x_2 are deflections projected on the planes of emergence or incidence, since the tracks are slightly inclined with respect to the plane of projection. But the tracks used in this experiment were long ($\geq$8000 microns) compared to the thickness of the emulsion ($\leq$600), so that the angle of dip was less than 4°, and the error caused by it negligible.

[13] Δ is a difference operator: $\Delta y_t=y_{t+1}-y_t$, $\Delta^2y_t=\Delta(\Delta y_t)=y_{t+2}-2y_{t+1}+y_t$, etc.

[14] Actually K is slightly dependent on s and (pv). Hence it seems preferable to use the term "scattering factor" instead of the term "scattering constant" often found in the literature.

[15] B. Rossi and K. Greisen, Revs. Modern Phys. **13**, 241 (1941); this paper contains a presentation of the calculations of Fermi, and of a simplified version of the theory of E. J. Williams.

curacy of the multiple scattering measurements in photographic emulsions is such, however, that it becomes necessary to use the more detailed theories such as those of Williams,[16] Moliere,[17] and Snyder and Scott,[18,19] which also take into account the single scattering tail. In these theories the increased accuracy had to be obtained at the expense of increased complexity, their results being given to a large extent in numerical rather than analytical form. All of the treatments mentioned so far give the distribution of the angular multiple scattering deflection; the distribution of the lateral deflections is only given by the theories of Fermi, and of Snyder and Scott (who have not worked it out in quite the same numerical detail as the angular distribution, however). In order to make the exact angular distributions applicable to the method of track analysis in terms of second coordinate differences or chord-angles as described in the previous section, one may use the relation between the mean square angular and lateral projected multiple scattering deflections

$$s^2\langle\phi^2\rangle=3\langle x^2\rangle, \tag{4}$$

predicted by the theory of Fermi and generally valid in the small angle approximation. In Fig. 1, the projected angular deflections corresponding to the lateral deflections $-x_1$ and x_2 are designated $-\phi_1$ and ϕ_2. It follows readily from (1) and (4) that $\langle\alpha^2\rangle^{\frac{1}{2}}=(2/3)^{\frac{1}{2}}\langle\phi^2\rangle^{\frac{1}{2}}$. The same factor of proportionality $(2/3)^{\frac{1}{2}}$ has in previous multiple scattering determinations using photographic plates also been assumed to hold for the mean angular and lateral deflections (both with and without a cutoff). This is an accurate approximation justified by the theory of Snyder and Scott.

B. An Extension of the General Theory of Goudsmit and Saunderson

Both the theories of Moliere, and of Snyder and Scott, have been shown to be special cases (small angle approximation) of the more general theory of Goudsmit and Saunderson.[20] This theory, which gives the angular multiple scattering distribution, has the following advantages: (1) The theory can be used in conjunction with an arbitrary underlying single scattering law, which makes it easy to take into account—to any desired degree of approximation—the modifications introduced into the Coulomb scattering of charged particles by atoms as the result of screening of the nuclear charge by the atomic electrons, and of the interference effects caused by the finite size of the nucleus; (2) Goudsmit and Saunderson's theory can readily be extended to provide simple expressions for the average lateral multiple scattering deflection, and thus can be made applicable to the coordinate method of analyzing tracks in photographic emulsions.

Consider a particle traversing a path length s in a scattering medium, and let x be the lateral multiple scattering deflection as projected on a plane containing the direction of incidence (direction of motion at $s=0$). It is shown in the Appendix [see Eq. (A26)] that the following expression holds for the average deflection as function of the path length s, provided that the particle is scattered in a thin layer of matter and predominantly in the forward direction (as is the case for fast charged particles undergoing multiple Coulomb scattering in photographic emulsions):

$$\langle x^k|s\rangle=k\int_0^s(s-s')^{k-1}\langle\sin^k\theta\cos^k\varphi|s'\rangle ds'. \tag{5}$$

In Eq. (5) $\langle\sin^k\theta\cos^k\varphi|s\rangle$ is an average angular multiple scattering deflection; it can be found from the theory of Goudsmit and Saunderson, according to which the probability $F(\theta,\varphi|s)\sin\theta d\theta d\varphi$ that a particle moving initially (at $s=0$) in the z direction, will be deflected by multiple scattering into direction $(\theta,\theta+d\theta)$, and $(\varphi,\varphi+d\varphi)$, is given by

$$F(\theta,\varphi|s)\sin\theta d\theta d\varphi=\sum_{n=0}^{\infty}\frac{2n+1}{4\pi}P_n(\cos\theta)\times\exp\left\{-N\int_0^s ds'\int_0^\pi\sin\theta' d\theta' I(\theta')\times[1-P_n(\cos\theta')]\right\}\sin\theta d\theta d\varphi, \tag{6}$$

where P_n is the nth Legendre polynomial, N is the number of scattering centers per unit volume, and $I(\theta)$ is the underlying single scattering law.

In the Appendix, a more complicated expression is also derived for $\langle x^k|s\rangle$ which holds without any approximation. It can be used when one is dealing with slow particles in photographic emulsions, while it reduces to the simpler expression (5) for fast particles.

C. Application to Multiple Coulomb Scattering

In applying the foregoing results to multiple Coulomb scattering we shall follow Williams[16] in assuming for the interaction between charged particles and atoms a potential

$$V(r)=\frac{ZZ'e^2}{r}e^{-r/a}(1-e^{-2r/b}), \tag{7}$$

which can be shown to give, in the Born approximation, a single scattering cross section,

$$I(\theta)=\frac{2\pi Z^2Z'^2e^4}{p^2v^2}\times\left\{\frac{1}{\sin^2\frac{1}{2}\theta+\frac{1}{4}\omega_0^2}-\frac{1}{\sin^2\frac{1}{2}\theta+\frac{1}{4}\omega_0^2(1+2\omega_2/\omega_0)^2}\right\}^2. \tag{8}$$

[16] E. J. Williams, Proc. Roy. Soc. (London) **A169**, 531 (1939); Phys. Rev. **58**, 292 (1940).
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[19] W. T. Scott, Phys. Rev. **85**, 245 (1952).
[20] S. Goudsmit and J. L. Saunderson, Phys. Rev. **57**, 24 (1940); **58**, 36 (1940).

Notations: Z' and A: atomic number and weight of the scattering atom; Z, p, V and $2\pi\lambda\!\!\!^{-}$: charge number, momentum, velocity, and de Broglie wavelength of the scattered particle; e and m: charge and mass of the electron; $\hbar$: Planck's constant; c: velocity of light; a and b: maximum and minimum impact parameters.

$$\omega_0=\lambda\!\!\!^{-}/a \quad \text{and} \quad \omega_2=\lambda\!\!\!^{-}/b.$$

In the potential (7) the factor $\exp(-r/a)$ takes into account the screening of the nucleus by the atomic electrons, which sharply reduces the scattering cross section for angles less than ω_0; the factor $[1-e^{-2r/b}]$, derived on the assumption that the electric charge of the nucleus is distributed uniformly over a sphere of radius b, takes into account the interference effects resulting from the finite size of the nucleus which cut down the scattering for angles larger than ω_2. We shall set the maximum impact parameter

$$a=0.881a_0[1.13+3.76(ZZ'c/137v)^2]^{-\frac{1}{2}}, \qquad (9)$$

where $a_0=(\hbar^2/me^2)(Z')^{-\frac{1}{3}}$ is the first Bohr radius of the atom. Moliere[21] has shown that in the case $b=0$ (nuclear point charge) the use of an impact parameter of this form will assure the validity of the scattering law (8), even if the condition for the applicability of the Born approximation ($ZZ'c/137v\ll 1$) is not strictly fulfilled, a circumstance that may easily arise with photographic emulsions in which most of the scattering is due to silver ($Z'=47$) and bromine ($Z'=35$). It seems plausible that for the case $b\neq 0$ the impact parameter a given by (9) is also adequate. Finally, we set the nuclear radius $b=1.4\times 10^{-13}A^{\frac{1}{3}}$.

