Elastic Scattering of Protons and Neutrons by Helium*

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Proton-alpha elastic scattering experiments at 5.81 and 9.48 Mev are analyzed in terms of phase shifts, and the results, together with earlier ones at lower energies, are used to compute the logarithmic derivatives of the wave functions at a radius of 2.9×10^{-13} cm. These logarithmic derivaties, for $P_{1/2}$ and $P_{3/2}$ states, are found to be linear functions of the energy. From the behavior of the logarithmic derivatives the widths and positions of the $P_{1/2}$ and $P_{3/2}$ energy levels are determined, the most striking results being the large splitting of the two levels and the very broad width of the $P_{1/2}$ level. These results are compared with neutron-alpha experiments and are found to be in sufficiently good agreement to support the conclusions about the $P_{1/2}$ level which differ from those previously expected. Small negative D wave phase shifts are found in the proton-alpha experiments which also show evidence of an inverted doublet spin-orbit splitting.

I. INTRODUCTION

A CONSIDERABLE amount of experimental data[1–5] on the elastic scattering of nucleons by alpha-particles has become available in the last few years. Analysis and interpretation of some of this data have been presented by Critchfield and Dodder[6] in terms of phase shifts and by Adair[3] in terms of resonances associated with virtual energy levels of the compound nuclei He^5 and Li^5. From the information then available it has learned[3] that the S wave phase shifts are very nearly those expected in scattering from an impenetrable sphere of radius 2.6×10^{-13} cm, while the $P_{3/2}$ phase shifts correspond to a resonance level at a few Mev positive energy. The experiments did not make possible a precise determination of the location of the $P_{1/2}$ level or its width, but it was ascertained that any $P_{1/2}$ level must lie at least several Mev above the $P_{3/2}$ level.

It is the purpose of this investigation to extend and amplify the above interpretations by making use of more recent data,[4,5] especially to find the position and width of the $P_{1/2}$ state and to find out something about the D states.

II. PHASE SHIFT ANALYSIS OF PROTON-ALPHA ELASTIC SCATTERING

The differential cross section for elastic scattering as a function of angle scattering, incident energy, and phase shifts is given by the following expression (the notation is that of reference 6).

$$k^2\sigma(\theta)=\left|-\frac{\eta}{2}\csc^2\frac{\theta}{2}\exp\left(i\eta\ln\csc^2\frac{\theta}{2}\right)+\sum_{l=0}^{\infty}[(l+1)\exp(i\delta_l^+)\sin\delta_l^+ +l\exp(i\delta_l^-)\sin\delta_l^-]\exp(i\phi_l)P_l(\cos\theta)\right|^2+\left|\sum_{l=0}^{\infty}[\exp(i\delta_l^-)\sin\delta_l^--\exp(i\delta_l^+)\sin\delta_l^+]\exp(i\phi_l)\sin\theta P_l'(\cos\theta)\right|^2. \quad (1)$$

Because of the complexity of this expression when many angular momenta are included, numerical calculations were made using the I.B.M. card programmed electronic calculators available at the Los Alamos Scientific Laboratory. The criterion used in fitting the experimental data was the same as that of reference 6; namely, minimization with respect to the phase shifts of the sum of the squares of the percentage differences between the experimental and calculated values of the cross sections at the various scattering angles. The method of satisfying this in the present work is different, however, and is the following. Using a set of trial phase shifts, the cross section was computed at each angle where it was measured experimentally. At the same time the gradient with respect to the phase shifts of the sum of the squares of the percent differences between observed and calculated cross sections was computed. A new set of trial phase shifts was then selected by changing the old ones in a direction opposite to that of the gradient, the magnitude of the change being proportional to the magnitude of the gradient. This iterative procedure was carried out until the magnitude of the gradient was a few percent of its value when the agreement with the data was first of the same order of magnitude as the given experimental errors. The calculations were made for three cases: S, $P_{1/2}$, and $P_{3/2}$ waves only, S, $P_{1/2}$, $P_{3/2}$, and D waves (the $D_{5/2}$ and $D_{3/2}$ phase shifts were constrained to be identical), and S, $P_{1/2}$, $P_{3/2}$, $D_{3/2}$, and $D_{5/2}$ waves.

