where $D(B)$ is the spacing of levels of the same spin and parity as the initial state at energy B, and $\rho_\lambda(B-\epsilon)$ is the density of levels at energy $B-\epsilon$ that can be reached through the radiative transitions of multipole 2^λ. The radiation widths are obtained by integrating $\hbar P(\epsilon)$ from $\epsilon=0$ to B.

The ratio of radiation widths of states of spin J and given parity with excitations B and $B+E$ then takes the following form:

$$\frac{\Gamma_r^{(J)}(B+E)}{\Gamma_r^{(J)}(B)} \simeq \frac{D^{(J)}(B+E)}{D^{(J)}(B)} \cdot \frac{\int_0^{B+E} \epsilon^{2\lambda+1}\rho(B+E-\epsilon)d\epsilon}{\int_0^{B} \epsilon^{2\lambda+1}\rho(B-\epsilon)d\epsilon} \equiv \frac{1}{f_\lambda(E)} \cdot \frac{D^{(J)}(B+E)}{D^{(J)}(B)}, \qquad (16)$$

where $D^{(J)}(B)$ is the spacing of levels of spin J and given parity at excitation B. It is assumed that for all spins J the energy dependence of the level densities is as in (11), the constant preceding the exponential alone having J dependence. This allows the use of ρ, the density of levels of all types, instead of ρ_λ on the right side of (16).

Many authors, for example, Feshbach, Peaslee, and Weisskopf,[2] have shown that

$$\Gamma_N^{(J,l')}(E') \simeq T_{l'}(E') D^{(J)}/2\pi, \qquad (17)$$

where the value of $D^{(J)}$ is to be taken at the excitation energy of the compound nucleus. Equation (6) follows directly from (16) and (17).

ρ_λ has no obvious J dependence. Hence, from (15) it can be seen that the main dependence of $\Gamma_r^{(J)}(B)$ is in its proportionality to $D^{(J)}(B)$. This is the basis for taking ξ_J to be independent of J.

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The Spread of the Soft Component of the Cosmic Radiation

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The fundamental diffusion equations describing the three-dimensional development of the electron-photon component of the cosmic radiation, are formulated. These equations take into account exactly ionization loss and variation of density in the medium, and enable one to determine all angular and radial moments of the distribution functions, as explicit functions of depth and energy for arbitrary initial conditions.

The exact general solution of these equations obtained in this paper are readily adapted to any physical situation of interest. The method is similar to that devised by the authors in considering the three-dimensional development of the nucleon component. The only approximations involved are those inherent in the Bethe-Heitler cross sections in the full screening approximation, and the neglect of angular deflections in processes other than elastic Coulomb scattering.

It is shown that all previous work is subject to very large errors on the following counts: (1) the neglect of fourth and higher angular moments for the Coulomb scattering in Landau's equation and equivalent integral equations; (2) the neglect of variation of density in the atmosphere—which alone can lead to errors as high as 5000 percent; (3) elimination of the depth dependence either by integration over all depths, or evaluation in the neighborhood of the cascade maximum; (4) miscellaneous errors introduced in the evaluation of already approximate integrals; (5) use of results due to Moliere, hitherto unpublished in detail, which involve errors of several orders of magnitude in the higher moments; and also in the distribution functions concerned.

No calculation of the actual distribution functions in the atmosphere or elsewhere has yet been made on the basis of a realistic physical model. The results obtained in this paper will allow the authors to do this in the future.

1. INTRODUCTION

It is difficult to find any topic in theoretical physics which has received so much attention, with results so lacking in precision, as the spread of the soft component of the cosmic radiation in the atmosphere. Both the physical assumptions and the mathematical techniques employed have been of such a crude nature that no confidence whatever can be placed in either the qualitative or quantitative aspects of the theories advanced.

All theories hitherto put forward which have not consisted of purely qualitative considerations, have been based either on equations due to Landau[1] or on completely equivalent integral equations due to Roberg and Nordheim.[2] These equations are actually inapplicable to the atmosphere, since they relate only to media of constant density. Janossy[3] has argued that "no great error arises from the fact that the variation of the cascade unit with air density has been neglected," but our exact calculations will show that the error actually attains a maximum of 5000 percent. Another defect of the equations of Landau, and Roberg and Nordheim, is the neglect of the higher angular moments of the Coulomb scattering of the electrons. In consequence of this neglect, the mean square angular deviation of particles from the shower axis can be calculated accurately for a medium of constant density, but the mean fourth power obtained is in error by 18 percent, the mean sixth power by 45 percent, and higher moments are completely inaccurate. Similar corrections apply to

[1] L. Landau, J. Phys. (U.S.S.R.) 2, 234 (1940).
[2] J. Roberg and L. W. Nordheim, Phys. Rev. 75, 444 (1949).
[3] L. Janossy, *Cosmic Rays* (Oxford University Press, London, 1948).

the moments of the radial distribution function. The so-called "exact" values of the higher moments computed by Eyges and Fernbach[4] and Nordheim *et al.*[5] are thus subject to large corrections. Furthermore, the angular and radial distribution functions constructed from such moments are necessarily incorrect.