Substituting (8) into (6), computing from (6) the value of $\langle \sin^k\theta \cos^k\varphi \,|\, s\rangle$ and substituting it into (5), one finds by a straightforward calculation that

$$\langle x^2\,|\,s\rangle=\tfrac{1}{3}n\omega_0^2s^2\log\left[\frac{\omega_2}{\omega_0}\left(1+2\frac{\omega_0}{\omega_2}\right)^{-1}\frac{2}{e}\right], \qquad (10)$$

where $n=4\pi Na^2s(ZZ'c/137v)^2$ denotes the average number of individual collisions which the particle makes in traversing a distance s. This result is the same as that predicted by the theory of Fermi, except for the correction factor $(2/e)(1+2\omega_0/\omega_2)^{-1}$ in the logarithm. But the distribution of the lateral multiple scattering deflections will in general not be Gaussian, unless the particle traverses a very large thickness of material, even though the finite size of the nucleus provides a cutoff ω_2 for the size of the individual angular single scattering deflections. This can easily be seen by considering the ratio $M=\langle x^4\,|\,s\rangle/\langle x^2\,|\,s\rangle^2$ which for a Gaussian distribution has the numerical value 3. We find from (5), (6), and (8) that

$$M=\frac{2.7}{n}\left\{\log\left[\frac{\omega_2}{\omega_0}\left(1+2\frac{\omega_0}{\omega_2}\right)^{-1}\frac{2}{e}\right]\right\}^{-2}\left(\frac{\omega_2}{\omega_0}\right)^2+1.8. \qquad (11)$$

[21] G. Moliere, Z. Naturforsch. **2a**, 133 (1947).

Numerical example: 340-Mev protons in Ilford G-5 emulsion.

	Silver	Bromine
Density in emulsion	1.85 g/cm³	1.36 g/cm³
n=number of collisions in path length of s microns	$2.60s$	$2.22s$
ω_2/ω_0	1324	1834
$M(s)$	$1.8+3.85\times 10^4/s$	$1.8+7.88\times 10^4/s$
M(2000 microns)	21.0	41.2

D. Mean Square Lateral Deflection with Cutoff

The mean square lateral multiple scattering deflection,

$$\langle x^2\rangle=\frac{1}{k}\sum_{i=1}^{k}x_i^2,$$

estimated from k observed deflections $x_1, x_2\cdots, x_k$, has a standard deviation equal to $\langle x^2\rangle[(M-1)/k]^{\frac{1}{2}}$. Thus if M is large the statistical fluctuations are also large. As has already been mentioned, this situation is remedied and M reduced by the use of a weighted average in which multiple scattering deflections larger in absolute value than some chosen limit (say, four times the root mean square deflection)[22] are given weight zero. We proceed to derive a theoretical expression for a mean square deflection with such a cutoff.

According to (8), the magnitudes of the angular single scattering deflections are essentially confined to the range (ω_0, ω_2); we note that $\omega_2/\omega_0=a/b\gg 1$. Let us choose an angle ω, such that $\omega_0\ll\omega_1\ll\omega_2$. Then the probability p_k that a particle traversing a distance s in a scattering medium will be deflected exactly k times through angles $\theta>\omega$, is given by the Poisson law:

$$p_k=\frac{1}{k!}\left[n\left(\frac{\omega_0}{\omega_1}\right)^2\right]^k\exp\left[-n\left(\frac{\omega_0}{\omega_1}\right)^2\right],$$

where n is the average number of all deflections. Let $f_k(x)$ be the distribution of lateral multiple scattering deflections corresponding to this contingency. Then the general lateral multiple scattering distribution may be written:

$$f(x)=\sum_{k=0}^{\infty}f_k(x)p_k.$$

If we furthermore require that $n(\omega_0/\omega_1)^2\ll 1$,

$$f(x)\cong[1-n(\omega_0/\omega_1)^2]f_0(x)+n(\omega_0/\omega_1)^2f_1(x). \qquad (12)$$

If the multiple scattering deflections are small, it is a good approximation to set

$$\int x^2f_1(x)dx=\int x^2f_0(x)dx+\int x^2h(x)dx, \qquad (13)$$

where $h(x)$ is the lateral single scattering distribution

[22] Under the conditions of the present experiment, on the order of 1 percent of the data were lost by such a cutoff. A smaller cutoff would have resulted in a slight further reduction of M, which would have been more than counterbalanced, however, by the increased loss of data.

corresponding to one Coulomb collision resulting in an angular deflection in the range (ω_1, ω_2). Thus, according to (12) and (13),

$$\langle x^2\rangle = \int x^2 f(x)dx = \int x^2 f_0(x)dx + n\left(\frac{\omega_0}{\omega_1}\right)^2 \int x^2 h(x)dx. \quad (14)$$

This separation of the mean square deflection into a true multiple scattering component $\langle x^2\rangle_0 = \int x^2 f_0(x)dx$ and a single scattering component $\langle x^2\rangle_1 = n(\omega_0/\omega_1)^2 \times \int x^2 h(x)dx$ has the advantage that a properly chosen cutoff may be applied, which—as we shall see—will affect the single scattering component $\langle x^2\rangle_1$ only.

The moments of $f_0(x)$ can be calculated by means of the general multiple scattering theory. The only differences compared to the previous calculation leading to formulas (10) and (11) is the use of a sharp cutoff ω_1 for the individual angular deflections instead of the more gradual cutoff at ω_2. With the use of the single scattering law

$$I(\theta) = \frac{2\pi Z'^2 Z^2 e^4}{p^2 v^2} \frac{1}{(\sin^2\frac{1}{2}\theta + \frac{1}{4}\omega_0^2)^2} \quad \text{if} \quad 0 \leq \theta \leq \omega_1, \quad (15)$$

$$I(\theta) = 0 \quad \text{if} \quad \theta > \omega_1,$$

one finds

$$\langle x^2 | s\rangle_0 = \tfrac{1}{3} n\omega_0^2 s^2 [\log(\omega_1/\omega_0) - \tfrac{1}{2}] \quad (16)$$

and

$$M_0 = \langle x^4 | s\rangle_0 / \langle x^2 | s\rangle_0^2 = \frac{2.7}{n[\log(\omega_1/\omega_2) - \frac{1}{2}]^2}\left(\frac{\omega_1}{\omega_0}\right)^2 + 1.8. \quad (17)$$

The numerical value of M_0 is of interest. Let us assume, for the sake of concreteness, that ω_1 is chosen so that $n\omega_0/\omega_1 = 1/20$. This means that the probability of the occurrence of two or more collisions in the path length s, resulting in deflections larger than ω_1, is equal to $1-\exp(-1/20)-(1/20)\times\exp(-1/20) = 0.12$ percent. Corresponding to $n = 10^2$, 10^3, or 10^4 collisions one finds that $M_0 = 3.04$, 2.48, and 2.23, respectively. Now for a normal distribution ($M = 3$) a cutoff at four times the standard deviation will reduce the second moment by only 0.11 percent. But $f_0(x)$ has a dispersion (as indicated by M_0) less than a normal distribution for $n > 100$; hence $\langle x^2\rangle_0$ will practically not be affected by the introduction of such a cutoff.