The nature of the method is such that there is always a possibility that there are solutions which fit

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[1] Freier, Lampi, Sleator, and Williams, Phys. Rev. **75**, 1345 (1949).
[2] Bashkin, Mooring, and Petree, Phys. Rev. **82**, 378 (1951).
[3] R. K. Adair, Phys. Rev. **86**, 155 (1952).
[4] T. M. Putnam, University of California Radiation Laboratory Report UCRL-1447, unpublished.
[5] Kreger, Kerman, and Jentschke, Phys. Rev. **86**, 593 (1952).
[6] C. L. Critchfield and D. C. Dodder, Phys. Rev. **76**, 602 (1949).

TABLE I. $E=9.48$ Mev. S and P phase shifts only.

	A		B		C		D	
	$\delta_0=-57.5°$ $\delta_1^+=113.8°$ $\delta_1^-=61.7°$		$\delta_0=-57.3°$ $\delta_1^+=85.1°$ $\delta_1^-=137.2°$		$\delta_0=-90.6°$ $\delta_1^+=126.7°$ $\delta_1^-=84.95°$		$\delta_0=-90.6°$ $\delta_1^+=101.6°$ $\delta_1^-=143.3°$	
θ	$k^2\sigma$ calc.	% dev.	$k^2\sigma$ calc.	% dev.	$k^2\sigma$ calc.	% dev.	$k^2\sigma$ calc.	% dev.
12.2°	29.27	−11.89	29.36	−11.62	29.59	−10.93	29.61	−10.85
52.0°	6.292	− 6.88	6.281	− 7.04	6.328	− 6.34	6.325	− 6.39
94.5°	1.125	−10.28	1.123	−10.44	1.134	− 9.55	1.134	− 9.61
123.8°	0.7804	− 1.71	0.7806	− 1.68	0.7850	− 1.13	0.7850	− 1.14
174.5°	2.694	− 4.25	2.692	− 4.32	2.701	− 4.00	2.701	− 4.00
rms % dev.		7.94		7.94		7.32		7.32

the experimental values within the experimental error but which were not found because a combination of trial phase shifts sufficiently close to such a solution was not used. However, the procedure was started with a number of initial sets of values of the phase shifts and always converged towards one of a limited number of solutions.

The solutions found for the 9.48-Mev case with S and P waves only are given in Table I. Solutions A and B are mathematically equivalent[7] inverted and normal doublets, as are C and D. In the lower energy work solutions equivalent to all of these were found, but two of them, equivalent to the present C and D, were far worse than the others at small angles. The reasons that this is not so in the present work are that the low angles are given weight in determining the solution, which was not done in the preliminary solutions in the lower energy work, and that the presence of D waves is required by the present experiments, preventing any solution with P waves only from approaching the observed values too closely. The solutions A and C, corresponding to inverted doublets, now modified by the addition of a single D wave phase shift, are given in Table II. It is seen that the presence of D wave phase shifts is definitely indicated. In Table III are given the solutions with $D_{5/2}$ and $D_{3/2}$ phase shifts split. The experimental results do not definitely require splitting, but as discussed later, the direction of splitting indicated is reasonable. In Table IV, for solution A, the results are given at each angle at which the cross section was measured experimentally.

At 5.83 Mev the analysis was not as exhaustive. Initial values of phase shifts used were those given for S, $P_{1/2}$, and $P_{3/2}$ by Kreger *et al.*[5] Eleven angles were used to improve their preliminary fit. Table V gives the result of this. Solutions with a single D phase shift are given in Table VI, while Table VII gives the results with all five phase shifts. In the analysis for split D waves 26 angles were used. It is seen that presence of D waves, and to a lesser degree their splitting, is indicated by this experiment.

[7] The two solutions are found from the parameters of reference 6 by taking the plus and minus signs in the following formulas:
$2\delta_1^+=\tan^{-1}[\rho\sin\beta/(\rho\cos\beta+3)]$
$\pm\cos^{-1}[(\rho^2+6\rho\cos\beta+12)/4(\rho^2+6\rho\cos\beta+9)^{\frac{1}{2}}]$,
$2\delta_1=\sin^{-1}[\rho\sin\beta-2\sin2\delta_1^+]$.