Moliere[6–8] attempted to obtain the distribution functions by a method not requiring the previous determination of the moments. The details of the method have never been published, but since it involves the use of Landau's equation, it is necessarily inexact. Further, if Landau's equations are accepted, Moliere's results are still incorrect, both quantitatively and qualitatively. His calculation has so little relation to the physical problem that it is a strange coincidence that experimental evidence[9] should appear to support his results. This coincidence has unfortunately led to an almost universal acceptance[10] of an incorrect theory, which has obstructed further advance.

Errors of principle have been introduced in the treatment of the depth dependence of the angular and radial spread. Some authors[2–5,11] have averaged over all depths; however, it will be shown presently that the spread is by no means constant, whether one assumes constant or variable density. Others[6,7] have obtained results only for the cascade maximum, or attempted by very approximate methods to extrapolate from the cascade maximum.[12] In view of these and other approximations which have been introduced, it seems futile, as some authors have attempted[2–4,6] to take ionization loss into account; this may be done after a theory valid for high energies has been established.

The conclusion is inevitable that all work hitherto on the angular and lateral spread of the soft component is incorrect or so inexact as to be severely in need of amendment.

From the physical point of view, two distinct problems need to be considered: the determination of the spread of the electron and photon components of the cosmic radiation in the atmosphere, and in a medium of constant density such as lead. It is now almost certain that in the atmosphere the soft component is secondary to the nucleon-mesonic component, so that the spread of the nucleon component, previously determined by the present authors,[13] provides the initial condition required for the determination of the spread of the soft component. The problem thus raised will be considered by us in a subsequent publication in which we shall obtain the correct radial and angular distribution functions. In media of constant density, the angular and radial distributions of interest result from two initial conditions, corresponding to either incident electrons or incident photons at the top of the layer considered. This is the problem which we presently considered, since it is that which has been treated by all previous authors. It has applications to the spread of showers in slabs of material with uniform density, and also to the spread of atmospheric showers through *shallow layers well down in the atmosphere*, where the density is almost constant. For purposes of comparison, we have, however, determined the mean square lateral spread of showers initiated by electrons (though few, if any, of these exist) at the top of the atmosphere.

Our theoretical results are applicable to an arbitrary distribution of particles incident on the absorbing layer concerned, within which the density may vary in an arbitrary manner, and are thus completely general. However, for purposes of numerical discussion, we have always considered an incident power law spectrum for the particles concerned. Numerical results for all other types of spectra can be obtained by a single complex integration, which in most cases is readily performed by the method of steepest descents. We have confined our attention in this paper to the determination of nth moments of the angular and radial distribution functions; it is possible, however, to reconstruct the functions from the moments by a method given by us in a previous publication[13] and this we shall do presently. The present theory is valid only in the region of high energies where ionization loss is negligible, but the consideration of an additional term in our fundamental equations will enable us to extend the theory to low energies in the near future.

2. FUNDAMENTAL EQUATIONS FOR THE SPREAD OF THE SOFT COMPONENT

We denote by $f^{(i)}(\mathbf{p}, \mathbf{r}, t)d\mathbf{p}d\mathbf{r}/(2\pi p^2)$ the probability that a particle of the ith kind ($i=1$ refers to electrons; $i=2$ to photons) be found with momentum in the range $\mathbf{p}d\mathbf{p}$ at height t cm, and with horizontal displacement $\mathbf{r}$, $d\mathbf{r}$ from the shower axis, assumed to be vertical.[14] Further, we denote by $w^{(i)}(\mathbf{p}', \mathbf{p})d\mathbf{p}/(2\pi p^2)$ the well-known Bethe-Heitler cross sections[15] in the full screening approximation, and by $w(\mathbf{p}', \mathbf{p})d\mathbf{p}/(2\pi p^2)$ the corresponding differential cross section for the elastic scattering of an electron[16] by an atom of the absorber considered. The total cross sections corresponding to $w^{(i)}$ and w are represented by $\alpha^{(i)}$ and α, respectively;

