Quantum Corrections to Classical Nonlinear Meson Theory

DONALD R. YENNIE*†
Institute for Advanced Study, Princeton, New Jersey

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For a linear Einstein-Bose field of large amplitude it is well known that the classical approximation is quite good for treating certain problems; the validity of this conclusion for the case of a nonlinear field is investigated here. In order to compare the classical and quantized versions of a nonlinear meson theory, we consider the problem of calculating the energy of interaction of the field with a given static source distribution. The quantized theory is treated in such a way that the classical result appears as a first approximation; the quantum corrections then include infinite renormalizations of the original parameters of the theory plus finite corrections to the energy. Part of these corrections are estimated and are found to become increasingly important with increasing source strength, contrary to the usual assumption. Since only a small part of the total quantum correction can be estimated by the present methods, a complete calculation might give a value very much larger or very much smaller than that given here; nevertheless, it is possible to conclude that the nature of quantum corrections is such that they cannot be treated as small perturbations.

I. INTRODUCTION

RECENTLY Schiff[1] and Malenka[2] have suggested the possibility of accounting for certain nuclear properties such as nuclear saturation and shell structure by means of forces derived from a nonlinear meson theory. In these calculations the meson field is treated classically under the assumption that quantum fluctuations can be neglected because the meson field has a large amplitude and obeys Einstein-Bose statistics. Such an assumption is certainly valid for a linear meson field because a large field amplitude in the classical theory corresponds to the presence of a large number of mesons in the quantized theory, and quantum fluctuations are unimportant in such a situation. In this paper we shall be concerned with the energy of interaction of the field with a static source; for a linear field it will be apparent that this energy is actually the same in the quantum theory as in the classical theory, independent of the source strength. In the nonlinear meson theory, however, new features arise because of the infinities introduced by the interaction of the field with itself. From the power series expansion of this interaction, we know that these infinities lead to renormalizations of the original parameters of the theory plus finite residues which are taken to be physically meaningful. There seems to be no *a priori* reason to expect that finite quantities obtained in this way should be small compared to the corresponding quantities calculated with the classical theory. The object of the present paper is to treat the quantized theory in such a way that the classical theory appears as a first approximation and to investigate the relative importance of the quantum corrections to this classical approximation.

Let us first review briefly the general features of the classical theory. We consider as a particular example the following Lagrangian density[3]

$$\mathcal{L}=\tfrac{1}{2}(\partial\phi/\partial t)^2-\tfrac{1}{2}(\nabla\phi)^2-\tfrac{1}{2}\kappa^2\phi^2-\tfrac{1}{4}\alpha^2\phi^4+f\phi, \quad (1)$$

where $f(\mathbf{r})$ is the nucleon source density, which we take to be time-independent (thus neglecting the dynamics of the source). We use units in which $\hbar=c=1$. From this Lagrangian density are deduced the Hamiltonian density and the Hamiltonian

$$\mathcal{H}=\tfrac{1}{2}\pi^2+\tfrac{1}{2}(\nabla\phi)^2+\tfrac{1}{2}\kappa^2\phi^2+\tfrac{1}{4}\alpha^2\phi^4-f\phi, \quad (2)$$

$$H=\int\mathcal{H}(dx)^3, \quad (3)$$

where $\pi=\partial\phi/\partial t$ is the momentum canonically conjugate to ϕ. The wave equation deduced from the Lagrangian or Hamiltonian is

$$\partial^2\phi/\partial t^2-\nabla^2\phi+\kappa^2\phi+\alpha^2\phi^3=f. \quad (4)$$

For a given source distribution $f(\mathbf{r})$, one has to solve the wave equation to obtain the static field ϕ_s (in practice this may be done only approximately). Inserting ϕ_s into (3), the classical energy associated with the given source distribution is obtained. By considering different source distributions, information about nuclear forces, the potential energy of a nucleon in a nucleus, etc., is deduced. The present note is concerned not with any of these particular nuclear problems, but only with the general method of calculation.

In the next section the field is quantized according to the usual commutation rules, and by means of a canonical transformation the Hamiltonian is expressed in a form which clearly reveals the classical approximation. Part of the quantum corrections to the energy are then easily estimated by calculating the zero point

* National Research Fellow.
† Now at Stanford University, Stanford, California.
[1] L. I. Schiff, Phys. Rev. **80**, 137 (1950); **83**, 239 (1951); **84**, 1 (1951), referred to here as SI; Phys. Rev. **84**, 10 (1951).
[2] B. J. Malenka, Phys. Rev. **85**, 686 (1952); **86**, 68 (1952).
[3] This particular Lagrangian density has been considered classically by Schiff in SI (see reference 1) and by Malenka (see reference 2); as shown by Malenka (see reference 2) the form of the nonlinearity in (1) is suggested by the pseudoscalar interaction of a quantized nucleon field with a classical pseudoscalar field. The sign of the nonlinearity in (1) has been chosen so that its contribution to the energy will be positive definite. This excludes from our consideration the case in which there is a classical particle-like solution, as treated by Finkelstein, LeLevier, and Ruderman, Phys. Rev. **83**, 326 (1951).

energy of the part of the Hamiltonian which is quadratic in the meson variables; this estimation involves the dropping of infinite κ- and α-renormalizations and the interpretation of the finite remainder as physically meaningful. These quantum corrections are found to increase in relative importance with increasing source strength, thus showing that quantum corrections are important and may not be treated as a perturbation. In order to see how to calculate higher order corrections and to show that the renormalization prescription used in this estimation is correct, we turn in Sec. III to a proof of the existence of a consistent renormalization program by means of a power series expansion. The main result of this program is that the energy is a functional of a variable ϕ_0 which obeys a complicated nonlinear differential-integral equation. The energy functional and the equation for ϕ_0 are related by the condition that a solution of the equation minimizes the energy functional. To the zeroth order in the quantum corrections, ϕ_0 is, of course, the same as ϕ_s, and the energy functional is the same as (3). In Sec. IV, the results of Sec. II are shown to result from summing a certain subset of the power series expansion. To investigate whether quantum corrections will be reduced by including a larger portion of the total quantum contribution, we sum the contribution from a somewhat larger subset of the power series expansion. It is found that the effect of including this larger subset is to increase the estimate of the quantum corrections, thus confirming the conclusions of Sec. II.