It is difficult to set precise limits on the accuracy of the phase shifts. First, if the S phase shift, for example, is varied while the others are held fixed, it might be found that it could be changed only by a few tenths of a degree without destroying the agreement with the experimental data. However, by changing it together with the other phase shifts, it might be possible to change it by say two degrees without damage to the fit. Second, were the experimental cross sections high by say 5 percent at all angles, the S wave phase shifts might be in error, while were the forward angles subject to experimental error, the D wave phase shifts would be in doubt. It seems more pertinent to examine a phase shift as a function of energy for smoothness and reasonableness. However, as a rough estimate based on an examination of the changes in the phase shifts as the procedure described above converged to the final fit, the P phase shifts are accurate to within 2°, while the S phase shifts are somewhat less accurate (5°). Comparison of the results for D wave phase shifts locked and split shows that they are definitely small and negative. As discussed later, this seems reasonable, and even the direction of splitting seems reasonable. But until experiments are performed at still higher energies, conclusions about the D states must remain preliminary, and the values quoted for the D wave

TABLE II. $E=9.48$ Mev. $\delta(D_{5/2})=\delta(D_{3/2})$.

	A		C	
	$\delta_0=-65.4°$ $\delta_1^+=106.1°$ $\delta_1^-=47.4°$ $\delta_2=-7.0°$		$\delta_0=-94.9°$ $\delta_1^+=117.2°$ $\delta_1^-=78.5°$ $\delta_2=-4.03°$	
θ	$k^2\sigma$ calc.	% dev.	$k^2\sigma$ calc.	% dev.
12.2°	31.77	−4.37	30.98	−6.73
12.8°	28.39	−1.93	27.78	−4.03
19.0°	15.12	2.18	15.44	4.33
31.3°	10.29	−1.22	10.95	5.09
52.0°	6.341	−6.15	6.787	0.45
63.7°	4.479	−7.72	4.718	−2.80
72.7°	3.255	−6.39	3.353	−3.57
83.8°	2.045	−1.73	2.027	−2.59
94.5°	1.222	−2.51	1.170	−6.73
114.4°	0.6251	−2.32	0.6473	1.14
123.8°	0.7408	−6.70	0.8042	1.29
149.4°	1.889	−2.96	1.891	−2.89
174.5°	2.867	1.90	2.703	−3.91
rms % dev.		4.29		3.98

TABLE III. $E=9.48$ Mev. D phase shifts split.

	A		C	
	$\delta_0 = -70.8°$		$\delta_0 = -98.9°$	
	$\delta_1^+ = 106.2°$		$\delta_1^+ = 115.3°$	
	$\delta_1^- = 53.8°$		$\delta_1^- = 76.6°$	
	$\delta_2^+ = -5.03°$		$\delta_2^+ = -7.16°$	
	$\delta_2^- = -8.84°$		$\delta_2^- = -2.95°$	
θ	$k^2\sigma$ calc.	% dev.	$k^2\sigma$ calc.	% dev.
12.2°	31.48	−5.22	30.96	−6.81
12.8°	28.22	−2.52	27.77	−4.08
19.0°	15.51	4.81	15.48	4.58
31.3°	10.83	3.93	10.98	5.44
52.0°	6.662	−1.41	6.808	0.76
63.7°	4.653	−4.15	4.745	−2.25
72.7°	3.339	−3.98	3.387	−2.57
83.8°	2.059	−1.07	2.067	−0.67
94.5°	1.213	−3.31	1.204	−3.98
114.4°	0.6416	0.25	0.6506	1.66
123.8°	0.7737	−2.56	0.7950	0.12
149.4°	1.900	−2.41	1.899	−2.44
174.5°	2.817	0.12	2.760	−1.92
rms % dev.		3.18		3.44

phase shifts should be thought of as correct in little more than order of magnitude.

