The Slope of Logarithmic Plots of the Fowler-Nordheim Equation*

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The slope of logarithmic plots of the Fowler-Nordheim equation for electron field emission is expressed in terms of the surface work-function and a tabulated function, $s(y)$. This new function, $s(y)$, is derived from Nordheim's $v(y)$ which appears in the exponent of the Fowler-Nordheim equation. Values of $v(y)$ are also tabulated and are believed to be more exact than Nordheim's original values.

THE Fowler-Nordheim equation gives a theoretical relation between the current density of field emission electrons and the electric field at the surface of the emitter. According to the theory the only controlling quantity that depends on the emission surface used is the work-function. It is the purpose of this paper to make available a tabulated function which will aid in the evaluation of surface work-function from field emission measurements.

The Fowler-Nordheim equation can be written in the following form.[1]

$$J=1.55\times10^{-6}F^2\phi^{-1}\exp[-6.86\times10^7\phi^{\frac{3}{2}}F^{-1}v(y)],$$

where J = current density in amp/cm^2, F = electric field at surface in volts/cm, ϕ = surface work-function in volts, $y=3.62\times10^{-4}F^{\frac{1}{2}}/\phi$ (reference 1), and $v(y)$ = elliptic function of y (reference 2).

In experimentally testing the above equation one usually plots $\log_{10}(J/F^2)$ *versus* $(1/F)$. Because $v(y)$ is a slowly varying function of F, the Fowler-Nordheim equation predicts for this plot a curve closely approximating a straight line over a small range of variation in F. It is useful to know the slope of this curve for two reasons: (1) It gives one an exact method of interpreting the slope of experimentally measured plots of $\log_{10}(J/F^2)$ *versus* $(1/F)$. (2) One can predict how far the experimental plot should depart from a straight line.

To find the slope of a $\log_{10}(J/F^2)$ *versus* $(1/F)$ plot of the Fowler-Nordheim equation, one takes the derivative with the following result:

$$\text{Slope}=\frac{d(\log_{10}J/F^2)}{d(1/F)}=-2.98\times10^7\phi^{\frac{3}{2}}\left(v-\frac{y}{2}\frac{dv}{dy}\right)$$
$$=-2.98\times10^7\phi^{\frac{3}{2}}s(y),$$

where $s(y)=v-\frac{1}{2}y\,dv/dy$.

The function $s(y)$ is tabulated in Table I. Also the values of $v(y)$ are listed because my calculated values differ slightly (up to 3 percent) from the original values published by Nordheim.[2] The values of $v(y)$ given here are believed to be correct to four figures, and $s(y)$ correct to three figures.

As F is increased, the absolute value of the slope at first decreases until approximately $y=0.7$ and then increases to infinity at $y=1$. The maximum of the potential barrier at the metal surface has been pulled down to the Fermi level when $y=1$.

As an example, consider field emission from a tungsten hemisphere of radius equal to one micron. The minimum current which can be measured is about 10^{-16} ampere, which corresponds to $J=1.6\times10^{-9}$ amp/cm^2, $F=1.5\times10^7$ volt/cm, and $y=0.31$. The maximum current density which has been reported[3] is about $J=10^7$ amp/cm^2, corresponding to $I=0.6$ amp, $F=7.5\times10^7$ volt/cm, and $y=0.70$. Over this range the slope changes by about 3 percent.

If experimental results are plotted in the form $\log_{10}J$ *versus* $1/F$, or $\log_e J$ *versus* $1/F$, the function $s(y)$ can still be used to interpret the slope:

$$(\log_{10}J)\ vs\ (1/F):\ \text{slope}=-0.8686F-2.98\times10^7\phi^{\frac{3}{2}}s(y),$$
$$(\log_e J)\ vs\ (1/F):\ \text{slope}=-2F-6.86\times10^7\phi^{\frac{3}{2}}s(y).$$

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TABLE I. Values of $v(y)$ and $s(y)$.

y	$v(y)$	$s(y)$
0	1.0000	1.000
0.1	0.9874	0.999
0.2	0.9565	0.995
0.3	0.9109	0.989
0.4	0.8525	0.982
0.5	0.7822	0.973
0.6	0.7002	0.964
0.65	0.6546	0.962
0.7	0.6056	0.961
0.75	0.5529	0.965
0.8	0.4956	0.976
0.85	0.4320	1.010
0.9	0.3587	1.085
0.95	0.2648	1.348
1	0	∞

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[1] H. Sommerfeld and H. A. Bethe, *Handbuch der Physik* (J. Springer, Berlin, 1934), Vol. 24, Part 2, p. 436.

[2] L. Nordheim, Proc. Roy. Soc. (London) **121**, 626 (1928).

[3] W. P. Dyke and J. K. Trolan, Linfield College Technical Report No. 1 (September, 1951).