It remains to determine $h(x)$. Employing now the small angle approximation, we consider a particle traveling in the direction of the z axis, which while traversing a layer of matter of thickness s makes one collision with a resulting angular deflection in the range (ω_1, ω_2) at a random point in the layer, after it has traveled a distance $s-t$. The lateral deflection resulting from such a collision is $x = \theta t\cos\varphi$. The joint probability density of ϑ, φ, and t is, approximately,

$$g(\theta, \varphi, t)d\theta d\varphi dt = \frac{\omega_1^2}{\pi\{1-(\omega_1/\omega_2)^2\}} \frac{d\theta}{\theta^3} d\varphi dt, \quad (18)$$

$$\omega_2 \geq \theta \geq \omega_1, \quad s \geq t \geq 0, \quad \pi \geq \varphi \geq -\pi.$$

By transforming (18) into a joint probability density for x, φ, and t, and integrating over t and φ, with due regard for the inequalities for the variables in (18), one finds

$$h(x)dx = dx \int dt \int d\varphi g\left(\frac{x}{t\cos\varphi}, \varphi, t\right)\frac{\partial(\theta, \varphi, t)}{\partial(x, \varphi, t)}$$

$$= \frac{1}{3\pi\{1-(\omega_1/\omega_2)^2\}} \frac{dx}{\omega_1 s}\left\{H\left(\frac{x}{\omega_1 s}\right) - \left(\frac{\omega_1}{\omega_2}\right)^3 H\left(\frac{x}{\omega_2 s}\right)\right\} \quad \text{if} \quad |x| \leq \omega_1 s,$$

$$= \frac{1}{3\pi\{1-(\omega_1/\omega_2)^2\}} \frac{dx}{\omega_1 s}\left\{\frac{\pi}{2}\left(\frac{\omega_1 s}{x}\right)^3 - \left(\frac{\omega_1}{\omega_2}\right)^3 H\left(\frac{x}{\omega_2 s}\right)\right\} \quad \text{if} \quad \omega_1 s \leq |x| \leq \omega_2 s, \quad (19)$$

$$= 0 \quad \text{if} \quad |x| \geq \omega_2 s,$$

where

$$H(x) = \log\left[\frac{1+(1-x^2)^{\frac{1}{2}}}{1-(1-x^2)^{\frac{1}{2}}}\right] + \tfrac{1}{3}[\sin^{-1}x - x(1-x^2)^{\frac{1}{2}}].$$

According to (19),

$$\int_{-u}^{u} x^2 h(x)dx = \frac{\omega_1^2 s^2}{6}\left[\log\left(\frac{2u}{s\omega_1}\right)^2 - \frac{1}{3}\right], \quad (20)$$

provided that $u \geq \omega_1 s$. The final result, obtained by combining (14), (13), and (20), is

$$\langle x^2 | s, u\rangle = \int_{-u}^{u} x^2 f(x)dx = \frac{1}{3}\omega_0^2 s^2 n\left(\log\frac{2u}{s\omega_0} - \frac{2}{3}\right). \quad (21)$$

In accordance with experimental practice, u is then chosen so as to satisfy the implicit relation $u = u_0 = 4\langle x^2 | s, u_0\rangle^{\frac{1}{2}}$.

It is a satisfactory feature that the angle ω_1 which was introduced as an artifice to facilitate the calculation but which has no clear-cut physical significance, has disappeared from the final result.

Formula (21) has been compared with the predictions of other theories. Calculations based on numerical values given by Moliere, and by Snyder and Scott for the angular scattering distribution, and on the relation $3\langle x^2\rangle = s^2\langle\phi^2\rangle$ between lateral and angular mean square deflections, gave results for $\langle x^2 | s_1 u_0\rangle$ that agreed to within the limits of error of the computation (1–2 percent) with Eq. (21). This equivalence was found to hold

under a wide range of conditions (n=av. no. of collisions ranging from 100 to 84000). Expression (21) thus provides a convenient substitute for numerical calculations. Moreover, the same approach may readily be adapted for use in conjunction with some other assumed form of the underlying single scattering law.

A calculation of $\langle x^4|s_1u_0\rangle$ (with $u_0=4\langle x^2|s_1u_0\rangle^{\frac{1}{2}}$), either by a method similar to that used above for $\langle x^2|s_1u_0\rangle$, or with the use of the numerical tables given by Snyder and Scott, indicates that $M(u_0)=\langle x^4|s_1u_0\rangle/\langle x^2|s_1u_0\rangle^2\sim 3.7$, so that the cutoff will considerably reduce the statistical fluctuations.

For scattering by a mixture of substances, as in photographic emulsions, we shall follow Snyder and Scott in using an effective average number of collisions n_{eff}, and an effective minimum angle $\omega_{0\text{eff}}$, defined by

$$n_{\text{eff}}=\sum_i n_i, \tag{22}$$

$$n_{\text{eff}}\omega_{0\text{eff}}{}^2=\sum_i n_i\omega_{0i}{}^2, \tag{23}$$

where n_i and ω_{0i} are the parameters appropriate for the i'th substance. A cutoff may then be applied after the substitution of n_{eff} and $\omega_{0\text{eff}}$ in (21).

IV. EXPERIMENTAL PROCEDURES

A. Source of Data

The proton tracks to be analyzed were obtained by the exposure of Ilford G-5 plates that were dropped through the proton beam of the Berkeley 184-in. cyclotron, which has an energy of 340.0±0.8 Mev.[23] In some of the exposures, the protons were first slowed down by a block of aluminum with a thickness of 42.210 g/cm^3 before entering the plates, which reduced their energies to 222±2 Mev.[24] The direction of entry of the protons into the plates was almost parallel to the emulsion surface. Most of the tracks were longer than 8000 microns. In order to avoid the effects of possible emulsion distortion in the edge zones of the plates, the multiple scattering was determined only in a portion of the tracks, averaging 5000 microns in length, that followed the first 1500 microns of track as measured from the point of entry of the protons into the emulsion, whereby sections of track lying closer than 50 microns to the surface of the emulsion were practically also excluded. Otherwise the selection of tracks was random. The direction of motion of the protons in the emulsion could easily be determined from the occasional stars which they produced. Energy loss in the emulsion was calculated to reduce the mean proton energy in the measured portion of each track to 337 Mev without, and 218 Mev with the intermediate aluminum layer. Nine different plates were used, with thicknesses of 200, 400, and 600 microns and surface areas of 2 in.×4 in. and 3 in.×4 in.

[23] R. L. Mather, Phys. Rev. **83**, 895 (1951).

[24] This follows from Bethe's theoretical expression for energy loss by ionization as evaluated numerically by J. H. Smith [Phys. Rev. **71**, 32 (1947)]. The increase in the energy spread is due to straggling.