[4] L. Eyges and S. Fernbach, Phys. Rev. **82**, 23 (1951).
[5] Nordheim, Osborne, and Blatt, unpublished Echo Lake Proceedings (1949).
[6] G. Moliere and W. Heisenberg, *Cosmic Radiation* (Dover Publications, New York, 1946).
[7] G. Moliere, Naturwiss. **30**, 87 (1942); Z. Physik **125**, 250 (1948).
[8] G. Moliere, Phys. Rev. **77**, 715 (1949).
[9] Cocconi, Cocconi-Tongiorgi, and Greisen, Phys. Rev. **76**, 1020 (1949).
[10] J. M. Blatt, Phys. Rev. **75**, 1584 (1949) is one of the few authors who appear to have appreciated that Landau and Moliere's theory was incorrect, though he was not able to amend it.
[11] S. Z. Belenky, J. Phys. (U.S.S.R.) 8, 9 (1944).
[12] S. Fernbach, Phys. Rev. **82**, 288 (1951).
[13] H. Messel and H. S. Green, Phys. Rev. **87**, 378 (1952).
[14] The method of generalization to the case where the shower axis is not vertical was described in our previous paper (reference 13).
[15] Multiplied by 2 for $i=2$.
[16] B. Rossi and K. Greisen, Revs. Modern Phys. **13**, 240 (1941).

the fact that $\alpha^{(1)}$ is mathematically divergent leads to no difficulty in this instance. The ionization loss for the electrons per cascade unit is represented by β; the depth in the absorber, measured in cascade units is denoted by $l(t)$. In a medium of constant density δ g/cm^3, $l=-\delta t/k$, where k is the radiation length in g/cm^2 ($k=43$ for air). In an isothermal atmosphere, on the other hand, $l=(p_0/43g)\exp(-g\delta_0 t/p_0)$, where p_0 and δ_0 are the surface pressure and density, and g the acceleration due to gravity.

The rate of change of $f^{(i)}$ per unit path length is $\mathbf{p}/p\cdot\partial f^{(i)}/\partial\mathbf{r}$, where $r_3=-t$; this results from a loss $(\alpha^{(i)}+\alpha\delta_{i,1})(dl/dt)f^{(i)}$ from the momentum range $\mathbf{p}$, $d\mathbf{p}$ to the other momenta, a gain

$$\frac{dl}{dt}\int[w^{(3-i)}(\mathbf{p}',\mathbf{p})f^{(3-i)}(\mathbf{p}')+\{w^{(i)}(\mathbf{p}',\mathbf{p}'-\mathbf{p})+w(\mathbf{p}',\mathbf{p})\}f^{(i)}(p')\delta_{i,1}]d\mathbf{p}'/(2\pi p'^2),$$

from other momenta, and a loss $\beta(dl/dt)\times(\partial f^{(i)}/\partial p)\delta_{i,1}$ due to ionization. Hence,

$$-\frac{dt}{dl}\frac{\mathbf{p}}{p}\cdot\frac{\partial f^{(i)}}{\partial\mathbf{r}}=-(\alpha^{(i)}+\alpha\delta_{i,1})f^{(i)}+\beta\frac{\partial f^{(i)}}{\partial p}\delta_{i,1}+\int[w^{(3-1)}(\mathbf{p}',\mathbf{p})f^{(3-i)}(\mathbf{p}')+\{w^{(i)}(\mathbf{p}',\mathbf{p}'-\mathbf{p})+w(\mathbf{p}',\mathbf{p})\}f^{(i)}(\mathbf{p}')\delta_{i,1}]d\mathbf{p}'/(2\pi p'^2). \quad (1)$$

Transforming to polar coordinates by writing

$$r=(r_1^2+r_2^2)^{\frac{1}{2}},\quad E=p=(p_1^2+p_2^2+p_3^2)^{\frac{1}{2}},$$
$$p_1r_1+p_2r_2=ErS\cos\Psi,\quad C=p_3/p,\quad S=(1-C^2)^{\frac{1}{2}}, \quad (2)$$

one obtains

$$\frac{dt}{dl}\left(C\frac{\partial f^{(i)}}{\partial t}-S\cos\Psi\frac{\partial f^{(i)}}{\partial r}+\frac{S\sin\Psi}{r}\frac{\partial f^{(i)}}{\partial\Psi}\right)+(\alpha^{(i)}+\alpha\delta_{i,1})f^{(i)}$$
$$=\beta\frac{\partial f^{(i)}}{\partial E}\delta_{i,1}+\int_E^\infty[w^{(3-i)}(E/E')f^{(3-i)}(E')+w^{(i)}(1-E/E')f^{(i)}(E')\delta_{i,1}]dE'/E'$$
$$+\int_0^{2\pi}\frac{d\Psi'}{2\pi}\int_{-1}^1dC'\int_E^\infty dE'w\{E',E,CC'+SS'\cos(\Psi-\Psi')\}f^{(i)}(E',C',\Psi')\delta_{i,1}. \quad (3)$$