III. ENERGY DEPENDENCE OF PROTON-ALPHA PHASE SHIFTS

The energy dependence of the phase shifts is conveniently examined in terms of the energy dependence of the logarithmic derivative Y of the wave function evaluated at a radius taken equal to, or greater than, the distance at which specifically nuclear events become negligible. The logarithmic derivative of a wave function at a given radius plays a central role in the formal description of nuclear reactions of Wigner and Eisenbud[8] being just the reciprocal of their R function if the simplest choice of boundary conditions defining their internal wave functions is made. In the case of the one level approximation the logarithmic derivative is then a linear function of the energy. Breit and Bouricius[9] have discussed the behavior of Y at various radii for the wave function of the S state of two protons. They find that it is possible to find radii at which $\partial Y/\partial E$ can have a wide range of positive and negative values which are independent of energy over the energy region extending up to 3.5 Mev. In the nucleon-alpha P states it is also possible to find a radius at which $\partial Y/\partial E$ is energy independent. However, the range of radii for which this is possible seems to be less than it is in the case of the S state of two protons. The radius a used in the present calculations is 2.9×10^{-13} cm, the same as that of Adair; a radius of 3.5×10^{-13} cm, on the other hand, gave an energy dependent $\partial Y/\partial E$.

The value of the logarithmic derivative at a radius a is given in terms of δ, the phase shift, and F and G, the regular and irregular radial wave functions in the external field, by the well-known formula

$$aY=-\frac{ka}{1+F/G\cot\delta}+\frac{kaF'}{F}. \tag{2}$$

The prime represents differentiation with respect to kr, where k is the wave number. For the case where the external field is Coulombic, aY can be expressed in terms of functions tabulated by Breit and collaborators,[10] and for orbital angular momentum L one obtains

$$aY=\frac{\phi_L^*}{\phi_L}-\frac{(2L+1)}{(2L+1)C_L^2\rho^{2L+1}\phi_L^2\cot\delta_L+\phi_L\theta_L}. \tag{3}$$

Equation (2) can be solved for δ_L, giving

$$\delta_L=\tan^{-1}\frac{ka/(F^2+G^2)}{aY-ka[(FF'+GG')/(F^2+G^2)]}-\tan^{-1}\frac{F}{G}.$$

TABLE IV. $E=9.48$ Mev.

A:	$\delta_0=-70.5°$, $\delta_1^+=105.8°$, $\delta_1^-=53.3°$, $\delta_2^+=-3.83°$, $\delta_2^-=-8.42$				
θ	$k^2\sigma$ calc.	% dev.	θ	$k^2\sigma$ calc.	% dev.
12.2°	30.57	−7.97	94.5°	1.216	−3.01
12.8°	27.27	−5.44	99.6°	0.9530	1.71
15.3°	19.62	2.21	104.7°	0.7765	−4.02
15.9°	18.54	1.20	106.8°	0.7281	−0.31
18.4°	15.55	0.13	111.7°	0.6682	−0.25
19.0°	15.06	1.79	114.4°	0.6652	3.94
21.5°	13.57	−0.93	116.4°	0.6760	0.91
25.2°	12.18	2.79	123.8°	0.8032	1.11
27.7°	11.49	4.42	128.3°	0.9388	−3.12
31.3°	10.64	2.13	132.7°	1.105	−3.13
33.9°	10.09	2.47	134.5°	1.181	−4.05
37.4°	9.377	4.09	141.2°	1.492	−2.75
40.0°	8.867	−0.25	145.4°	1.699	3.93
43.5°	8.196	0.44	149.4°	1.897	−2.55
49.5°	7.076	1.65	151.1°	1.980	−0.80
52.0°	6.622	−1.99	157.3°	2.264	−2.99
61.3°	5.025	0.46	159.0°	2.335	−0.79
63.7°	4.642	−4.38	165.0°	2.553	2.76
72.7°	3.335	−4.09	166.7°	2.604	3.23
80.6°	2.385	−0.30	172.6°	2.738	2.04
83.8°	2.058	−1.13	174.2°	2.761	−1.87
86.1°	1.844	−2.97	rms % dev.		2.87

TABLE V. $E=5.81$ Mev. S and P phase shifts only.