B. Apparatus and Measurements

The measurements were made with a Zeiss "Lumipan" microscope using a 90-power oil immersion lens. The total magnification was 1350, and the field of view had a diameter of 140 microns. The ocular of the microscope was provided with a filar micrometer, by means of which distances of the order of 0.001 micron could be measured. The regular microscope stage was mounted on a specially built stage[25] that could be moved rectilinearly in two directions at right angles to each other by means of two precision screws. By means of measurements with an interferometer it was found that the deviation from rectilinearity of the motion of the stage was less than 0.03 micron when the stage was moved over a distance of one centimeter. The microscope was mounted on a solid bench anchored to the ground in order to eliminate vibrations, and was located in a well-insulated cork-lined basement room in which the temperature was found to vary not more than 2° over periods of several days.

The procedure used for measuring the track deflections was the following. The track under consideration was lined up so that it was very nearly parallel to one of the directions of motion of the microscope stage. The thread of the filar micrometer of the eyepiece, also approximately parallel to the track, was lined up to lie across the center of a grain of the track. The stage was then moved through a distance equal to the cell length, which resulted in a displacement of the thread from the track. The distance through which the thread had to be shifted in order to be again centered on the track gave the lateral displacement or change in the y coordinate of the track with respect to the reference line (the direction of motion of the stage) (see Fig. 1). Since the ionization density of the tracks was low, it was sometimes necessary to center the thread not on a grain but on the estimated position of the track between two adjacent grains. The basic cell length was chosen to be 250 microns, while the analysis of the measurements was carried out for multiples of this unit: 250, 500, 750, and 1000 microns.

The experimentally measured scattering includes both the true multiple scattering and spurious scattering. This spurious scattering may in turn be considered to consist of two components of different origin. One of them, usually designated by the term noise level, is approximately random, and results from errors in the experimental determination of the trajectory of the particle in the photographic plate. The other component is more or less systematic and results from emulsion distortion.

The noise level, expressed in terms of a spurious average second coordinate difference, is found to be approximately independent of the cell length. It can be determined, therefore, by repeated measurements of the same track, using different cell lengths, provided

[25] Used in a previous investigation by Dr. J. Lord.

that the dependence of the true multiple scattering on the cell length is quite accurately known. Since the determination of this dependence was one of the objects of this experiment, this approach was not used. Instead, reliance was placed on measurements carried out on the tracks of particles with energies so high that any apparent scattering was certain to be spurious.[26] These measurements had to be made on plates other than those containing the proton tracks, but the experimental evidence indicates that the noise level is essentially determined by the measuring apparatus and is not affected by the idiosyncracies of individual emulsions. Numerous noise level calibrations were made periodically throughout the course of the experiment in order to make sure that the apparatus was in good working order.

The spurious second coordinate difference due to noise was found to have a root mean square value $E=0.182$ micron, and an absolute mean value $E'=0.150$ micron. The ratio $E'/E=0.824$ is compatible with the assumption that the noise level deflections have a Gaussian distribution. Variations of the cell length from 125 to 1000 microns did not result in a significant change of the noise level. The correction applied to the proton data consisted of subtracting E^2 from the observed $\langle(\Delta^2 y)^2\rangle$, and E'^2 from the observed $\langle|\Delta^2 y|\rangle^2$ and $\langle|\Delta^2 y|\rangle_c^2$.

In order to detect the possible presence of emulsion distortion, the distribution of the algebraic signs of the observed second coordinate differences was examined. Under the hypothesis that there is no emulsion distortion, positive and negative $\Delta^2 y_t$'s are equally likely; moreover, alternate $\Delta^2 y_t$'s are independent of each other; and the probability that an actually observed distribution of algebraic signs may have arisen under this hypothesis due to chance fluctuations may be calculated with the aid of formulas for the probability distribution of the number of "runs" of alternate $\Delta^2 y_t$'s.[27]

[26] The tracks used for this purpose included those of mesons with energies estimated to be 250 Bev or more that were produced in a nucleon-nucleon collision [Lord, Fainberg, and Schein, Phys. Rev. **80**, 970 (1950)], and the track of an oxygen nucleus with an estimated energy of more than 15 Bev, found in the cosmic radiation by M. Schein and J. Lord (private communication).

[27] In the sequence of second coordinate differences $\Delta^2 y_t$ $(t=1,2,\cdots)$ obtained from a track, successive $\Delta^2 y_t$'s are correlated:

$$\langle\Delta^2 y_t\cdot\Delta^2 y_{t+1}\rangle=\langle(x_{t+1}-x_t)(x_{t+2}-x_{t+1})\rangle=-\langle x_{t+1}^2\rangle$$

But in subsequence of alternate $\Delta^2 y_t$'s (e.g., $t=2T$, $T=1,2\cdots$) all elements are uncorrelated and independent; moreover, each is equally likely to be positive or negative. Let $P(r,k|R)$ denote the probability that in an observed ordered sequence of R alternate $\Delta^2 y_t$'s there are exactly $\Delta^2 y_t$'s with a positive sign, and that the sequence contains exactly k runs. The number of runs is equal to the number of blocks containing $\Delta^2 y_t$'s of like sign, into which the sequence can be divided; e.g., in the sequence $-++---+$, there are four runs: $-$, $++$, $---$, $+$. It can be shown [see A. M. Mood, Ann. Math. Stat. **11**, 367 (1940)] that

$$\text{if } k=2m \quad P(r,k|R)=2^{1-R}\binom{R-r-1}{m-1}\binom{r-1}{m-1};$$

$$\text{if } k=2m+1 \quad P(r,k|R)$$

$$=2^{-R}\left\{\binom{R-r-1}{m}\binom{r-1}{m-1}+\binom{R-r-1}{m-1}\binom{r-1}{m}\right\}.$$

Given R, one can compute with the aid of these formulas the probability that k and r will lie in a specified range.

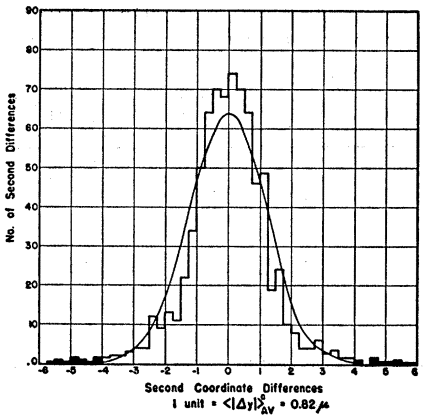

FIG. 2. Distribution of 795 uncorrelated second coordinate differences obtained from the tracks of 337-Mev protons divided into cells with a length of 500 microns.

Such a statistical analysis was applied individually to the results obtained with each of the nine plates used in this experiment. For eight of these plates the hypothesis of no distortion had a probability of 75 percent or better. For one plate (2 in.×4 in., 600 microns thick) this probability was as low as 15 percent so that, because of the suspicion of emulsion distortion, the results from this plate were not used for the computation of the scattering factors.