Expanding $f^{(i)}(E',C',\Psi')$ about the value $C'=C$, $\Psi'=\Psi$, by means of associated Legendre operators, and substituting[16]

$$w(E',E,c)=\delta(E'-E)V/E^2(1-c)^2,$$
$$V=\tfrac{1}{2}\pi[137m_e^2/\ln(181z^{-\frac{1}{3}})], \quad (4)$$

the last term in (3) reduces to

$$\sum_{l=0}^{\infty}4^{-l}(l!)^{-2}E^{-2l}w_lP(P+1.2)\cdots\times\{P+l(l-1)\}f^{(i)}(E)\delta_{i,1}, \quad (5)$$

where P is the operator

$$P=\left(\frac{\partial}{\partial C}S^2\frac{\partial}{\partial C}+\frac{1}{S^2}\frac{\partial^2}{\partial\Psi^2}\right), \quad (6)$$

and

$$w_l=2^lV\int_{c_0}^{c_1}(1-c)^{l-2}E^{2l-2}dc,$$
$$c_0=1-\tfrac{1}{2}(137m_ez^{-\frac{1}{3}})^2/(0.57E)^2, \quad (7)$$
$$c_1=1-\tfrac{1}{2}(m_ez^{\frac{1}{3}})^2/(137E)^2.$$

The term with $l=0$ in (5) yields the total cross section for elastic scattering, and cancels the equivalent term $(\alpha\delta_{i,1}f^{(i)})$ on the left-hand side of (3). The latter is then

$$\frac{dt}{dl}\left(C\frac{\partial f^{(i)}}{\partial t}-S\cos\Psi\frac{\partial f^{(i)}}{\partial r}+\frac{S\sin\Psi}{r}\frac{\partial f^{(i)}}{\partial\Psi}\right)+\alpha^{(i)}f^{(i)}$$
$$=\beta\frac{\partial f^{(i)}}{\partial E}\delta_{i,1}+\int_E^\infty[w^{(3-i)}(E/E')f^{(3-i)}(E')+w^{(i)}(1-E/E')f^{(i)}(E')\delta_{i,1}]dE'/E'$$
$$+\sum_{l=1}^{\infty}4^{-l}(l!)^{-2}E^{-2l}w_lP(P+2)\cdots\times\{P+l(l-1)\}f^{(i)}(E)\delta_{i,1}. \quad (8)$$

This equation is exact, within the limitations of the assumed Bethe-Heitler cross sections. One may legitimately approximate C by 1, and P by $\partial^2/\partial\theta_1^2+\partial^2/\partial\theta_2^2$, where

$$\theta_1=S\cos\Psi\quad\text{and}\quad\theta_2=S\sin\Psi, \quad (9)$$

since this merely amounts to dropping terms of order $(m_ec^2/E)^2$ compared with unity. At sufficiently high energies (above the critical energy) one may also set $\beta=0$, neglecting ionization loss. If in addition, all terms with $l>1$ are dropped from the summation in (8), and dt/dl is replaced by the constant $-k/\delta$ (appropriate only to a medium of constant density), then (8) reduces to Landau's equations.[1] The last two procedures mentioned account for the substantial errors involved in the use of Landau's equations, or the equivalent integral equations of Nordheim and Roberg. It might be supposed that the error in neglecting the term with $l=2$ in (8) in comparison with the first is of the same order of magnitude as that resulting from the substitution $C=1$, for example; this is not so, however, as it makes a direct contribution to the fourth angular and radial moments. This is evident on physical grounds, and will appear so in subsequent mathematical developments.

3. SOLUTION OF THE FUNDAMENTAL EQUATIONS

If the Mellin transform $f^{(i)}(v)$ of $f^{(i)}(E)$ is defined by

$$f^{(i)}(v)=\int_0^\infty E^{v-1}f^{(i)}(E)dE, \tag{10}$$

then (8) reduces to

$$\begin{aligned}\frac{dt}{dl}\left\{C\frac{\partial}{\partial t}-S\cos\Psi\frac{\partial}{\partial r}+\frac{S\sin\Psi}{r}\frac{\partial}{\partial\Psi}\right\}f^{(i)}(v)\\
=-(v-1)\beta f^{(i)}(v-1)\delta_{i,1}+B^{(i)}(v)f^{(3-i)}(v)\\
-D^{(i)}(v)f^{(i)}(v)+\sum_{l=1}^{\infty}4^{-l}(l!)^{-2}w_lP(P+2)\cdots\\
\times\{P+l(l-1)\}f^{(i)}(v-2l)\delta_{i,1},\end{aligned} \tag{11}$$

where

$$\begin{aligned}B^{(1)}(v)&=2\left\{\frac{1}{v}-\frac{(4/3+\alpha_0)}{(v+1)(v+2)}\right\},\\
B^{(2)}(v)&=\frac{1}{v+1}+\frac{(4/3+\alpha_0)}{v(v-1)},\\
D^{(1)}(v)&=(4/3+\alpha_0)\{\Psi(v)-\Psi(1)\}+\tfrac{1}{2}-\frac{1}{v(v+1)},\\
D^{(2)}(v)&=7/9-\alpha_0/6,\end{aligned} \tag{12}$$

and

$$\Psi(v)=\frac{d}{dv}\ln(v!),\quad\text{while}\quad 9\alpha_0=\{\ln(181z^{-\frac{1}{3}})\}^{-1}.$$