II. QUANTIZATION OF THE FIELD

The field is quantized according to the usual rules

$$[\pi(\mathbf{x}),\phi(\mathbf{x}')]=-i\delta(\mathbf{x}-\mathbf{x}'), \tag{5}$$

where π and ϕ are Schrödinger representation operators. We make a canonical transformation[4]

$$S=\exp\left[i\int\phi_0(\mathbf{x})\pi(\mathbf{x})(dx)^3\right], \tag{6}$$

where ϕ_0 is a c-number function of position to be specified later. Under this transformation, the field quantities and the Hamiltonian are transformed in the following way

$$S\phi S^*=\phi_0+\phi,\quad S\pi S^*=\pi,$$
$$SHS^*=H_{cl}+H_1+H_2, \tag{7}$$

$$H_{cl}=\int\{\tfrac{1}{2}(\nabla\phi_0)^2+\tfrac{1}{2}\kappa^2\phi_0^2+\tfrac{1}{4}\alpha^2\phi_0^4-f\phi_0 +\phi(-\nabla^2\phi_0+\kappa^2\phi_0+\alpha^2\phi_0^3-f)\}(dx)^3,$$

$$H_1=\int\{\tfrac{1}{2}\pi^2+\tfrac{1}{2}(\nabla\phi)^2+\tfrac{1}{2}\kappa^2\phi^2+\tfrac{3}{2}\alpha^2\phi_0^2\phi^2\}(dx)^3,$$

$$H_2=\int\{\tfrac{1}{4}\alpha^2\phi^4+\alpha^2\phi_0\phi^3\}(dx)^3.$$

[4] This transformation was used previously by L. I. Schiff, Phys. Rev. **86**, 625 (1952).

The choice of ϕ_0 will be discussed in more detail in Sec. III. In this section we shall make the working assumption that quantum corrections are small when the classical field strengths are large. Under this assumption, ϕ_0 is approximately equal to ϕ_s, the classical static field. In the Hamiltonian the term linear in ϕ is then small and H_{cl} is approximately the usual classical energy. The quantized field interacts with the source mainly through the dependence of H_1 and H_2 on ϕ_0. Since we are assuming that ϕ_0 is in some sense large compared to ϕ, we try to treat H_1 exactly, with H_2 as a perturbation.

The field equation derived from H_1 is

$$\partial^2\phi/\partial t^2-\nabla^2\phi+\kappa^2\phi+3\alpha^2\phi_0^2\phi=0. \tag{8}$$

In this equation it is seen that $3\alpha^2\phi_0^2$ serves as an effective potential through which the ϕ field moves. Considered as a classical equation with periodic boundary conditions, the field equation has fundamental solutions of the form

$$\exp(\pm i\omega_n t)\phi_n(\mathbf{x}),\quad \omega_n>0 \tag{9}$$

satisfying

$$(-\omega_n^2-\nabla^2+\kappa^2+3\alpha^2\phi_0^2)\phi_n=0. \tag{10}$$

This enables us to define the operator $\omega=(-\nabla^2+\kappa^2+3\alpha^2\phi_0^2)^{\frac{1}{2}}$ by the equation

$$\omega\phi_n=\omega_n\phi_n. \tag{11}$$

We may now expand ϕ and π in terms of the complete set of functions ϕ_n:

$$\phi=\textstyle\sum_n(2\omega_n)^{-\frac{1}{2}}(a_n\phi_n+a_n{}^*\phi_n{}^*),$$
$$\pi=\textstyle\sum_n(\omega_n/2)^{\frac{1}{2}}i(a_n{}^*\phi_n{}^*-a_n\phi_n), \tag{12}$$

where the a's obey the following commutation rules

$$[a_n,a_{n'}{}^*]=\delta_{nn'}. \tag{13}$$

Expressing H_1 in terms of these expansions, we obtain the usual result

$$H_1=\textstyle\sum_nN_n\omega_n+\frac{1}{2}\sum_n\omega_n,$$
$$N_n=a_n{}^*a_n, \tag{14}$$

where the operators N_n have the eigenvalues 0, 1, 2, 3, $\cdots$. The vacuum is defined as the state of lowest energy: all $N_n=0$. Normally the infinite zero-point energy $(\frac{1}{2}\sum\omega_n)$ is neglected because it is an infinite additive constant. In the present case, however, the zero-point energy may not be neglected because it depends on the source strength through ϕ_0; in fact, it varies by an infinite amount when the source is changed. We shall show that part of this infinite dependence on the source can be interpreted as a change in the values of κ and α appearing in H_{cl}. After this renormalization, we shall have a finite remainder which may be interpreted as physically meaningful (within the spirit of the usual renormalization treatment) and compared with terms already occurring in the classical energy. The zero-point energy is

$$\tfrac{1}{2}\sum \omega_n=\tfrac{1}{2}Tr\omega$$
$$=\frac{1}{2}\int\int \delta(\mathbf{x}-\mathbf{x}')(-\nabla^2+\kappa^2+V_0)^{\frac{1}{2}}\delta(\mathbf{x}'-\mathbf{x})\times(dx)^3(dx')^3$$
$$=\frac{1}{16\pi^3}\int\int\int \delta(\mathbf{x}-\mathbf{x}')(-\nabla^2+\kappa^2+V_0)^{\frac{1}{2}}\times\exp[i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')](dx)^3(dx')^3(dk)^3, \quad (15)$$

where $V_0=3\alpha^2\phi_0^2$.

We cannot evaluate this expression exactly because ∇^2 may act on the V_0 under the square root; we accordingly make the approximation that V_0 is a slowly varying function of position and that we may neglect its derivatives. Even though this approximation may not be good in general, we expect it to give us an estimate of the order of magnitude of the correction to the energy. In the Appendix we use the WKB approximation to estimate the fundamental frequencies for the case of a one-dimensional source distribution and show that it agrees with the following result except for possible ambiguities in the renormalization; surface effects arising from sharp discontinuities in V_0 are also estimated there. With this approximation, the zero-point energy becomes

$$\tfrac{1}{2}\sum \omega_n=\frac{1}{16\pi^3}\int\int (dx)^3(dk)^3(k^2+\kappa^2+V_0)^{\frac{1}{2}}$$
$$=\lim_{K\to\infty}\frac{1}{4\pi^2}\int (dx)^3\int_0^K k^2dk(k^2+\kappa^2+V_0)^{\frac{1}{2}}$$
$$=\int (dx)^3\{A+B\alpha^2\phi_0^2+C\alpha^4\phi_0^4+\frac{1}{64\pi^2}\times[(V_0+\kappa^2)^2\ln(1+V_0\kappa^{-2})-\kappa^2V_0-1.5V_0^2]\}, \quad (16)$$

where A, B, and C are infinite constants, independent of α and ϕ_0. A is the zero-point energy of the field in the absence of a source and B and C give renormalizations of the κ and α appearing in the first line of H_{cl}. The finite part has been made unique by the requirement that its series expansion in powers of ϕ_0 contains no terms in ϕ_0^2 or ϕ_0^4. The consistency of this renormalization prescription will be discussed in Secs. III and IV. Interpreting the finite part as physically meaningful, the energy of the vacuum is given approximately by

$$E_0\cong\int\{\tfrac{1}{2}(\nabla\phi_0)^2+\tfrac{1}{2}\kappa^2\phi_0^2+\tfrac{1}{4}\alpha^2\phi_0^4-f\phi_0+\frac{1}{64\pi^2}[(3\alpha^2\phi_0^2+\kappa^2)^2\ln(1+3\alpha^2\phi_0^2\kappa^{-2})-3\kappa^2\alpha^2\phi_0^2-13.5\alpha^4\phi_0^4]\}(dx)^3. \quad (17)$$

It is easily seen that for small field strengths the new contribution is unimportant since the leading term in a power series expansion is proportional to ϕ_0^6. For large field strengths, however, the logarithmic term becomes more important than other terms in the energy. As an example, we take the following parameters for the interior of a nucleus from Schiff's paper:[1] $\kappa=1$; $\alpha=7.96$, $\phi_0=0.149$. With these parameters the nonlinear term in the energy has the value $\frac{1}{4}\alpha^2\phi_0^4=0.0078$, while the quantum correction just obtained has the value 0.023. For larger field strengths, quantum corrections will become relatively more important.