	$\delta_0=-46.6°$, $\delta_1^+=112.5°$, $\delta_1^-=40.7°$	
θ	$k^2\sigma$ calc.	% dev.
16.22°	17.50	−3.24
28.64°	8.509	−3.01
43.32°	6.186	1.68
55.28°	4.708	0.43
66.88°	3.426	−0.12
80°	2.233	−1.62
95°	1.321	−1.22
115°	0.9196	−0.74
125°	1.017	−2.37
140°	1.390	2.74
154°	1.809	−2.08
rms % dev.		2.01

[8] E. P. Wigner and L. Eisenbud, Phys. Rev. **72**, 29 (1947). See also Feshbach, Peaslee, and Weisskopf, Phys. Rev. **71**, 145 (1947).

[9] G. Breit and W. G. Bouricius, Phys. Rev. **75**, 1029 (1949). Y in the present article is equal to Y/r of this reference.

[10] Bloch, Hull, Broyles, Bouricius, Freeman, and Breit, Revs. Modern Phys. **23**, 147 (1951). References to earlier work and to other tables of coulomb functions are given here.

Using the simple boundary condition in the Wigner-Eisenbud formalism, $Y=1/R$, and the aY term becomes, in the one-level approximation, just $(a/\gamma_\lambda^2)(E_\lambda-E)$ where E_λ is the characteristic energy of the level and γ_λ^2 is the reduced width of the level; the term $-ka(FF'+GG')/(F^2+G^2)$ becomes $a\Delta_\lambda/\gamma_\lambda^2$ where Δ_λ is the level shift. This Δ_λ differs by $\gamma_\lambda^2 L/a$ from that evaluated by Thomas[11] because of the different choice of boundary conditions defining the E_λ and the resulting different relationship between R and Y.

Graphs of aY as a function of the center-of-mass energy for the $P_{1/2}$ and $P_{3/2}$ states in Li^5 are given in Fig. 1. The points for energies less than or equal to 3.5 Mev were obtained using the phase shift analysis of Critchfield and Dodder.[6] The two most noticeable features are the markedly different slopes, corresponding to different reduced widths, γ_λ^2, and the large splitting between the E_λ's for the two states. This is much greater than that previously indicated, and these results may be hard to reconcile with experiments showing two sharply defined P levels a few Mev apart.[12,13] The best values of E_λ and γ_λ^2 are: $P_{1/2}$: $E_\lambda=2.7$ Mev, $\gamma_\lambda^2=85\times10^{-13}$ Mev cm and $P_{3/2}$: $E_\lambda=-3.3$ Mev, $\gamma_\lambda^2=20.0\times10^{-13}$ Mev cm. The $P_{3/2}$ E_λ is different from that of Adair because of the different Δ_λ. Figure 2 is a corresponding graph for the S waves. It is seen that due to the large uncertainties in the values of the points, it is hard to determine exactly the energy dependence of aY.

Wigner and Eisenbud have shown that under very general assumptions regarding the interaction in the internal region, $R=\sum_\lambda \gamma_\lambda^2/(E_\lambda-E)$. From this it is seen that $\partial Y/\partial E=(\partial 1/R)/\partial E$ is always a negative quantity. The graph for aY for the S wave shows either an approximately constant value or a slightly negative slope. This is to be interpreted that the S wave scattering is nearly that expected from an impenetrable sphere and that there is no evidence for a nearby S resonance level. This agrees with the conclusion of Adair.

TABLE VI. $E=5.81$ Mev. $\delta(D_{5/2})=\delta(D_{3/2})$.

$\delta_0=-47.6°$, θ	$\delta_1^+=112.1°$, $k^2\sigma$ calc.	$\delta_1^-=39.9°$, $\delta_2=-0.52°$ % dev.
16.22°	17.91	−0.96
28.64°	8.643	−1.49
43.32°	6.226	2.34
55.28°	4.714	0.56
66.88°	3.416	−0.40
80°	2.220	−2.19
95°	1.315	−1.64
115°	0.9271	0.06
125°	1.029	−1.28
140°	1.402	3.65
154°	1.818	−1.56
rms % dev.		1.75

TABLE VII. $E=5.81$ Mev. D phase shifts split.