Introducing the polar Fourier transform

$$\begin{aligned}g^{(i)}(v,k,\beta')=\int_0^{2\pi}d\Psi\int_0^\infty rdr\\
\times\exp\{ikr\cos(\Psi-\beta')\}f^{(i)}(v,r,\Psi),\end{aligned} \tag{13}$$

whose inverse is

$$\begin{aligned}f^{(i)}(v,r,\Psi)=(2\pi)^{-2}\int_0^{2\pi}d\beta'\int_0^\infty kdk\\
\times\exp\{-ikr\cos(\Psi-\beta')\}g^{(i)}(v,k,\beta'),\end{aligned} \tag{14}$$

one has

$$\begin{aligned}\frac{dt}{dl}\left\{C\frac{\partial}{\partial t}+ikS\cos\beta'\right\}g^{(i)}(v)\\
=-(v-1)\beta g^{(i)}(v-1)\delta_{i,1}+B^{(i)}(v)g^{(3-i)}(v)\\
-D^{(i)}(v)g^{(i)}(v)+\sum_{l=1}^{\infty}4^{-l}(l!)^{-2}w_lQ(Q+2)\cdots\\
\times\{Q+l(l-1)\}g^{(i)}(v-2l)\delta_{i,1},\end{aligned} \tag{15}$$

where

$$Q=\left(\frac{\partial}{\partial C}S^2\frac{\partial}{\partial C}+\frac{1}{S^2}\frac{\partial^2}{\partial\beta'^2}\right). \tag{16}$$

In order to solve the simultaneous Eqs. (15) with $\beta=0$ (the general case $\beta\neq0$ is postponed for future consideration), we formulate the equation satisfied by

$$h(v)=g^{(1)}(v)+\mu(v)g^{(2)}(v), \tag{17}$$

where $\mu(v)$ is independent of depth. If $\mu(v)$ satisfies the quadratic equation

$$B^{(2)}(v)\mu^2(v)+\{D^{(2)}-D^{(1)}(v)\}\mu(v)-B^{(1)}(v)=0, \tag{18}$$

one obtains

$$\left\{C\frac{\partial}{\partial l}+\frac{iKS\cos\beta'}{l'(t)}\right\}h(v)+a(v)h(v)=R(v,l), \tag{19}$$

where

$$\begin{aligned}R(v,l)=\sum_{l=1}^{\infty}4^{-l}(l!)^{-2}w_lQ(Q+2)\cdots\\
\times\{Q+l(l-1)\}g^{(1)}(v-2l),\end{aligned} \tag{20}$$

and

$$a(v)=D^{(1)}(v)-\mu(v)B^{(2)}(v). \tag{21}$$

Inserting the two admissible values

$$\begin{aligned}\mu_1(v)=[(D^{(1)}-D^{(2)})+\{(D^{(1)}-D^{(2)})^2\\
+4B^{(1)}B^{(2)}\}^{\frac{1}{2}}]/2B^{(2)},\end{aligned} \tag{22}$$

$$\begin{aligned}\mu_2(v)=[(D^{(1)}-D^{(2)})-\{(D^{(1)}-D^{(2)})^2\\
+4B^{(1)}B^{(2)}\}^{\frac{1}{2}}]/2B^{(2)},\end{aligned} \tag{23}$$

of $\mu(v)$ into (19), and writing

$$\begin{aligned}G_1^{(1,1)}(v)&=\frac{D^{(2)}-a_1(v)}{a_2(v)-a_1(v)}, & G_2^{(1,1)}(v)&=\frac{a_2(v)-D^{(2)}}{a_2(v)-a_1(v)},\\
G_1^{(1,2)}(v)&=\frac{B^{(2)}(v)}{a_2(v)-a_1(v)}, & G_2^{(1,2)}(v)&=-G_1^{(1,2)}(v),\\
G_1^{(2,1)}(v)&=\frac{B^{(1)}(v)}{a_2(v)-a_1(v)}, & G_2^{(2,1)}(v)&=-G_1^{(2,1)}(v),\\
G_1^{(2,2)}(v)&=G_2^{(1,1)}(v), & G_2^{(2,2)}(v)&=G_1^{(1,1)}(v),\end{aligned} \tag{24}$$

one obtains

$$\begin{aligned}g^{(i)}(v)=g^{(j)}(v,l=\lambda)G_m^{(j,i)}(v)\exp(-C^{-1}[a_m(v)(l-\lambda)\\
+iSk\cos\beta'\{t(l)-t(\lambda)\}])\\
+C^{-1}G_m^{(1,i)}(v)\int_\lambda^1\exp(-C^{-1}[a_m(v)(l-l')\\
+iSk\cos\beta'\{t(l)-t(l')\}])R(v,l')dl',\end{aligned} \tag{25}$$