In addition to these energy corrections, we may also desire to determine the corresponding quantum corrections to the equation which ϕ_0 must satisfy. To the order of approximation of the present section, these corrections arise from the term $\alpha^2\phi_0\phi^3$ which occurs in H_2. Before this term can be used in a perturbation expansion, it is necessary to rewrite it as an ordered product[5] plus another term which is linear in ϕ. Because of (12), the definition of the creation and destruction operators, and hence of the ordered product, depends on ϕ_0. The ordered product is used in the perturbation expansion, while the other term gives a contribution to the equations which ϕ_0 must satisfy. The correction to the equation of ϕ_0 is

$$3\alpha^2\phi_0\langle\phi^2\rangle_0=3\alpha^2\phi_0\sum_n(1/2\omega_n)\phi_n{}^*\phi_n=(\tfrac{3}{2})\alpha^2\phi_0(\mathbf{x}|(-\nabla^2+\kappa^2+V_0)^{-\frac{1}{2}}|\mathbf{x}). \quad (18)$$

Now it is easily derived from (10) that

$$\omega_n\delta\omega_n/\delta\phi_0(\mathbf{x})=3\alpha^2\phi_0(\mathbf{x})\phi_n{}^*(\mathbf{x})\phi_n(\mathbf{x}). \quad (19)$$

Hence, the corrected equation for ϕ_0 may be written

$$-\nabla^2\phi_0+\kappa^2\phi_0+\alpha^2\phi_0^3+\tfrac{1}{2}\delta(\sum\omega_n)/\delta\phi_0=f. \quad (20)$$

From this equation it is apparent that the κ- and α-renormalizations occurring in the equation for ϕ_0 are the same ones that occurred in the energy, Eq. (17). The renormalization is thus consistent to this order of approximation. The finite correction to the equation satisfied by ϕ_0 is obtained simply by taking the functional derivative of the corresponding finite correction to the classical energy. In the following section, this relation will be seen to be generally true, although it appears there as a simplifying choice rather than a necessary requirement.

It is now evident that our working assumption is not valid; for having assumed that quantum corrections were small, we have found that to a certain order of approximation they are large. In fact, the quantum corrections seem to become relatively more important as the classical field strength increases, contrary to the usual assumptions that the effect of lack of commutativity may be neglected when the field magnitudes

[5] In an ordered product, all creation operators are written to the left of all destruction operators; the use of the ordered product was introduced by A. Houriet and A. Kind, Helv. Phys. Acta 22, 319 (1949).

are large. We have not actually proved that quantum corrections are large, because we have not treated the whole Hamiltonian exactly. It is conceivable, although it seems to be very unlikely, that the net effect of all the perturbations we have neglected would be to reduce the quantum corrections to a small value. In any case, we have shown that we may not assume at the start that quantum corrections will be small enough to be calculable by the sort of perturbation techniques used here.

III. THE RENORMALIZATION PROGRAM

In the preceding section, a finite result was extracted from an infinite quantity by applying a certain prescription for renormalizing the original parameters of the theory. This renormalization took place with respect to the lowest possible order of the parameter of nonlinearity. In order that the result obtained in this way should have a physical meaning, it is necessary to show that the result of the preceding section is the first stage of some renormalization program which can be carried through consistently to all orders of approximation. It must be shown, for instance, that each of the five quantities α^2 appearing in the Hamiltonian will be renormalized in exactly the same way; similarly for the three quantities κ^2. In carrying out this demonstration, it will be necessary to make a power series expansion of the energy so that the techniques developed for handling the renormalization of the S-matrix may be employed.[6,7] We shall rely heavily on the results of Ward's paper[7] because the set of interacting fields treated by him includes, as a special case, the nonlinear field of this paper. Only the features arising because of the presence of a source will be treated in this note.

Before proceeding, we note that the canonical transformation (6) is time-independent and therefore does not alter the eigenvalues of H. This means that if we calculate the energy from (7) using perturbation theory, the net effect of all contributions involving ϕ_0 will be identically zero; this will be seen explicitly when the perturbation method is presented. For the quantum-mechanical calculation, it is therefore simplest to set ϕ_0 equal to zero at the start and proceed with a perturbation treatment of the original Hamiltonian (3). In the perturbation treatment it will be seen that the various contributions to the energy can be represented by Feynman graphs. The classical approximation to the energy is represented by the set of graphs in which there are no closed loops; the contribution of the last section is represented by the set of graphs with one closed loop; etc. The results of Ward's paper[7] can be applied immediately, and with a suitable renormalization of the source, they lead to a well-defined power series expansion for the energy. However, because of the closer connection with the classical theory, we prefer to make the canonical transformation involving ϕ_0. To avoid having all the contributions from ϕ_0 cancel each other out, we calculate the part of the energy represented by a small (but infinite) subset of the Feynman graphs and then choose ϕ_0 in such a way that the remaining contribution vanishes identically. In this way the quantum corrections appear as a modification of the classical problem.

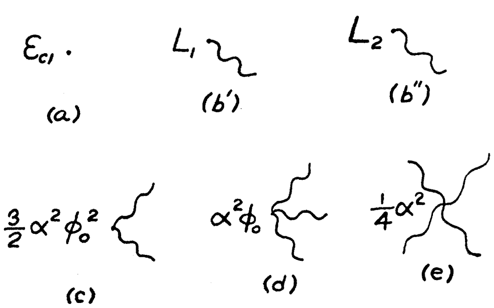

FIG. 1. Types of vertices which may appear in Feynman graphs.

[6] F. J. Dyson, Phys. Rev. 75, 1736 (1949).
[7] J. C. Ward, Phys. Rev. 84, 897 (1951).