$\delta_0=-48.2°$, θ	$\delta_1^+=112.3°$, $k^2\sigma$ calc.	$\delta_1^-=39.8°$, % dev.	$\delta_2^+=-0.42°$, θ	$\delta_2^-=-1.7°$ $k^2\sigma$ calc.	% dev.
16.22°	17.98	−0.58	72.6°	2.863	−1.98
18.70°	13.82	−1.14	80°	2.230	−1.75
21.80°	11.20	−0.50	85°	1.870	−1.42
24.93°	9.747	−0.18	95°	1.326	−0.85
28.64°	8.675	−1.12	105°	1.019	−2.14
31.11°	8.145	−0.02	115°	0.9317	0.57
35.05°	7.448	0.39	125°	1.026	−1.54
37.25°	6.802	−3.20	130°	1.124	1.06
43.32°	6.243	2.62	134°	1.221	0.64
49.30°	5.463	2.63	140°	1.384	2.34
55.28°	4.726	0.81	145°	1.530	−1.02
61.13°	4.047	0.44	150°	1.674	−0.12
66.88°	3.426	−0.12	154°	1.785	−3.36
			rms % dev.		1.58

At both of the present energies the D wave phase shifts are small and negative and show a splitting, the $D_{5/2}$ wave being less negative than the $D_{3/2}$. The potential scattering associated with a radius of $a=2.9\times10^{-13}$ cm is small and negative, and any nonpotential effect caused by an inverted doublet type of effective spin-orbit forces similar to those acting in the P states would make the $D_{5/2}$ phase shift more positive than the $D_{3/2}$. The apparent agreement with this picture is not conclusive, however, since a distant level with a large width could produce more effect at a given energy than a closer one with a narrow width, and the fact that the widths can be quite different is seen from the P-state widths.

The $D_{?/2}$ state in the mirror nucleus He^5 is also effective in producing the resonance in the D-T reaction

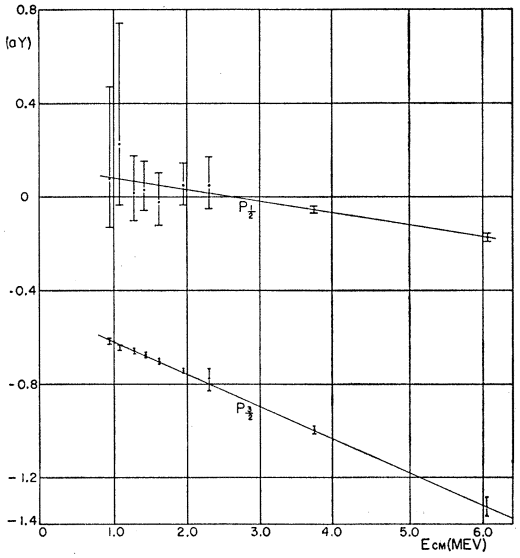

FIG. 1. Logarithmic derivative times radius evaluated at a radius of 2.9×10^{-13} cm for the P states in proton-alpha elastic scattering as a function of center-of-mass energy. For points below 3.5 Mev, the errors are those of reference 6. Above 3.5 Mev, an error of $\pm1°$ is shown.

[11] R. G. Thomas, Phys. Rev. **80**, 136 (1950).
[12] W. J. Leland and H. M. Agnew, Phys. Rev. **82**, 558 (1951).
[13] W. E. Titterton and T. A. Brinkley, Proc. Phys. Soc. (London) **A64**, 212 (1951).

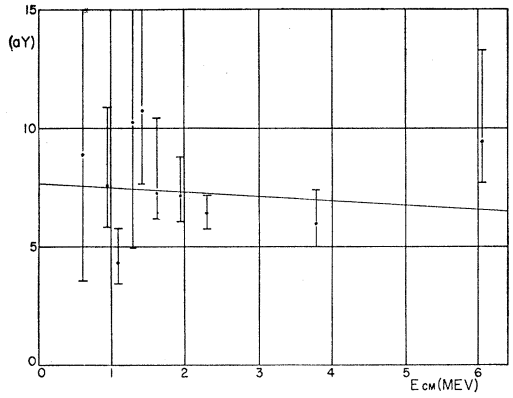

FIG. 2. Logarithmic derivative times radius evaluated at a radius of 2.9×10^{-13} cm for the S state in proton-alpha-elastic scattering as a function of center-of-mass energy. For points below 3.5 Mev, the errors are those of reference 6. Above 3.5 Mev, an error of $\pm2°$ is shown.