where $t(l)$ is the height, expressed as a function of l, and the repeated affixes j and m are summed over the values 1 and 2, as in relativity theory. In practice an iteration procedure using (20) and (25) is employed in solving these equations. The integrals which arise may be simplified by expanding the exponential factor

$$\exp[-iSk\cos\beta'\{t(l)-t(l')\}/C]$$

in a power series, and integrating by parts a sufficient number of times to replace $t(l)$ everywhere by its derivative. The result may be expressed in terms of the operator

$$Y_i = C^{-1}G_m{}^{(1,i)}(v)\exp\{-C^{-1}a_m(v)l\} \times \sum_{u=0}^{\infty}(SC^{-1}\cos\beta')^u \sum_{s=1}^{\infty} w_s 4^{-s}(s!)^{-2}Q(Q+2)\cdots \times\{Q+s(s-1)\}\left\{-ik\int_{t(\lambda)}^{t(l)} dt(l)\right\}^u \times\int_\lambda^l dl\exp\{C^{-1}a_m(v)l\}E_v{}^{-2s}, \quad (26)$$

where E_v is the incremental operator which raises the value v by unity. One has, in fact, the solutions

$$g^{(1)}(v) = \left(\sum_{n=0}^{\infty} Y_1{}^n\right) g^{(j)}(v, l=\lambda) \times G_m{}^{(j,1)}(v)\exp(-C^{-1}[a_m(v)(l-\lambda) + iSk\cos\beta'\{t(l)-t(\lambda)\}]), \quad (27)$$

and

$$g^{(2)}(v) = g^{(j)}(v, l=\lambda)G_m{}^{(j,2)}(v)\exp(-C^{-1}[a_m(v)(l-\lambda) + iSk\cos\beta'\{t(l)-t(\lambda)\}]) + Y_2 g^{(1)}(v). \quad (28)$$

This gives the exact values of the $g^{(i)}(v)$ throughout the atmosphere or other medium, for an arbitrary given distribution of electrons and photons at depth λ, for all depths and for an arbitrary distribution of density.

4. SIMPLE APPLICATIONS

The nature of the solutions (27) and (28) depends on the initial conditions supplied by the physical situation considered. In the atmosphere, a correct initial condition for the soft component is the distribution function for the neutral π-mesons, which is governed by the distribution function of the nucleonic component. Since we have already determined the distribution function for the nucleonic component of the cosmic radiation in the atmosphere, this problem is ready for solution. In the present paper, however, we are concerned with assessing the merits of previous work on the soft component, in which it was customary to take initial conditions corresponding to a particle or spectrum of particles incident on the top of the atmosphere, and therefore adopt the same initial conditions ourselves, namely,

$$g^{(j)}(v, l=0) = g_0{}^{(j)}(v)\delta(1-C), \quad (29)$$

where $g_0{}^{(j)}(v)$ is independent of C. When this is substituted into (27) and (28), one obtains a result of the form (see reference 13)

$$g^{(i)}(v) = \sum_{l,m,n} \frac{(-2)^{-n}}{l!m!n!}(ik\cos\beta')^l(ik\sin\beta')^m \times\delta^{(n)}(1-C)f_{(l,m,n)}{}^{(i)}(v). \quad (30)$$

This gives, on applying the transform defined by (13) and (14),

$$f^{(i)}(v) = \sum_{l,m,n}\frac{(-1)^{l+m+n}2^{-n}}{l!m!n!}f_{(l,m,n)}{}^{(i)}(v) \times\delta^{(l)}(r\cos\Psi)\delta^{(m)}(r\sin\Psi)\delta^{(n)}(1-C), \quad (31)$$

from which it may be deduced by integration that

$$f_{(l,m,n)}{}^{(i)} = 2^n\int\int\int f^{(i)}(r_1, r_2, C)r_1{}^l r_2{}^m(1-C)^n dr_1 dr_2 dC, \qquad r_1 = r\cos\Psi, \quad r_2 = r\sin\Psi, \quad (32)$$

is a typical moment of $f^{(i)}(v)$ with respect to r_1, r_2, and $2(1-C)$.

By comparison of (27), (28), and (29) with (30) we obtain for the $2n$th pure angular moment of the electron distribution

$$f_{(0,0,n)}{}^{(1)}(v) = (n!)^2 g_0{}^{(j)}(v-2n)\exp\{-a_m(v-2n)\} \times\sum{}'\prod_k\left\{\frac{w_{n(k)}}{\{n(k)!\}^2}G_{m(k)}{}^{(1,1)}\left\{v-2\sum_{q=1}^{k}n(q-1)\right\} \times\exp\left[-a_{m(k)}\left\{v-2\sum_{q=1}^{k}n(q-1)\right\}l\right] \times\int_0^1\exp\left[a_{m(k)}\left\{v-2\sum_{q=1}^{k}n(q-1)\right\}l\right]\right\}, \quad (33)$$