In order to apply perturbation theory, we separate the Hamiltonian (7) into a free field part H_0 and a perturbation H_i

$$SHS^* = H_0 + H_i,$$

$$H_0 = \int \{\tfrac{1}{2}\pi^2 + \tfrac{1}{2}(\nabla\phi)^2 + \tfrac{1}{2}\kappa_c^2\phi^2\}(dx)^3,$$

$$H_i = H_{cl} + \int \{\tfrac{1}{4}\alpha^2\phi^4 + \alpha^2\phi_0\phi^3 + (\tfrac{3}{2})\alpha^2\phi_0^2\phi^2 + (\kappa^2 - \kappa_c^2)\phi^2\}(dx)^3. \quad (21)$$

The quantities κ^2, α^2, f, and ϕ_0 appearing in (21) are not finite but must be related to the corresponding finite quantities κ_c^2, α_c^2, f_c, and ϕ_{0c} in such a way that the final results are unique and finite (aside from any possible lack of convergence of the perturbation expansion itself). Notice that H_0 contains the finite mass κ_c; the last term of H_i cancels out mass corrections explicitly as they arise from the perturbation expansion of the other parts of H_i. It is easily seen that the mass corrections for H_{cl} cancel in the same way term by term, so we shall take no further account of mass renormalization.

We are interested in the energy of the state in which there are no freely propagating mesons present. This energy is easily calculated by averaging the rate of change of phase of the U-matrix over a long period of time and is found to be

$$E_{\text{vac}} = \sum_G \sum_{n=0}^{\infty} [(-i)^n/(n+1)!] \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} (dx_0)^3 \times (dx_1)^4 \cdots (dx_n)^4 P[H_i(x_0),\ H_i(x_1)\cdots H_i(x_n)]. \quad (22)$$

The sum is over all Feynman graphs G which are connected with the point x_0 and which have no external meson lines. The operators occurring in $H_i(x_j)$ are interaction representation operators; that is, they have the time dependence of free field operators and reduce to Schrödinger representation operators at zero time. P is the usual chronological ordering operator. Since ϕ_0 is independent of time, the result of the integrations in (22) is independent of the time assigned to $H_i(x_0)$; for definiteness, we set this time equal to zero.

The types of vertices which may occur in Feynman graphs are shown in Fig. 1. We have used the abbreviations:

$$\begin{aligned}\mathcal{E}_{cl}&=\tfrac{1}{2}(\nabla\phi_0)^2+\tfrac{1}{2}\kappa_c^2\phi_0^2+\tfrac{1}{4}\alpha^2\phi_0^4-f\phi_0,\\ L_1&=(-\nabla^2+\kappa_c^2)\phi_0,\\ L_2&=\alpha^2\phi_0^3-f.\end{aligned}\tag{23}$$

Terms involving mass renormalizations have already been dropped. The two types of vertices (b') and (b'') have been distinguished to aid in proving the statement made previously that the net effect of all contributions involving ϕ_0 will be identically zero. The vertex (a) appears in one Feynman graph, consisting only of itself $(n=0)$.

Corresponding to each line of a graph there is a factor

$$\begin{aligned}\langle P[\phi(x),\phi(y)]\rangle_0&=\tfrac{1}{2}\Delta_F(x-y)\\ &=[(-i)/(2\pi)^4]\int_C (dk)^4(k^2+\kappa_c^2)^{-1}\\ &\qquad\times\exp[ik_\mu(x-y)_\mu],\end{aligned}\tag{24}$$

where the contour C is defined by imagining κ_c to have a small negative imaginary part which displaces the poles from the real axis.

It is now easy to show that the net effect of all contributions involving ϕ_0 is zero. Consider a particular graph which does not contain any vertices of type (b'), but has the quantity ϕ_0 occurring a certain number of times. Any ϕ_0 may be replaced by a contribution involving a vertex of type (b'), yielding a new graph which must be considered in the enumeration. The net result is that if ϕ_0 occurs at the place x in the original graph, in the new graph it is to be replaced by

$$\tfrac{1}{2}(-i)\int\Delta_F(x-y)(-\nabla^2+\kappa_c^2)\phi_0(y)(dy)^4=-\phi_0(x).$$

Since this replacement takes place independently of all others, all contributions involving ϕ_0 are canceled out identically. The only contributions which do not cancel in this way are those which are made up purely of the perturbation expansion of $(-f\phi+\frac{1}{4}\alpha^2\phi^4)$.

In order to make the use of the canonical transformation (6) nontrivial, it is necessary to choose ϕ_0 in such a way that the energy may be evaluated from a subset of all possible Feynman graphs. It may be possible to choose such a subset in more than one way, but we make

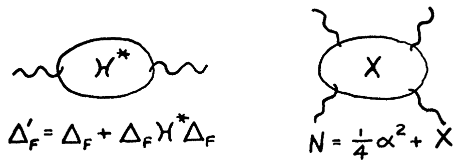

FIG. 2. The two types of primitive divergences.

the following choice which seems to make the subset as small as possible, at the expense of complicating the equation which ϕ_0 must satisfy. We define the subset Ω' to consist of all graphs G' which have the property that they cannot be bisected by cutting a single meson line. ϕ_0 must then be chosen so that the net contribution from all remaining graphs is zero. This is easily done, for each of these graphs must contain at least two subgraphs of the type (β), which have the properties that they are connected to the remainder of the graph by a single line and may not be bisected by cutting a single internal line. The simplest examples of this type of subgraph are the vertices (b') and (b''). In order that the net contribution from graphs not in Ω' be zero, we require simply that the sum of all contributions represented by the subgraphs (β) vanish. This gives a complicated nonlinear differential integral equation which ϕ_0 must satisfy; the classical field equation is the first approximation to this more general equation.

There is obviously a close structural relationship between the graphs G' and the subgraphs (β); for if we replace a ϕ_0 acting at some point in G' by an external line attached to the point, the result is a contribution to one of the subgraphs (β). If this is done in all possible ways, several subgraphs of the type (β) will be obtained from the same G', and inspection shows that all these contributions will have the proper weight factor associated with them. This graphical process is a representation of the mathematical process of taking the functional derivative with respect to $\phi_0(x)$ of the integral corresponding to G'. Letting $A[\phi_0]$ be the contribution to the energy arising from graphs in Ω', the condition that the sum of all contributions from subgraphs of type (β) vanishes is then expressed by the equation

$$\delta A[\phi_0]/\delta\phi_0(x)=0.\tag{25}$$

We are now ready to discuss the renormalization program. Except for the terms involving ϕ_0, Ward[7] has already discussed a more general case than that of this paper; and we shall make full use of his results, which we review briefly. In this theory there are two types of primitive divergences, self-energy parts and four-vertex parts, shown in Fig. 2. Every graph G' has a "skeleton" which may be obtained by omitting all self-energy and four-vertex parts from G'. Corresponding to a given skeleton there is a whole set of graphs which may be built up by all possible insertions of self-energy and four-vertex parts. The net effect of all these insertions is to modify the propagation and scattering of mesons in the following way

$$\Delta_F(x-x')\rightarrow\Delta_F'(x-x')$$

$$\tfrac{1}{4}\alpha^2\delta(x_1-x_2)\delta(x_1-x_3)\delta(x_1-x_4)\rightarrow$$
$$\tfrac{1}{4}\alpha^2\delta(x_1-x_2)\delta(x_1-x_3)\delta(x_1-x_4)+X(x_1,x_2,x_3,x_4)=N(x_1,x_2,x_3,x_4). \quad (26)$$