cross section at 165-kev triton energy, which can be identified with a state of $J=3/2$ and even parity.[14] The values of E_λ and of the reduced widths which agree with the energy dependence of the D-T cross section are not uniquely determined; however, the range of values of E_λ and the neutron width γ_λ^2 are in agreement with a small deviation from potential scattering for the n-He^4 $D_{3/2}$ phase shift at energies corresponding to the 5.81 and 9.48 p-He^4 energies. If the $D_{5/2}$ level does lie lower there should be a resonance associated with this state lying between 10 Mev and 22 Mev, the energy corresponding to the D-T reaction resonance. This could be detected either by an experiment measuring the total cross section for n-He^4 elastic scattering as a function of energy over the region, or by an angular distribution of the elastic scattering of protons or neutrons by helium at an energy sufficiently close to the resonant energy. A determination of the proton-alpha angular distribution at 32 Mev has been made[15] and is now being investigated.

IV. NEUTRON-ALPHA SCATTERING

Since the width of the $P_{1/2}$ level is quite different from the one chosen by Adair, the predicted contribution to the total cross section for the n-He^4 elastic scattering from this level at energies above 2 Mev would be somewhat different. Using Adair's assumptions that the E_λ's differ for the mirror nuclei by about 1 Mev, the difference being chosen so that the $P_{3/2}$ phase shift for the n-He^4 scattering will be 90° near the maximum cross section, and that the γ_λ^2 are the same for the two nuclei, the n-He^4 elastic scattering total and differential cross sections were computed as a function of energy. Figure 3 shows the total cross section together with the experimental points and Adair's[3] calculated total cross section. From this it is clear that the experiment is in better agreement with the broader $P_{1/2}$ level, corroborating the results from the p-He^4 scattering. The calculations at 14.1 Mev, using S and P phase shifts obtained from extrapolating the lower energy values, assuming the $D_{3/2}$ phase shift to be that due to potential scattering, and assuming the $D_{5/2}$ phase shift to be zero, gives a total cross section of 1.05 barns as compared with an experimental value of 1.02 ± 0.03 barns.[16] The angular distribution is in qualitative agreement with preliminary experiments of Seagrave.[17] The choice of zero for the $D_{5/2}$ phase shift was made assuming that the effect of a $D_{5/2}$ level would just cancel the potential scattering effect at this energy. This is in rough agreement with the trend shown by the p-He^4 experiments at 5.81 and 9.48 Mev.

V. CONCLUSION

It is seen that the experiments on the elastic scattering of protons and neutrons by He^4 are very well explained on the basis of the resonance theory of nuclear interactions, and afford considerable information on the level scheme of the He^5 and Li^5 compound nuclei. Due to the wide spacing of the levels they can be studied almost individually, and due to their large widths their shapes can be investigated. These seem so far to be quite well described by the expressions of Wigner and Eisenbud and others. The widths of the P state levels are of particular interest, the $P_{3/2}$ level being just about as broad as expected from the single particle model, and the $P_{1/2}$ level being surprisingly broader as if due either to some type of "surface" reaction where the internal wave function is concentrated closer to the boundary than is a particle in a potential well, or to a much smaller internal region for the interaction. Fortunately it does not seem impossible to investigate the dependence of the five nucleon

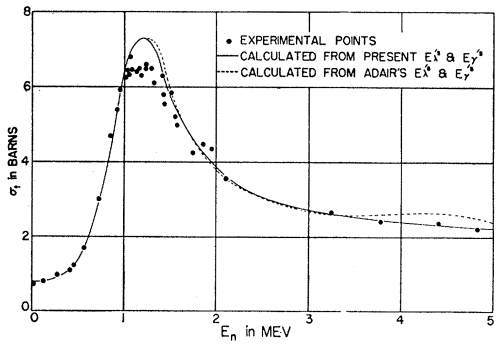

FIG. 3. Neutron-alpha total elastic cross section as a function of neutron energy in the laboratory system.

[14] Argo, Adair, Agnew, Hemmendinger, Leland, and Taschek, Phys. Rev. 87, 205 (1952); Argo, Taschek, Agnew, Hemmendinger, and Leland, Phys. Rev. 87, 612 (1952).