V. RESULTS AND CONCLUSIONS

A. Distribution of Lateral Multiple Scattering Deflections

Each second coordinate difference $\Delta^2 y_t$ is the difference of two independent lateral multiple scattering deflections. The histogram in Fig. 2 shows the distribution of 795 $\Delta^2 y_t$'s obtained from measurements on the tracks of 337-Mev protons divided into 500-micron cells. Only alternate $\Delta^2 y_t$'s were used in plotting the histogram in order to avoid correlation effects, successive $\Delta^2 y_t$'s being correlated since they have one lateral deflection in common. The scale of the abscissa is in units of $\langle|\Delta^2 y|\rangle_c$; the shaded areas to the left of -4 and to the right of $+4$ on the abscissa represent the 1.4 percent of the data eliminated when the cutoff was introduced. The curve in Fig. 2 represents a Gaussian distribution with an absolute mean value equal to $\langle|\Delta^2 y|\rangle_c$. It is seen that the experimental histogram exceeds this curve in the center and at the tails, while it is lower in the intermediary region. This is consistent with the theoretically predicted non-Gaussian shape of the distribution curve of the lateral deflections.

B. Scattering Factors

The results of the measurements are summarized in Table I, which gives, for cell lengths $s=250$, 500, 750, and 1000 microns, the number of cells included in the analysis, and the multiple scattering in Ilford G-5 emulsions, as expressed in terms of the average second

TABLE I. Multiple scattering of protons in Ilford G-5 photographic plates.

Energy T (Mev)	Cell length s (microns)	No. of cells k	Mean deflections (no cutoff) $\langle\|\Delta^2y\|\rangle$ (microns)	K_1	Mean deflections (with cutoff) $\langle\|\Delta^2y\|\rangle_c$ (microns)	K_1^c	rms deflections (with cutoff) $\langle(\Delta^2y)^2\rangle_c^{\frac{1}{2}}$ (microns)	K_2^c
337	250	3620	0.310	26.2±0.9	0.289	24.4±0.7	0.383	32.4±0.9
	500	1682	0.878	26.3±1.0	0.820	24.5±0.8	1.101	32.9±1.1
	750	1140	1.636	26.6±1.1	1.512	24.6±0.9	2.035	33.1±1.2
	1000	815	2.602	27.5±1.2	2.395	25.3±1.0	3.189	33.7±1.3
218	250	1607	0.464	26.5±1.0	0.437	25.0±0.8	0.579	33.1±1.2
	500	756	1.347	27.2±1.3	1.257	25.4±1.0	1.683	34.0±1.4
	750	496	2.509	27.6±1.5	2.337	25.7±1.2	3.127	34.4±1.6
	1000	360	3.933	28.1±1.7	3.681	26.3±1.4	4.913	35.1±1.9

coordinate differences and the corresponding scattering factors. The last two of these quantities are listed for the following three cases (see Sec. II): (a) average of absolute values without cutoff ($\langle|\Delta^2y|\rangle$ and K_1); (b) average of absolute values with cutoff at four times the experimental mean value ($\langle\Delta^2y|\rangle_c$ and K_1^c); (c) root mean square average with cutoff at four times the experimental root mean square value ($\langle(\Delta^2y)^2\rangle_c^{\frac{1}{2}}$ and K_2^c). The values given are those obtained after correcting for spurious scattering (noise level) according to the procedure described in Sec. IV B, by subtracting E'^2 from the observed $\langle|\Delta^2y|\rangle^2$ and $\langle|\Delta^2y|\rangle_c^2$, and E^2 from the observed $\langle(\Delta^2y)^2\rangle_c$. ($E'=0.150$ micron, $E=0.182$ micron.) The indicated standard errors of the scattering factors were calculated on the assumption that the total error has the following three independent components whose squares are to be added: (1) the statistical error, arising from the circumstance that the scattering factors are determined from a limited number of cells;[28] (2) an error of 2 percent resulting from the expected variation of the composition of the emulsion from the nominal composition (density variations; humidity effects); (3) the error resulting from the uncertainty of the proton energy, which was assumed to be 1 percent for the 337-Mev protons, and 2 percent for the 218-Mev protons, considering the spread in the proton beam energy as well as energy loss in the emulsions.

In Fig. 3, the experimental values of the scattering factors K_1 and K_1^c are compared with the theoretical values predicted both by the theory of Moliere,[17] and the equivalent theory of Snyder and Scott (as modified by Scott).[19] It can be seen that the experimental values are in fair agreement with, though somewhat lower than, those predicted theoretically, the discrepancies being just outside the limits of error (3–4 percent) for 337-Mev protons, and within the limits of error (3–6 percent) for the 218-Mev protons. The rise of the scattering factors with increasing cell length is less than predicted. The simple version of the theory of Williams (as presented by Rossi and Greisen) would give $K_1^c=32.7$ (regardless of the cell length and proton energy), which is 24 percent to 34 percent higher than the experimental results. According to calculations by Goldschmidt,[3] and Voyvodic and Pickup,[7] the more exact form of the theory of Williams is in close agreement with that of Moliere, and therefore is also consistent with the results of this experiment.

In Fig. 4, the experimental values of the scattering factor K_2^c are compared with the corresponding theoretical values. The latter were obtained by first computing $\langle x^2|s,u_0\rangle$ with the use of (21), multiplying this result by a factor of two to obtain $\langle(\Delta^2y)^2\rangle_c$, from which K_2^c is then found according to (3).[29] Again the experimental values are seen to lie somewhat below the theoretical curve, the discrepancy being within or just outside the probable experimental error.

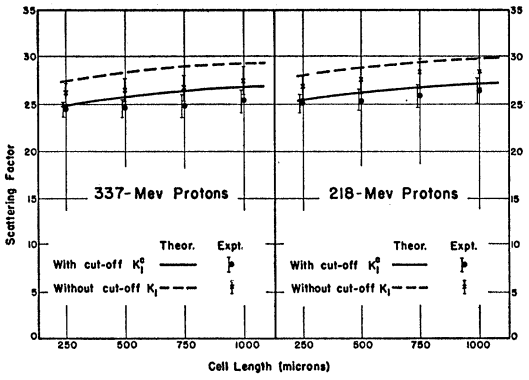

FIG. 3. Scattering factors K_1 and K_1^c for Ilford G-5 emulsions. Comparison of the experimental results with the theory of Moliere and the equivalent theory of Snyder and Scott.

[28] The following expression can be shown to hold for the statistical error:

$$\delta K_2^c/K_2^c=\tfrac{1}{2}[(k-\tfrac{3}{2})(M-3)+(3k-4)(1+\mu^4/s^2)]^{\frac{1}{2}}(k-1)^{-1},$$

where δK_2^c = standard deviation of K_2^c; k = number of cells; s = cell length (microns); $M=3.7$; $\mu=10E\ pv/K_2^cZs$; E = noise level (microns). The ratios $\delta K_1^c/K_1^c$ and $\delta K_1/K_1$, being based on averages of absolute values, cannot be easily calculated, but may be assumed to be of the same order of magnitude as $\delta K_2^c/K_2^c$.

[29] This step in the calculation involves a slight approximation. For the cutoff is applied to x_i theoretically, and to Δx_i experimentally. But it may be shown that the following relation holds for the distribution functions of Δx_i and x_i

$$f(\Delta x_i/A)d(\Delta x_i/A)\sim f(x_i/A')d(x_i/A'),$$

for $|\Delta x_i|\geq 4A$ and $|x_i|\geq 4A'$, where A and A' are the truncated root mean square values of Δx_i and x_i, respectively. Hence $A^2=2A'^2$. It was estimated that under the conditions of the present experiment, this approximation will introduce an error less than 1 percent into the calculated value of K_2^c.