Because of the divergences of the theory, these quantities are given by power series with infinite coefficients. Δ_F' and N are called the infinite functions. If at each stage of the power series expansion we define finite quantities by the unique subtraction procedure specified in Ward's paper, we arrive at a corresponding set of finite functions which we designate Δ_c and N_c. It then turns out that the infinite functions, expressed in terms of α^2, are simply numerical multiples of the finite functions, expressed in terms of α_c^2.

$$\Delta_F'=Z\Delta_c,\quad N=Z^{-2}N_c. \quad (27)$$

Z and α^2 are given by certain power series in α_c^2 (with infinite coefficients).

In order to apply these results to the present problem, we distinguish the various contributions to A according to the number of times that ϕ_0 acts in the contributing graphs; $A_m[\phi_0]$ is the total contribution from all graphs in which ϕ_0 acts m times. The cases $m=1$, $m=2$ and $m=4$ which occur in $\mathcal{E}_{cl}$ are somewhat special and will be discussed last. For $m\geqslant 6$, Ward's work shows us that

$$A_m[\phi_0]=Z^{-m/2}A_{mc}[\phi_0], \quad (28)$$

where A_{mc} is defined by inserting the finite functions Δ_c and N_c (expressed in terms of α_c and κ_c) into the various skeletons of graphs which contribute to A_m. From (28), it is obvious that the field strength should be renormalized in the following way

$$\phi_0=Z^{\frac{1}{2}}\phi_{0c}. \quad (29)$$

We have to show that (28) is also correct for $m=1$, $m=2$, and $m=4$. To make (28) correct for $m=1$, we are required to renormalize the source strength as follows

$$f=Z^{-\frac{1}{2}}f_c. \quad (30)$$

This is in agreement with the renormalization of the meson-nucleon interaction constant in Ward's paper.

The $m=2$ and $m=4$ contributions are special cases because they are primitively divergent. For $m=4$, it is obvious from (26) that the total contribution is given by

$$A_4=\int\cdots\int N(x_1,x_2,x_3,x_4)\phi_0(x_1)\phi_0(x_2)\phi_0(x_3)\times\phi_0(x_4)(dx_1)^3(dx_2)^4(dx_3)^4(dx_4)^4.$$

By virtue of (27), it is obvious that A_4 satisfies (28). For $m=2$, we may express the result in terms of the proper self-energy part $\mathcal{H}^*$

$$A_2=\int\tfrac{1}{2}\phi_0(-\nabla^2+\kappa_c^2)\phi_0(dx)^3+i\int\phi_0(x_1)\mathcal{H}^*(x_1,x_2)\phi_0(x_2)(dx_1)^3(dx_2)^4.$$

From Ward's paper

$$\mathcal{H}^*(x_1,x_2)=\tfrac{1}{2}i(Z-1)Z^{-1}(-\Box^2+\kappa_c^2)\delta(x_1-x_2)+Z^{-1}\mathcal{H}_c^*(x_1,x_2),$$

where $\mathcal{H}_c^*$ is the finite part of $\mathcal{H}^*$, expressed in terms of the finite parameters. When this is substituted into the expression for A_2, it is found that the result satisfies (28). This completes the proof of the consistency of the renormalization program.

We may summarize the results of this section by the following set of rules for calculating the energy: (1) calculate the functional $A_c[\phi_{0c}]$ by summing the contributions from all graphs of type G' and dropping the infinite κ and α renormalizations in the manner specified by Ward;[7] (2) minimize this functional by requiring ϕ_{0c} to satisfy

$$\delta A_c[\phi_{0c}]/\delta\phi_{0c}(x)=0. \quad (31)$$

This minimum value of the functional is the energy associated with the given source distribution.

IV. NEARLY-UNIFORM SOURCE DISTRIBUTION

In this section we shall make the approximation, used previously in Sec. II, that the source is such a slowly varying function of position that its derivatives may be neglected. Such an approximation is probably not valid in practical problems; nevertheless, features which arise in such an approximation will probably have a counterpart in a more exact calculation. Strictly speaking, the results of this section will apply only to a source which is independent of position. Because of the great complexity that graphs of type G' may have, we shall sum the contributions from only a certain simple subset of all possible graphs. The method of summation is as follows: for sufficiently small f, the series expansion for A_c may converge, defining an analytic function; the analytic continuation of this function along the real f axis, if it exists and is unique, is taken to represent the sum of the infinite series, even outside the original radius of convergence.

It was mentioned in Sec. II that the quantity[8] $V_0=3\alpha^2\phi_0^2$ plays the role of a potential through which the ϕ-field moves; we shall now calculate some of the quantum corrections to this potential. The simplest type of correction is shown in Fig. 3(b). The potential V_0 must act one or more times on the closed loop because the graph with no such interactions corresponds

[8] In this section we shall drop the subscript c to denote the finite quantities; infinite quantities will be dropped as they arise according to the prescriptions of Ward.

to a mass renormalization which must be dropped. If V_0 acts only once, it yields a renormalization of the α^2 occurring in the contribution from Fig. 3(a). It is obvious that the result of summing all contributions (including zero interactions of V_0) will be, before renormalization, simply the propagation function for a particle in a potential V_0, which we call Δ_{V_0}. The net effect of all these contributions, including renormalizations, is the following correction to the original potential V_0

$$V_1(x)=(\tfrac{3}{2})\alpha^2\Delta_{V_0}(x,x)-(\tfrac{3}{2})\alpha^2\Delta_F(x-x)-(-3i/4)\alpha^2V_0\int[\Delta_F(x-y)]^2(dy)^4. \quad (32)$$

(Note that this corresponds to the addition of a term $\frac{1}{2}V_1\phi^2$ to the Hamiltonian.) Expressing Δ_{V_0} in terms of the normal mode expansion (12), we obtain

$$V_1(x)=3\alpha^2\sum_n(1/2\omega_n)\phi_n{}^*\phi_n-[3\alpha^2/2(2\pi)^3]\int(dk)^3(k^2+\kappa^2)^{-\frac{1}{2}}+[3\alpha^2/4(2\pi)^3]\int(dk^3)(k^2+\kappa^2)^{-\frac{3}{2}}V_0. \quad (33)$$

This is in agreement with Eq. (18), with the proper renormalization rule now specified. This shows that the results of Sec. II correspond to the sum of contributions from all graphs with one closed loop.