[15] Bruce Cork, University of California Report UCRL 1673, unpublished.

[16] J. Coon, private communication.

[17] John Seagrave, private communication.

system on the nucleon-nucleon forces. It is hoped that such an investigation will be able to explain the observed results. Further experiments at higher energies should reveal in a similar manner detailed information about the higher levels.

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Iterative Procedures and the Helium Wave Equation

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In an attempt to solve the helium wave equation exactly, study of a 180-point mesh shows the need for an improved difference operator to approximate the Laplacian. Such an operator has been developed, allowing one to express the value of the function at one point in terms of the values at 26 neighboring points (in three dimensions).

SINCE the wave equation for the ground state of the helium atom is nonseparable, analytic methods for obtaining a solution have been so far unsuccessful, and it has seemed worthwhile to try to solve the equation by numerical methods. If such a solution can be obtained, it will demonstrate whether or not the formulation of the many-body problem of quantum mechanics is correct. Also, a solution for the first excited state ($^3S_1$) would afford an opportunity to check the theory of hyperfine structure, using the spectrum of He^3 I.

Until a successful analog machine for partial differential equations in more than two dimensions is developed, it is probably necessary to replace the continuum of points by a mesh, and the differential equation by a difference equation. A prototype calculation was made[1] in one dimension for the hydrogen radial wave equation, and we shall show how these methods can be extended to several dimensions, with prospects of considerable accuracy.

The wave equation that is to be solved is of the form

$$\nabla^2\psi+\frac{1}{z}\frac{\partial\psi}{\partial z}+\frac{1}{4r}(E-V)\psi=0, \tag{1}$$

where, if r_1 and r_2 are the electron-nucleus distances and θ is the angle between the corresponding radius vectors, the coordinates x, y, and z are given by the relations $4x=2r_1r_2\cos\theta$, $4y=r_1^2-r_2^2$, $4z=2r_1r_2\sin\theta$. (Note that $z\geq 0$.) The radius in this system is $r=\frac{1}{4}(r_1^2+r_2^2)$, and V is the potential energy. If the substitution $\psi=Wz^{-\frac{1}{2}}$ is made, Eq. (1) is transformed into the equation

$$\nabla^2W+\left(\frac{E-V}{4r}+\frac{1}{4z^2}\right)W=0, \tag{2}$$

which is of the type $\nabla^2u+f(xyz)u=0$, where f is singular along the $+x$ axis and the whole y axis. However, we expect ψ to be finite everywhere and, consequently, W to be zero when $z=0$. Also, the solution is to be symmetric with respect to exchange of the electron space coordinates, so that $\psi(-y)=\psi(y)$. Hence the boundary conditions are

$$W=0, \quad z=0,$$
$$\partial W/\partial y=0, \quad y=0,$$
$$W\to 0, \quad r\to\infty.$$

In an attempt to construct an approximate solution of Eq. (2), the differential equation was replaced by a difference equation. This was done by equating the second partial derivative in each direction to the corresponding second partial difference, i.e., $\partial^2W/\partial x^2\cong\Delta_{xx}W$. A coarse mesh consisting of 180 points was chosen as follows:

$$x=0,\pm1,\pm3,\pm7,\pm15; \quad y=0,1,3,7,15; \quad z=1,3,7,15.$$

(W was taken to equal zero for values of x, y, or z equal to 31.) Then the iterative procedure of Liebmann,[2] regarding each point as influenced by its six neighbors along the axes, was applied, and the solution of the 180-point mesh was obtained.[3] The eigenvalue turned out to be $E=-1.14$, but this could be lowered to $E=-1.33$ by drawing smooth curves through the mesh points and integrating by planimeter. This indicates that the procedure is much too crude and that one must use a better way of approximating the Laplacian. That this is possible will now be shown.

[1] G. E. Kimball and G. H. Shortley, Phys. Rev. **45**, 815 (1934).

[2] See G. H. Shortley and R. Weller, J. Appl. Phys. **9**, 334 (1938).

[3] This was done on the SEAC, the digital computer at the National Bureau of Standards by Mr. Joseph H. Wegstein, who also remarked that Eq. (2) might be simpler than Eq. (1) to solve numerically.