TABLE II. Scattering factors: results of various calibrations.

Particle	Cell length (microns)	K_1^c	K_1	Reference
Electrons				
Pairs from 14.8- and 17.6-Mev Be^8 gamma-rays	15–70	21.3±1.0		Voyvodic and Pickup[a]
Electrons and positrons				
40 to 283 Mev (weighted average)		25.1±0.6		Corson[b]
Positrons				
105 Mev	200	26.2±0.6	26.7±0.6	Gottstein *et al.*[c]
185 Mev	400	24.0±0.8	24.9±0.8	Gottstein *et al.*[c]
Protons				
336 Mev	600	29.2±1.0	30.7±1.0	Gottstein *et al.*[c]
337 Mev	500	24.5±0.7	26.2±0.9	This work
218 Mev	500	25.4±1.0	27.2±1.3	This work

[a] See reference 7.
[b] See reference 6.
[c] See reference 8.

It should be mentioned that the results obtained for different plates were consistent; no significant variations of the value of the scattering constants were found beyond those to be expected on account of statistical fluctuations.

In Table II a listing is given of the scattering factors for Ilford G-5 emulsions found by various investigators under conditions such that the particle energy was controlled and approximately constant. The values obtained in this experiment which are quoted for comparison are those for a cell length of 500 microns; these are regarded as the most reliable values because their statistics are good while the noise level corrections were quite small (1.4 percent for the 337-Mev protons, and 0.7 percent for the 218-Mev protons). As previous results for photographic emulsions have been given in terms of the mean rather than the root mean square deflections, the comparison is confined to the factors K_1^c and K_1.

C. Accuracy of Energy Determinations by the Scattering Method

The measurement of a large number of tracks of particles with identical properties provided an opportunity of making an experimental check of the accuracy with which energies can be determined by the multiple scattering technique. For this purpose a group of 128 tracks of 337-Mev protons, all in one plate with dimensions 3 in.×4 in.×400 microns, were selected. Each track had a measured portion 5000 microns long and was divided into twenty cells with a length of 250 microns. With the use of the experimentally determined scattering factor K_2^c an apparent energy was computed for each track from the observed root mean square second coordinate difference (with cutoff). The resultant energy spectrum is shown in Fig. 5. The peak of the spectrum is at approximately 330 Mev, and 50 percent of the tracks underwent deflections indicating apparent energies in the range from 285 Mev to 395 Mev. The slight asymmetry of the histogram is not due to an asymmetry of the scattering deflections, but arises from the transformation to a kinetic energy scale. The histogram in Fig. 5 may also be interpreted to represent the inherent uncertainty of an energy determination based on the knowledge of the true scattering factor and measurements made on a single track 5000 microns long and divided into 20 cells, the probable error in this case being approximately 20 percent.

ACKNOWLEDGMENTS

The author takes pleasure in acknowledging the encouragement and valuable advice received from Professor Marcel Schein. He is also indebted to Professors E. O. Lawrence and W. Barkas for arranging for the exposure of the plates to the beam of the Berkeley cyclotron, to Dr. J. J. Lord for help and advice in matters of photographic plate technique, and to Mr. Roy Berg and Mr. Lewis Leder for their assistance with the measurements.

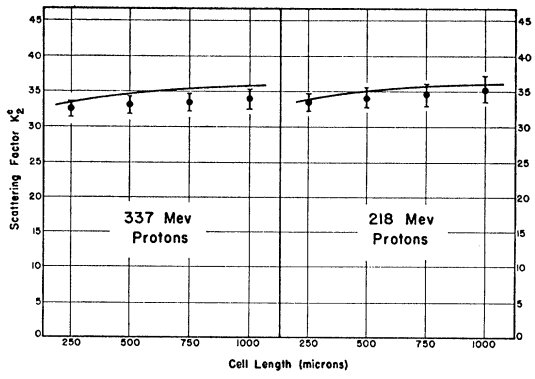

FIG. 4. Scattering factor K_2^c for Ilford G-5 emulsions. Comparison of the experimental results with the theoretical value calculated from Eq. (21).

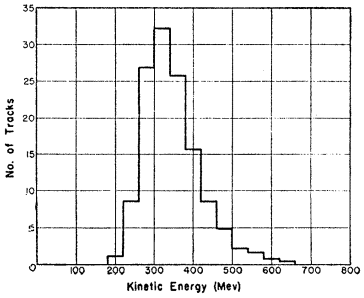

FIG. 5. Apparent energy spectrum for 128 protons with a true energy of 337 Mev, obtained by the multiple scattering method.

APPENDIX: LATERAL MULTIPLE SCATTERING DEFLECTIONS

A. General Theory

Let us consider a particle traveling in an infinite medium in which scattering centers are distributed at random. We describe the state of the particle by the probability density $F(\theta,\varphi,x,y,z|s)$ which is conditional on the path length s traversed by the particle; s is measured from a reference point on the trajectory which is also taken to be the origin of a Cartesian coordinate system (x,y,z) specifying the position, and a spherical coordinate system (θ,φ) specifying the direction of motion of the particle. If the particle is initially, at $s=0$, moving in the z direction, the x coordinate may be used to represent the lateral deflection as projected on the $x-z$ plane (a plane parallel to the surface of the emulsion as discussed in Sec. II). The dependence of F on y and z is then of no interest to us, and we shall ignore these coordinates from now on. The stochastic equation governing the probability density F is

$$\begin{aligned}F(\theta,\varphi,x|s+\Delta s)&=[1-NQ(s)\Delta s]F(\theta,\varphi,x-\sin\theta\cos\varphi\Delta s|s)\\&+NQ(s)\Delta s\int_0^{2\pi}d\varphi'\int_0^{\pi}d\theta' F(\theta',\varphi',x\\&-\sin\theta'\cos\varphi'\Delta s|s)\cdot\psi(\theta',\varphi';\theta,\varphi|s).\end{aligned}\quad(A1)$$

The initial condition is assumed to be

$$F(\theta,\varphi,x|0)=(2\pi)^{-1}\delta(x)\delta(\theta)\quad(A2)$$

where $\delta(x)$ and $\delta(\theta)$ are delta-functions. The first term on the right side of Eq. (A1) takes into account the case in which the particle makes no collision in the interval $(s,s+\Delta s)$, while the second term refers to the case that the particle makes one collision by which it is deflected into the direction (θ,φ). The probability of more than one collision in the interval $(s,s+\Delta s)$ is assumed to be negligibly small. N is the number of scattering centers per unit volume of the medium, $Q(s)$ is the total cross section for the underlying single scattering process, and $\psi(\theta',\varphi';\theta,\varphi|s)$ is the transition probability density for a change of direction from (θ',φ') to (θ,φ) in a single collision. ψ is assumed to obey the following conditions:

$$\psi(\theta',\varphi';\theta,\varphi|s)=\psi(\theta,\varphi;\theta',\varphi'|s)\quad\text{(symmetry)},\quad(A3)$$

$$\int_0^{2\pi}d\varphi\int_0^{\pi}d\theta\psi(\theta',\varphi':\theta,\varphi|s)=1\ \text{(normalization)}.\quad(A4)$$

The dependence of Q and ψ on s may be used to express their energy dependence, if the energy for some value of s and the energy loss as a continuous function of s are known.