We may now extend this result to a somewhat larger class of graphs by a simple iteration procedure. For example, we may replace the quantities V_0 of Fig. 3(b) in arbitrary ways by the quantities V_1; the result will be the quantity V_2, given by an expression like (32), but with V_0 replaced by V_0+V_1. It is obvious that V_2 consists of V_1 plus additional contributions, so the next step of the iteration procedure is to replace the quantities V_0 of Fig. 3(b) in arbitrary ways by V_2 (not V_1+V_2), thus producing the quantity V_3. In this way we define V_{n+1} in terms of V_n, as indicated in Fig. 3(c). It is obvious that the renormalization procedure is precisely analogous to that employed in (31), so that we have the following recursion relation for $W_n=V_0+V_n$.

$$W_{n+1}=V_0+\tfrac{3}{2}\alpha^2(\mathbf{x}|(-\nabla^2+\kappa^2+W_n)^{-\frac{1}{2}}|\mathbf{x})-\tfrac{3}{2}\alpha^2(\mathbf{x}|(-\nabla^2+\kappa^2)^{-\frac{1}{2}}|\mathbf{x})+\tfrac{3}{4}\alpha^2(\mathbf{x}|(-\nabla^2+\kappa^2)^{-\frac{3}{2}}|\mathbf{x})W_n, \quad (34)$$
$$W_0=V_0.$$

So far this is an exact expression for the graphs considered in the iteration process. In order to proceed further, we take advantage of the approximation that the quantity ϕ_0 is a slowly varying function of position. Then the various W_n will also have this property and

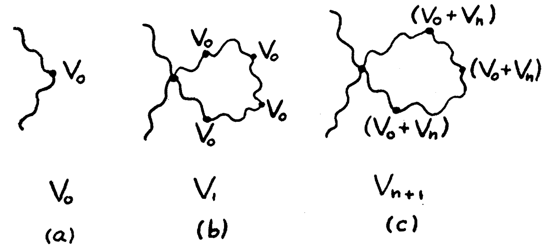

FIG. 3. Some contributions to the effective potential for the meson field.

Eq. (34) can be evaluated by the methods of Sec. II, yielding

$$W_{n+1}=V_0+(3\alpha^2/16\pi^2)[(\kappa^2+W_n)\times\ln(1+\kappa^{-2}W_n)-W_n], \quad (35)$$
$$W_0=V_0.$$

Examination of this recursion relation shows that for $V_0>0$:

$$0<W_0<W_1\cdots<W_n<\cdots. \quad (36)$$

Thus, either W_n approaches a finite limit W, or it approaches infinity. In the first case, W must satisfy the equation

$$V_0=W-(3\alpha^2/16\pi^2)[(\kappa^2+W)\ln(1+\kappa^{-2}W)-W]. \quad (37)$$

If, for a given V_0, no real solutions of this equation exist, the iteration process is divergent. Considered as a function of W, V_0 is zero for W equal to zero, has a maximum value when

$$\ln(1+\kappa^{-2}W)=16\pi^2/3\alpha^2, \quad (38)$$

and approaches minus infinity when W approaches infinity. This implies that the iteration process diverges when V_0 is greater than a certain value, which is the maximum of the right side of (37)

$$(V_0)_{\max}=\kappa^2[(3\alpha^2/16\pi^2)\{\exp(16\pi^2/3\alpha^2)-1\}-1]. \quad (39)$$

For $V_0<(V_0)_{\max}$, there are two solutions of Eq. (37); inspection shows that the recursion relation (35) converges to the smaller of these two solutions.

We may now calculate the energy and the correction to the classical field equation associated with graphs of the type shown in Fig. 3. Neglecting derivatives of ϕ_0, the corrected field equation is simply

$$\kappa^2\phi_0+\alpha^2\phi_0{}^3+\phi_0(W-V_0)=f. \quad (40)$$

Equations (37) and (40) are to be solved simultaneously for ϕ_0 and W. Equation (37) may be used to eliminate ϕ_0 from Eq. (40), thus giving f as an analytic function of W. As W increases from zero, this function increases from zero to a maximum at a value of W somewhat greater than given by (38), and then decreases to zero at the value of W for which the right side of (37) vanishes. The inverse function, which gives W as a function of f, is obviously analytic from zero to the maximum value of f where it has a branch point; the proper branch is chosen by the condition that W must vanish when f does. For values of f larger than this maximum, no real solution to these equations exists and the contribution from this subset of graphs presumably diverges. This does not necessarily mean that the complete power series expansion diverges however, because if other graphs are included in the calculation in a suitable way the result as a whole may be finite.

We calculate the contribution to the energy by applying (31) in reverse; we now know a term in the equation for the classical field and wish to find the corresponding contribution A', which must satisfy

$$\delta A'/\delta\phi_0 = \phi_0(W - V_0). \tag{41}$$

Assuming

$$A' = \int [B(W) - (\tfrac{3}{4})\alpha^2\phi_0^4](dx)^3, \tag{42}$$

we find

$$(dB/dW)(dW/d\phi_0) = \phi_0 W.$$

But from (37)

$$6\alpha^2\phi_0 = (dW/d\phi_0)[1 - (3\alpha^2/16\pi^2)\ln(1+\kappa^{-2}W)]. \tag{43}$$

Therefore,

$$dB/dW = (\tfrac{1}{6}\alpha^2)W[1 - (3\alpha^2/16\pi^2)\ln(1+\kappa^{-2}W)]. \tag{44}$$

Integrating this equation, we find

$$B = (W^2/12\alpha^2) - (1/64\pi^2)[(W^2 - \kappa^4)\ln(1+\kappa^{-2}W) - \tfrac{1}{2}W^2 + \alpha^2 W]. \tag{45}$$

This calculation of A' by means of (41) is much simpler than by the original definition of A' because of the difficulty of properly enumerating the various contributions to A' in the direct calculation.

These results seen to reinforce the views of Sec. II that quantum corrections to the energy associated with a source distribution beçome relatively more important as the source strength is increased. It certainly seems to be unlikely that the net effect of all the contributions omitted would be just sufficient to make the total effect very small, although such a possibility cannot definitely be ruled out. In any case, we are justified in saying that quantum corrections are not so small that they may be treated as a perturbation. The results of this section also suggest the possibility that the relation of the energy to the source strength may be changed qualitatively as well as quantitatively. For example, the contribution of A' to the energy increases with f until a critical value of f is reached beyond which the problem can no longer be solved. For this value of f, the energy is still finite, which means that such a source could be built up without an infinite expenditure of energy; if, instead, the energy became infinite as f approached its critical value, then we would have a saturation effect which would prevent the source from being built up. Since we are considering only one class of graphs out of an infinite number of classes, it is hard to predict whether a final complete calculation would show either of these effects, or possibly other effects not yet revealed in the calculation to this stage of approximation; however, we should be prepared to expect such effects.

