Spontaneous Magnetization of a Triangular Ising Lattice

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The spontaneous magnetizations for the anisotropic rectangular and triangular Ising lattices are obtained by generalizing Yang's result for the isotropic square Ising lattice.

1. INTRODUCTION

ALTHOUGH exact treatments have been given for the statistics of the rectangular and triangular Ising lattices, no exact results have been obtained for these lattices in the presence of an external field. However, it has been known for some time that Onsager has obtained the long-range order (or spontaneous magnetization) for the square Ising lattice; and now that the derivation of his result has finally appeared,[1] it is appropriate to state the corresponding results for the rectangular and triangular lattices.

2. RECTANGULAR LATTICE

The surprisingly simple result for the square lattice is (in a form slightly different to that given by Yang)

$$I(S)^8=1-16x^4/(1-x^2)^4, \tag{1}$$

where $I(S)$ is the spontaneous magnetization and $x=\exp(-2H)$, using the usual notation of kTH for the energy of interaction between neighboring units. For the square (or isotropic) lattice this energy is the same whether the neighbors are in a row or in a column.

One can immediately generalize the result (1) for the case of a rectangular (anisotropic) lattice in which the energy of interaction between neighbors in a row is different from that for neighbors in a column. If this anistropy is distinguished by the variables H_x and H_y and $x=\exp(-2H_x)$, $y=\exp(-2H_y)$, then the spontaneous magnetization for the rectangular lattice is

$$I(R)^8=1-16x^2y^2/[(1-x^2)^2(1-y^2)^2]. \tag{2}$$

This result is, of course, symmetric in x and y, and it has been checked using the series solution given by Domb.[2]

Equation (2) can be written in an alternative form which permits generalization to the triangular lattice. The inversion transformation $T\rightarrow T'$ for the rectangular lattice may be written,[3] in terms of hyperbolic functions,

$$\frac{\sinh^2 2H_x'}{\sinh^2 2H_x}=\frac{\sinh^2 2H_y'}{\sinh^2 2H_y}=\frac{(\cosh^2 2H_x')(\cosh^2 2H_y')}{(\cosh^2 2H_x)(\cosh^2 2H_y)}=\frac{16x^2y^2}{(1-x^2)^2(1-y^2)^2}. \tag{3}$$

Hence the result for $I(R)$ may be written

$$I(R)^8=1-\sinh^2 2H_x'/\sinh^2 2H_x. \tag{4}$$

3. TRIANGULAR LATTICE

For the anisotropic triangular Ising lattice, with variables H_x, H_y, H_z, and $x=\exp(-2H_x)$ etc., the inversion transformation is[3,4]

$$\frac{\sinh^2 2H_x'}{\sinh^2 2H_x}=\frac{\sinh^2 2H_y'}{\sinh^2 2H_y}=\frac{\sinh^2 2H_z'}{\sinh^2 2H_z}=\frac{[(\cosh 2H_x')(\cosh 2H_y')(\cosh 2H_z')+(\sinh 2H_x')(\sinh 2H_y')(\sinh 2H_z')]^2}{[(\cosh 2H_x)(\cosh 2H_y)(\cosh 2H_z)+(\sinh 2H_x)(\sinh 2H_y)(\sinh 2H_z)]^2}$$
$$=\frac{16x^2y^2z^2}{(1+xy+yz+zx)(1+xy-yz-zx)(1-xy+yz-zx)(1-xy-yz+zx)}. \tag{5}$$

This transformation is symmetrical in x, y, and z and reduces to (3) when one of H_x, H_y, or H_z, is zero.

The generalization of (4) is now obvious and leads to the result that the spontaneous magnetization of the triangular lattice is given by

$$I(T)^8=1-\sinh^2 2H_x'/\sinh^2 2H_x, \tag{6}$$

or from (5),

$$I(T)^8=1-\frac{16x^2y^2z^2}{(1+xy+yz+zx)(1+xy-yz-zx)(1-xy+yz-zx)(1-xy-yz+zx)}. \tag{7}$$

For the special case of an isotropic $(H_x=H_y=H_z)$ triangular lattice,

$$I(T)^8=1-16x^6/[(1+3x^2)(1-x^2)^3], \tag{8}$$

which is not an obvious generalization of (1) or (2). The formula (8) has been checked by the low temperature series solution,

$$I(T)=1-2(x^6+6x^{10}-x^{12}+39x^{14}-12x^{16}+274x^{18}-114x^{20}+2025x^{22}+\cdots), \tag{9}$$

derived previously[3] as a consequence of an investigation of the rectangular lattice with first and second interactions.[5]

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