From the difference equation (A1) one obtains, by going to the limit $\Delta s\to 0$, the integro-differential equation

$$\begin{aligned}\frac{\partial F(\theta,\varphi,x|s)}{\partial s}&=NQ(s)\int_0^{2\pi}d\varphi'\int_0^{\pi}d\theta'[F(\theta',\varphi',x|s)\\&\times\psi(\theta',\varphi';\theta,\varphi|s)]-NQ(s)F(\theta,\varphi,x|s)\\&-\sin\theta\cos\varphi\frac{\partial F(\theta,\varphi,x|s)}{\partial x}.\end{aligned}\quad(A5)$$

The kernel ψ, positive and symmetric in (θ,φ) and (θ',φ'), can be expanded in terms of its eigenfunctions:

$$\psi(\theta',\varphi';\theta,\varphi|s)=\sum_{n=0}^{\infty}c_n(s)\phi_n(\theta',\varphi')\phi_n(\theta,\varphi).\quad(A6)$$

The eigenfunctions ϕ_n are solutions of the integral equation

$$\begin{aligned}\phi_n(\theta,\varphi)=\frac{1}{c_n(s)}\int_0^{2\pi}d\varphi'\int_0^{\pi}d\theta'\phi_n(\theta',\varphi')\\\times\psi(\theta',\varphi';\theta,\varphi|s).\end{aligned}\quad(A7)$$

They form an orthogonal system, and will be assumed to be normalized. All the eigenvalues are real and positive. One eigenfunction, $\phi_0=1/(4\pi)^{\frac{1}{2}}$, can be found by inspection with the help of the normalization equation (A4) of the text. The corresponding eigenvalue $c_0=1$ is the largest eigenvalue since it belongs to the eigenfunction with no node.

Multiplying Eq. (A5) by x^k, substituting the expansion (A6), and integrating with respect to θ,φ and x, one finds

$$\begin{aligned}\int_0^{2\pi}d\varphi\int_0^{\pi}d\theta\int_{-s}^{s}dx\left[\frac{\partial}{\partial s}+NQ(1-c_n)\right]x^k\phi_n F\\=-\int_0^{2\pi}d\varphi\int_0^{\pi}d\theta\int_{-s}^{s}dx\,x^k\sin\theta\cos\varphi\phi_n\frac{\partial F}{\partial x}\\(n=0,1,2,\cdots).\end{aligned}\quad(A8)$$

The initial condition (A2) implies that

$$F(\theta,\varphi,\pm s\,|\,s)=0, \tag{A9}$$

so that

$$\frac{\partial}{\partial s}\int_{-s}^{s} x^k F dx=\int_{-s}^{s} x^k \frac{\partial F}{\partial S}dx$$

and

$$\int_{-s}^{s} x^k \frac{\partial F}{\partial x}dx=-k\int_{-s}^{s} x^{k-1}F dx.$$

Hence

$$\left[\frac{\partial}{\partial s}+NQ(1-c_n)\right]\int_0^{2\pi}d\varphi\int_0^{\pi}d\theta\int_{-s}^{s}x^k\phi_n F dx$$

$$=k\int_0^{2\pi}d\varphi\int_0^{\pi}d\theta\int_{-s}^{s}x^{k-1}\sin\theta\cos\varphi\phi_n F. \tag{A10}$$

It is convenient to use the notation

$$\langle f\,|\,s\rangle=\int_0^{2\pi}d\varphi\int_0^{\pi}d\theta\int_{-s}^{s}dx f(\theta,\varphi,x,s)F(\theta,\varphi,x\,|\,s),$$

$$R_n(s)=\exp\left\{-N\int_0^{s}Q(s')[1-c_n(s')]ds'\right\},$$

in terms of which the solution of (A10) may be written

$$\langle x^k\phi_n\,|\,s\rangle=kR_n(s)\int_0^{s}\frac{ds_1}{R_n(s_1)}\langle x^{k-1}\sin\theta\cos\varphi\phi_n\,|\,s_1\rangle. \tag{A11}$$

If the expansion

$$\sin\theta\cos\varphi\phi_n=\sum_{j_1}A_{j_1}{}^n\phi_{j_1} \tag{A12}$$

is substituted in (A11), the result is

$$\langle x^k\phi_n\,|\,s\rangle=kR_n(s)\sum_{j_1}A_{j_1}{}^n\int_0^{s}\frac{ds_1}{R_n(s_1)}\langle x^{k-1}\phi_{j_1}\,|\,s_1\rangle. \tag{A13}$$

Repeated application of this recursion relation yields

$$\langle x^k\phi_n\,|\,s\rangle=k!R_n(s)$$

$$\times\sum_{j_1 j_2\cdots j_k}\left\{A_{j_1}{}^nA_{j_2}{}^{j_1}\cdots A_{j_k}{}^{j_{k-1}}\cdot\int_0^{s}ds_1\frac{R_{j_1}(s_1)}{R_n(s_1)}\right.$$

$$\left.\times\int_0^{s_1}ds_2\frac{R_{j_2}(s_2)}{R_{j_1}(s_2)}\cdots\int_0^{s_{k-1}}ds_k\frac{\langle\phi_{j_k}\,|\,s_k\rangle}{R_{j_k}(s_k)}\right\}. \tag{A14}$$

The remaining problem is to determine $\langle\phi_{j_k}\,|\,s\rangle$. Let

$$G(\theta,\varphi\,|\,s)=\int_{-s}^{s}dx F(\theta,\varphi,x\,|\,s).$$

It follows from (A5) and (A9), that

$$\frac{\partial G(\theta,\varphi\,|\,s)}{\partial s}=NQ(s)\int_0^{2\pi}d\varphi'\int_0^{\pi}d\theta'[G(\theta',\varphi'\,|\,s)$$

$$\times\psi(\theta',\varphi';\theta,\varphi\,|\,s)]-NQ(s)G(\theta,\varphi\,|\,s). \tag{A15}$$

Substituting in (A15) the expansion

$$G(\theta,\varphi\,|\,s)=\sum_{n=0}^{\infty}A_n(s)\phi_n(\theta,\varphi), \tag{A16}$$

and the eigenfunction expansion (A6) for the kernel ψ, one obtains the differential equations

$$[d/ds+NQ(1-c_n)]a_n=0. \tag{A17}$$

Solving for the coefficients a_n and substituting the solutions in (A16), one obtains

$$G(\theta,\varphi\,|\,s)=\sum_{n=0}^{\infty}\langle\phi_n\,|\,0\rangle\phi_n(\theta,\varphi)R_n(s), \tag{A18}$$

where

$$\langle\phi_n\,|\,0\rangle=\int_0^{2\pi}d\varphi\int_0^{\pi}d\theta G(\theta,\varphi\,|\,0)\phi_n(\theta,\varphi). \tag{A19}$$