where $\sum'$ indicates summation over all products for which $\sum_k n(k) = n$, with $n(0)=0$. The corresponding formula for the photons is obtained by changing $G_{m(1)}{}^{(1,1)}(v)$ to $G_{m(1)}{}^{(1,2)}(v)$ in (33). These formulas are independent of the variation of density in the absorber, except insofar as it affects the cascade unit. The formulas for the radial moments, however, depend very sensitively on the law of variation of density adopted. One has

$$f_{(2,0,0)}{}^{(i)}(v) = 2w_1 g_0{}^{(j)}(v-2)G_m{}^{(j,1)}(v-2)G_n{}^{(1,i)}(v) \times\exp\{-a_n(v)l\}\chi_{m,n}(v,l), \quad (34)$$

where

$$\chi_{m,n}(v,l) = \int_0^l dlt'(l)\int_0^l dlt'(l)\int_0^l dl \times\exp[\{a_n(v)-a_m(v-2)\}l]. \quad (35)$$

Substituting, for an isothermal atmosphere, $t'(l) = -p_0/(g\delta_0 l)$ one has

$$\chi_{m,n}(v,l) = \left(\frac{p_0}{g\delta_0}\right)^2 \frac{J_1[\{a_n(v)-a_m(v-2)\}l]}{a_n(v)-a_m(v-2)} \times \exp[\{a_n(v)-a_m(v-2)\}l], \tag{36}$$

$$J_1(x) = e^{-x}\sum_{n=1}^{\infty} x^n/n^2 n!$$

and for an absorber of constant density, with $t'(l) = -k/\delta$, one has

$$\chi_{m,n}(v,l) = \left(\frac{k}{\delta}\right)^2 \frac{J_2[\{a_n(v)-a_m(v-2)\}l]}{\{a_n(v)-a_m(v-2)\}^3} \times \exp[\{a_n(v)-a_m(v-2)\}l], \tag{37}$$
$$J_2(\tau) = e^{-x}(e^x - 1 - x - x^2/2).$$

The moments $f_{(0,2,0)}{}^{(i)}(v)$ which describe the horizontal spread in a direction normal to the instantaneous motion, are of order E^{-2} compared with $f_{(2,0,0)}{}^{(i)}(v)$. and make no effective contribution to the total radial moments,

$$f_{(0,2,0)}{}^{(i)}(v) + f_{(2,0,0)}{}^{(i)}(v).$$

We have written down corresponding expressions for the higher radial moments, but as these expressions are somewhat complicated we shall not reproduce them here.

For a single incident particle of energy E_0, $g_0{}^{(j)}(v-2n) = E_0{}^{v-2n-1}\delta_{j,i}$, where $i=1$ for an incident electron and $i=2$ for an incident photon. For an incident power law spectrum of electrons or photons,

$$g_0{}^{(j)}(v-2n) = \{\gamma E_c{}^{v-2n-1}/(\gamma+2n+1-v)\}\delta_{j,i},$$

where E_c is the geomagnetic cut-off energy and $\gamma = 1.5$ is the primary power law exponent; it should, however, be remembered that this substitution has little relation to the physical reality. With this latter substitution, the inverse Mellin transform which has to be applied to the formulas of this section is readily evaluated for energies $E > E_c$; the pole at $v = \gamma+2n+1$ is the only one within the contour completed by a semicircle in the right-hand half of the complex plane, and one has therefore only to take the residue at this pole.

Table I. Values of the mean square distance of particles of energy E from the shower axis, in units of $(kE_s/\delta E)^2$, where k is the radiation length, $E_s = 21$ Mev, and δ is the constant density of the medium considered. In all cases, figures are for a primary power law spectrum of exponent $\gamma = 1.5$.

Primary	Secondary	$l=1$	$l=2$	$l=3$	$l=4$	$l=5$	$l=6$	$l=\infty$
Electron	Electron	0.1625	0.5252	0.7913	1.023	1.261	1.505	2.623
	Photon	0.07139	0.3962	0.9294	1.556	2.189	2.808	5.543
Photon	Electron	0.04386	0.1900	0.3721	0.5637	0.7674	0.9758	2.623
	Photon	0.00454	0.07028	0.2704	0.6092	0.053	1.542	5.543

Hence, one has

$$f_{(0,0,n)}{}^{(i)}(E,l) = \lim_{v=\gamma+2n+1} (\gamma+2n+1-v)E^{-v} f_{(0,0,n)}{}^{(i)}(v), \tag{38}$$

$$f_{(2,0,0)}{}^{(i)}(E,l) = \lim_{v=\gamma+3} (\gamma+3-v)E^{-v} f_{(2,0,0)}{}^{(i)}(v). \tag{39}$$