Unfortunately the results of this paper do not shed too much light on the question of convergence of the power series expansion. The functional A is a power series expansion in both α^2 and ϕ_0; each power of ϕ_0 has a coefficient which is an infinite series in powers of α^2; each power of α^2 has a coefficient which is a polynomial in ϕ_0. According to Hurst,[9] the coefficients of each power of ϕ_0 are divergent series for all values of α^2. This result is perhaps to be expected because the whole nature of the physical problem changes when α^2 changes sign, and this situation should not be capable of representation by a power series expansion with a nonvanishing radius of convergence (compare the remarks of Dyson[10] for the case of quantum electrodynamics). It is quite possible that the unusual result found in the calculation of A' is simply due to the fact that we have selected out a certain finite part of a divergent series. This divergent series as a whole may be meaningless, in which case the theory is meaningless; or it may be some sort of an asymptotic representation of an analytic function which is not expandable as a power series about the origin in the α^2 plane.

The order of magnitude of the corrections given by (42) is not to be taken too seriously; the particular form of the odd qualitative features in the calculation of A' is not to be taken seriously either, although the existence of some qualitative changes in the relation between the energy and the source strength does seem to be plausible. The present results therefore can be taken only as an indication that quantum corrections are relatively important for a nonlinear boson field with a large classical strength.

The author wishes to express his thanks to Dr. J. R. Oppenheimer for his interest and encouragement in this work. He is also indebted to Dr. M. Gell-Mann and Dr. K. V. Roberts for many stimulating discussions and for constructive criticism of the manuscript.

APPENDIX

It is the purpose of this Appendix to derive Eq. (16) for a one-dimensional source distribution by means of the WKB approximation for the normal mode frequencies. Consider such a source leading to a classical field $\phi_0(x)$ which depends only on the x-coordinate, but not on y and z, and assume periodic boundary conditions. The normal mode solutions (9) then take the form

$$\phi_n = \exp(i\omega_{n_1n_2n_3}t + ik_{n_1}y + ik_{n_2}z)u_{n_1}(x), \tag{A1}$$

[9] C. A. Hurst (private communication).

[10] F. J. Dyson, Phys. Rev. 85, 631 (1952).

where

$$[-\omega_{n_1n_2n_3}^2-(d^2/dx^2)+k_{n_2}^2+k_{n_3}^2+\kappa^2+3\alpha^2\phi_0^2]u_{n_1}(x)=0. \quad \text{(A2)}$$

The one-dimensional eigenvalue problem for the frequencies is therefore

$$[-(d^2/dx^2)+3\alpha^2\phi_0^2]u_n(x)=\Omega_n^2u_n(x), \quad \text{(A3)}$$

with

$$\omega_{n_1n_2n_3}=(\kappa^2+k_{n_2}^2+k_{n_3}^2+\Omega_{n_1}^2)^{\frac{1}{2}}. \quad \text{(A4)}$$

If the period of the functions is L, the zero-point energy is

$$\tfrac{1}{2}\sum\omega_n=\tfrac{1}{2}\left(\frac{L}{2\pi}\right)^2\int dk_2dk_3\sum_n(\kappa^2+k_2^2+k_3^2+\Omega_n^2)^{\frac{1}{2}}. \quad \text{(A5)}$$

The WKB approximation to the frequencies is

$$\int_R dx(\Omega_n^2-3\alpha^2\phi_0^2)^{\frac{1}{2}}+\beta_n=2\pi n, \quad \text{(A6)}$$

where R is the region of integration for which $\Omega_n^2>3\alpha^2\phi_0^2$. β_n is a phase factor which must be introduced if there are classical turning points for the given value of Ω_n^2. If ϕ_0^2 is a slowly varying function of position in the region of the turning point, the effects of β_n will be averaged out and β_n may be neglected in (A6). We shall treat this case first, and later modify it for the case in which there are discontinuities in ϕ_0^2. From (A6), the number of states in a range $d\Omega_n$ is given by

$$dn=\frac{2}{(2\pi)}\int_R dx\Omega_nd\Omega_n(\Omega_n^2-3\alpha^2\phi_0^2)^{-\frac{1}{2}}. \quad \text{(A7)}$$

The zero-point energy accordingly becomes

$$\tfrac{1}{2}\sum\omega_n=\frac{L^2}{(2\pi)^3}\int_{-\infty}^{\infty}dk_2dk_3\int_R dx\int_0^{\infty}\Omega d\Omega \times(\kappa^2+k_2^2+k_3^2+\Omega^2)^{\frac{1}{2}}(\Omega^2-3\alpha^2\phi_0^2)^{-\frac{1}{2}}. \quad \text{(A8)}$$

With the change of variable,

$$k_1^2=\Omega^2-3\alpha^2\phi_0^2,$$

this is seen to be identical with (16):

$$\tfrac{1}{2}\sum\omega_n=\frac{L^2}{2(2\pi)^3}\int dx\int_{-\infty}^{\infty}dk_1dk_2dk_3 \times(\kappa^2+k_1^2+k_2^2+k_3^2+3\alpha^2\phi_0^2)^{\frac{1}{2}}. \quad \text{(A9)}$$

We now turn to the case in which there are discontinuities in the function ϕ_0. For simplicity, we assume

$$\begin{aligned}3\alpha^2\phi_0^2&=0, \quad -L/2<x<-a;\\ &=U, \quad -a<x<a;\\ &=0, \quad a<x<L/2.\end{aligned} \quad \text{(A10)}$$

We further assume that the distance a is so large that for all except a negligible fraction of the frequencies we may assume

$$\begin{aligned}(\Omega_n^2-U)^{\frac{1}{2}}a\gg1, \quad \Omega_n^2>U;\\ (U-\Omega_n^2)^{\frac{1}{2}}a\gg1, \quad \Omega_n^2<U.\end{aligned} \quad \text{(A11)}$$

Under this assumption the phase shifts β_n for $\Omega_n^2>U$ will average out in a more or less random fashion, while those for $\Omega_n^2<U$ will not. Distinguishing the phase shifts for even and odd functions of x, we find for $\Omega_n^2<U$

$$\begin{aligned}\beta_{n\,\text{odd}}&\cong2\tan^{-1}[\Omega_n^2/(U-\Omega_n^2)]^{\frac{1}{2}},\\ \beta_{n\,\text{even}}&\cong-\pi+2\tan^{-1}[\Omega_n^2/(U-\Omega_n^2)]^{\frac{1}{2}}.\end{aligned} \quad \text{(A12)}$$

On the average,

$$\bar{\beta}_n\cong-\pi/2+2\tan^{-1}[\Omega_n^2/(U-\Omega_n^2)]^{\frac{1}{2}}. \quad \text{(A13)}$$