Thus, since the ϕ_n are orthogonal,

$$\langle\phi_{j_k}\,|\,s\rangle=\langle\phi_{j_k}\,|\,0\rangle R_n(s), \tag{A20}$$

so that finally

$$\langle x^k\phi_n\,|\,s\rangle=k!R_n(s)\sum_{j_1 j_2\cdots j_k}A_{j_1}{}^nA_{j_2}{}^{j_1}\cdots A_{j_k}{}^{j_{k-1}}$$

$$\times\langle\phi_{j_k}\,|\,0\rangle T_{j_1 j_2\ldots j_k}{}^n, \tag{A21}$$

where

$$T_{j_1 j_2\ldots j_k}{}^n=\int_0^{s}ds_1\frac{R_{j_1}(s_1)}{R_n(s_1)}\int_0^{s_1}ds_2\frac{R_{j_2}(s_2)}{R_{j_1}(s_1)}\cdots$$

$$\times\int_0^{s_{k-1}}ds_k\frac{R_{j_k}(s_k)}{R_{j_{k-1}}(s_k)}.$$

This expression gives not only the moments of x (for $n=0$), but also the product moments of x and the eigenfunctions $\phi_n(\theta,\varphi)$. It should be noted that (A18) is a generalization of the angular distribution function of Goudsmit and Saunderson [Eq. (6) of the text], to which it reduces if the underlying single scattering law

is axially symmetric so that $\psi=\psi(\cos\chi)$, where $\cos\chi=\cos\theta\cos\theta'-\sin\theta\sin\theta'\cos(\varphi-\varphi')$. In this case spherical harmonics

$$Y_{nm}=\frac{1}{2\pi}\left\{\frac{2n+1}{2}\frac{(n-m)!}{(n+m)!}\right\}^{\frac{1}{2}}P_n(\cos\theta)e^{im\varphi},$$

are the normalized eigenfunctions satisfying the integral Eq. (A5). Since these eigenfunctions are characterized by two indexes, m and n, the corresponding matrix elements in Eq. (A21) will have four indexes:

$$\left.\begin{aligned}
&A_{j_n}{}^{i_n}=\int_0^{2\pi}d\varphi\int_0^{\pi}d\theta\phi_{i_n}\sin\theta\cos\varphi\phi_{j_{n'}}\\
&\rightarrow A_{n'm'}{}^{n,m}=\int_0^{2\pi}d\varphi\int_0^{\pi}d\theta Y_{nm}\sin\theta\cos\varphi Y_{n'm'}{}^{*}\\
&A_{n+1,m+1}{}^{n,m}=\frac{1}{2}\left\{\frac{(n+m+1)(n+m+2)}{(2n+1)(2n+3)}\right\}^{\frac{1}{2}},\\
&A_{n-1,m+1}{}^{n,m}=-\frac{1}{2}\left\{\frac{(n-m)(n-m+1)}{(2n+1)(2n-1)}\right\}^{\frac{1}{2}}\\
&A_{n+1,m-1}{}^{n,m}=-\frac{1}{2}\left\{\frac{(n-m+1)(n-m+2)}{(2n+1)(2n+3)}\right\}^{\frac{1}{2}},\\
&A_{n-1,m-1}{}^{n,m}=\frac{1}{2}\left\{\frac{(n+m)(n+m-1)}{(2n+1)(2n-1)}\right\}^{\frac{1}{2}}\\
&A_{n',m'}{}^{nm}=0\quad\text{if}\quad\begin{cases}m\neq m'\pm1\\ n\neq n'\pm1\end{cases}.
\end{aligned}\right\}\quad(A22)$$

The moments $\langle x^k\rangle$ for the case of axial symmetry have also been dealt with by Lewis,[30] who expanded them into infinite series whose coefficients are shown to obey an infinite set of coupled differential equations that can be solved recursively. The explicit solution of these equations would yield a result equivalent to (A21) with the matrix elements (A22).

B. Approximation

If the underlying single scattering is predominantly in the forward direction (which implies that (c_n-1) is small), and if the layer of matter traversed by the particle is thin (s small), then we may set

$$R_n(s)=\exp\left\{-N\int_0^s Q(1-c_n)ds\right\}=1-\delta_n s,\quad(A23)$$

where δ_n is assumed to be sufficiently small that we may neglect its second and higher powers. Then

$$\begin{aligned}
T_{j_1j_2\cdots j_k}{}^{n}&\cong\int_0^s ds_1\int_0^{s_1}ds_2\cdots\int_0^{s_{k-1}}ds_k\\
&\times(1+\delta_n-\delta_{j_1})(1+\delta_{j_1}-\delta_{j_2})\cdots(1+\delta_{j_{k-1}}-\delta_{j_k})\\
&\cong\int_0^s ds_1\int_0^{s_1}ds_2\cdots\int_0^{s_{k-1}}ds_k(1+\delta_n-\delta_{j_k})\\
&\cong\int_0^s ds_1\int_0^{s_1}ds_2\cdots\int_0^{s_{k-1}}ds_k\frac{R_{j_k}(s_k)}{R_{j_n}(s_k)}.
\end{aligned}\quad(A24)$$

But because of (A12) and (A21),

$$\sum_{i_1i_2\cdots i_k}A_{j_1}{}^{n}A_{j_2}{}^{i_1}\cdots A_{j_k}{}^{i_{k-1}}\langle\phi_{j_k}|0\rangle R_{j_k}(s)=\langle\sin^k\theta\cos^k\varphi\phi_n|s\rangle,\quad(A25)$$

so that

$$\langle x^k\phi_n|s\rangle=k!\int_0^s ds_1\int_0^{s_1}ds_2\cdots\int_0^{s_{k-1}}ds_k\times\langle\sin^k\theta\cos^k\varphi\phi_n|s\rangle.\quad(A26)$$

By a simple integral transformation[31] one obtains from (A26) Eq. (5) of the text (which corresponds to the case $n=0$).

C. Reversibility

In addition to the "forward" stochastic Eq. (5) of the text, one can also write down the "backward" equation

$$\begin{aligned}
F(\theta,\varphi,x|s-\Delta s)&=(1-NQ\Delta s)F(\theta,\varphi,x+\sin\theta\cos\varphi\Delta s|s)\\
&+NQ\Delta s\int_0^{2\pi}d\varphi'\int_0^{\pi}d\theta'F(\theta',\varphi',x\\
&+\sin\theta'\cos\varphi'\Delta s|s)\psi(\theta',\varphi';\theta',\varphi'|s),
\end{aligned}\quad(A27)$$

and the corresponding integro-differential equation

$$-\frac{\partial F}{\partial s}=NQ\int_0^{2\pi}d\varphi'\int_0^{\pi}d\theta'[F(\theta',\varphi',x|s)\times\psi(\theta,\varphi;\theta',\varphi'|s)]-NQF+\cos\varphi\sin\theta\partial F/\partial x.\quad(A28)$$

Let us assume that ψ and Q do not depend on s (no energy loss). Since $\psi(\theta',\varphi';\theta,\varphi)=\psi(\theta,\varphi|\theta'\varphi')$, the substitution $s'\rightarrow(-s)$ and $x'\rightarrow(-x)$ will transform (A28) into the "forward" equation (A5). Hence the solutions of the "backward" equation will be the same as that of the "forward" equation, except that the "initial" conditions (at $s=0$) become "final" conditions (at $s'=0$), i.e., refer to the direction of emergence instead of the direction of incidence of the particle.

[30] H. W. Lewis, Phys. Rev. **78**, 526 (1950).

[31] See, for example, B. Van der Pol and H. Bremmer, *Operational Calculus* (Cambridge University Press, Cambridge, 1950), p. 52.