From (A6) we see that each Ω_n used in calculating the zero-point energy must be shifted by an amount $\delta\Omega_n$ given by

$$\int_R dx\frac{\Omega_n\delta\Omega_n}{(\Omega_n^2-U)^{\frac{1}{2}}}+\bar{\beta}_n=0. \quad \text{(A14)}$$

The resulting shift in the zero-point energy itself is then

$$\tfrac{1}{2}\sum\delta\omega_n=\frac{L^2}{(2\pi)^3}\int dk_2dk_3\int_R dx\int_0^{U^{\frac{1}{2}}}\Omega d\Omega(\Omega^2-U)^{-\frac{1}{2}} \times\Omega\delta\Omega(\kappa^2+k_2^2+k_3^2+\Omega^2)^{-\frac{1}{2}}. \quad \text{(A15)}$$

Combining these results, we obtain

$$\tfrac{1}{2}\sum\delta\omega_n=\left(\frac{L}{2\pi}\right)^2\lim_{K\to\infty}\int_0^K kdk\int_0^{U^{\frac{1}{2}}}\Omega d\Omega \times\left\{\frac{\pi}{2}-2\tan^{-1}\left(\frac{\Omega^2}{U-\Omega^2}\right)^{\frac{1}{2}}\right\}(\kappa^2+k^2+\Omega^2)^{-\frac{1}{2}}. \quad \text{(A16)}$$

Carrying out the k integration and using the fact that

$$\int_0^{U^{\frac{1}{2}}}\Omega d\Omega\left\{\frac{\pi}{2}-2\tan^{-1}\left(\frac{\Omega^2}{U-\Omega^2}\right)^{\frac{1}{2}}\right\}=0,$$

we find

$$\tfrac{1}{2}\sum\delta\omega_n=-\left(\frac{L}{2\pi}\right)^2\int_0^{U^{\frac{1}{2}}}\Omega d\Omega \times\left\{\frac{\pi}{2}-2\tan^{-1}\left(\frac{\Omega^2}{U-\Omega^2}\right)^{\frac{1}{2}}\right\}(\kappa^2+\Omega^2)^{\frac{1}{2}}. \quad \text{(A17)}$$

Making the substitution,

$$\Omega=U^{\frac{1}{2}}\sin(\tfrac{1}{2}\theta+\tfrac{1}{4}\pi),$$

we find after some simple manipulations

$$\tfrac{1}{2}\sum\delta\omega_n=\left(\frac{L}{2\pi}\right)^2\frac{U^2}{8}\int_{-\pi/2}^{\pi/2}\theta\cos\theta I(\sin\theta)d\theta, \quad \text{(A18)}$$

$$I(\sin\theta)=\sin\theta/[\kappa^2+\tfrac{1}{2}U+\tfrac{1}{2}U\sin\theta)^{\frac{1}{2}}+(\kappa^2+\tfrac{1}{2}U-\tfrac{1}{2}U\sin\theta)^{\frac{1}{2}}].$$

From the form of $I(\sin\theta)$, we are able to place the following upper and lower bounds on the change in the zero-point energy

$$\tfrac{1}{2}\sum\delta\omega_n<\left(\frac{L}{2\pi}\right)^2\frac{\pi U^2}{32}[(\kappa^2+U)^{\frac{1}{2}}+\kappa]^{-1}$$
$$>\left(\frac{L}{2\pi}\right)^2\frac{\pi U^2}{32}\cdot\tfrac{1}{2}(\kappa^2+U/2)^{-\frac{1}{2}}. \quad \text{(A19)}$$

For $U\gg\kappa^2$, $I(\sin\theta)$ reduces to

$$I(\sin\theta)=(2/U)^{\frac{1}{2}}\sin(\theta/2),$$

which gives

$$\tfrac{1}{2}\sum\delta\omega_n=(L/2\pi)^2(\pi U^{\frac{3}{2}}/32)(0.806). \quad \text{(A20)}$$

Under this condition, the surface energy per unit area is then approximately

$$(0.806/2^7\pi)U^{\frac{3}{2}}=[(3\alpha^2)^{\frac{3}{2}}(0.806)/2^7\pi]\phi_0^3. \quad \text{(A21)}$$

Since in actual practice ϕ_0 will not have sharp discontinuities, the actual surface energy will be somewhat less than (A21).

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The Angular Distribution of Prompt Neutrons Emitted in Fission

J. S. Fraser
Chalk River Laboratory, Atomic Energy of Canada Limited, Chalk River, Ontario, Canada
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The angular distribution, relative to the direction of motion of the fragments, of the prompt fast neutrons emitted in the thermal neutron fission of U^{233}, U^{235}, and Pu^{239} has been measured. Collimated fission fragments were selected in energy in a gridded ionization chamber and coincident prompt neutrons in a given direction were counted by proton recoils in an electron collecting chamber filled with methane. The distributions obtained selecting only light fragments have been compared with curves computed on the basis of the evaporation of the neutrons from the moving fragments of the most probable mode. Reasonably good agreement is obtained if one postulates that in the fission of each of the three nuclides studied, the neutron emission probability is about thirty percent greater for the light fragment than for the heavy one. An upper limit of 4×10^{-14} sec following fission may be placed on the time of emission of the neutrons.

I. INTRODUCTION

In 1945, Wilson[1] measured the correlation between the direction of the prompt neutrons and the fragments in the fission process. The results were consistent with the view that the neutrons are evaporated isotropically in the frame of reference of the moving fragments.

The angular dependence of coincidences between fission neutrons has been studied by De Benedetti *et al.*,[2] who concluded that there are twice as many neutron pairs emitted by opposite fragments than by the same fragment.

Prior to the work of Leachman[3] on the ionization yields of fission fragments, it was suggested by Brunton and Hanna[4] that preferential emission of neutrons from one group of fragments may contribute to the disagreement between the distributions in fission fragment mass derived from the ionization and chemical yield measurements. This discrepancy has been shown recently by Leachman[3,5] to be due to a variation in ionization yield with fragment mass and to a dispersion arising from instrumental errors and poor resolution.

The experiment to be described, essentially an extension of Wilson's experiment, was designed to investigate the possibility of preferential emission of neutrons by one of the fragments. It was also found possible to place a much lower limit on the time of emission of the neutrons than the figure of 8×10^{-9} sec given by Snyder and Williams.[6]

II. APPARATUS

A. The Fission Chamber

A cross section of the fission chamber is shown schematically in Fig. 1. A layer of fissile material, approximately 180 μg/cm^2 thick was deposited on a 2.5-cm diameter nickel or aluminum plate which was then cemented to the cathode. Over the source was placed a 0.081-in. thick Dural plate with $\frac{1}{32}$-in. holes drilled in an hexagonal array to act as a collimator. The average angle of emission of the fragments was, therefore, approximately 9° from the normal. The position of the neutron counter was such that the angular uncertainty of the neutron direction was equal to that of the fission fragments passing through the collimator. A

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