The expressions (38) and (39) relate to particles with energy E; to obtain results for particles with energies $>E$ one simply integrates the expressions from E to E_0. The values of the G's required for the numerical evaluation of (38) and (39) are:[17]

$$\begin{aligned} &G_1{}^{(1,1)}(2.5) = 0.30628, \quad G_1{}^{(1,1)}(4.5) = 0.049882,\\ &G_1{}^{(1,2)}(2.5) = 0.46833, \quad G_1{}^{(1,2)}(4.5) = 0.18567,\\ &G_1{}^{(2,1)}(2.5) = 0.45368, \quad G_1{}^{(2,1)}(4.5) = 0.25525,\\ &G_1{}^{(1,1)}(6.5) = 0.013816\\ &G_1{}^{(1,1)}(6.5) = 0.093836\\ &G_1{}^{(2,1)}(6.5) = 0.14520, \end{aligned} \tag{40}$$

and those of the a's

$$\begin{aligned} &a_1(2.5) = 0.35001, \quad a_1(4.5) = 0.70167,\\ &a_2(2.5) = 1.73328, \quad a_2(4.5) = 2.14525,\\ &a_1(6.5) = 0.74846\\ &a_2(6.5) = 2.57416. \end{aligned} \tag{41}$$

The numerical results will be presented with the discussion in the next section.

5. RESULTS AND DISCUSSION

To assess the error introduced by the neglect of the higher angular moments for the Coulomb scattering in Landau's equation, we have evaluated the exact asymptotic values of the second, fourth, and sixth angular moments for large depths, valid for inhomogeneous as well as homogeneous media.

For an incident electron power law spectrum one has

$$\frac{f_{(0,0,1)}{}^{(1)}(E, l=\infty)}{f_{(0,0,0)}{}^{(1)}(E, l=\infty)} = \frac{G_m{}^{(1,1)}(4.5)w_1 E^{-2}}{a_m(4.5)-a_1(2.5)} = 0.6711(E_s/E)^2, \tag{42}$$

where $E_s{}^2 = (4\pi)137 m_e{}^2$; and

$$\frac{f_{(0,0,2)}{}^{(1)}(E, l=\infty)}{f_{(0,0,0)}{}^{(1)}(E, l=\infty)} = \frac{G_m{}^{(1,1)}(6.5)(w_2 + 2.684 w_1{}^2)E^{-4}}{a_m(6.5)-a_1(2.5)} = 1.519(E_s/E)^4. \tag{43}$$

The ratio of w_2 to $2.684 w_1{}^2$ is 0.184, so the error involved in the use of Landau's equation (or other equivalent equations) is 18.4 percent for the mean fourth power of

[17] L. Janossy and H. Messel, Proc. Roy. Irish Acad. A54, 217 (1951).

the angular deviation from the shower axis. For the corresponding mean sixth power,

$$\frac{f_{(0,0,3)}{}^{(1)}(E, l=\infty)}{f_{(0,0,0)}{}^{(1)}(E, l=\infty)} = \frac{G_m{}^{(1,1)}(8.5)(w_3+10.34w_1w_2+11.55w_1{}^3)E^{-6}}{a_m(8.5)-a_1(2.5)}. \quad (44)$$

The ratio of $10.34w_1w_2$ to $11.55w_1{}^3$ is 0.443, so, even if one neglects w_3, the error in using Landau's equation exceeds 44 percent. This error varies somewhat with the initial condition applied; however, the order of magnitude remains unchanged. Thus, not only are the "exact" moments previously determined[4,5] subject to increasingly large corrections, but the determinations of the distribution function which employ these moments are seriously affected. Moliere[6,7] apparently used a different method to determine his angular distribution function from Landau's equation; however, according to Eyges and Fernbach,[4] the sixth angular moment (for example) derived from his distribution function is in error by an order of magnitude, and the distribution function itself is thus in even greater error than that determined by Eyges and Fernbach.

The depth dependence of the angular moments can be computed readily from our formula (33). However, the radial moments are of more interest in this connection, and we have selected the mean square distance from the shower axis for special consideration. Values for a medium of constant density derived from (34) are given in Table I.

It will be noticed that the spread approaches its asymptotic value only very slowly; there is thus no support for the assertion by Janossy[3] that "most of the scattering takes place in a layer of about one cascade unit above the observer."

Another interesting feature of the results is that the spread of the electrons at depths greater than 4 cascade units is less than that of the photons—in spite of the fact that only the elastic scattering of the electrons contributes appreciably to the spread of the shower. What happens is that the electrons are absorbed more quickly than the photons, with the result that the photons are ultimately left further from the shower axis. Only at depths less than 4 cascade units does the spread of the electrons exceed that of the photons. Moreover, this conclusion is independent of whether the primary particles are photons or electrons.

It is clear on physical grounds that the spread at great depth must be independent of the type of primary particle. A photon primary generates a pair at some lower depth, so that the spread is delayed throughout the medium. These conclusions are confirmed by our calculations, and are indeed elementary consequences of our Eqs. (27) and (28).

As can be seen from (34), in media of constant or

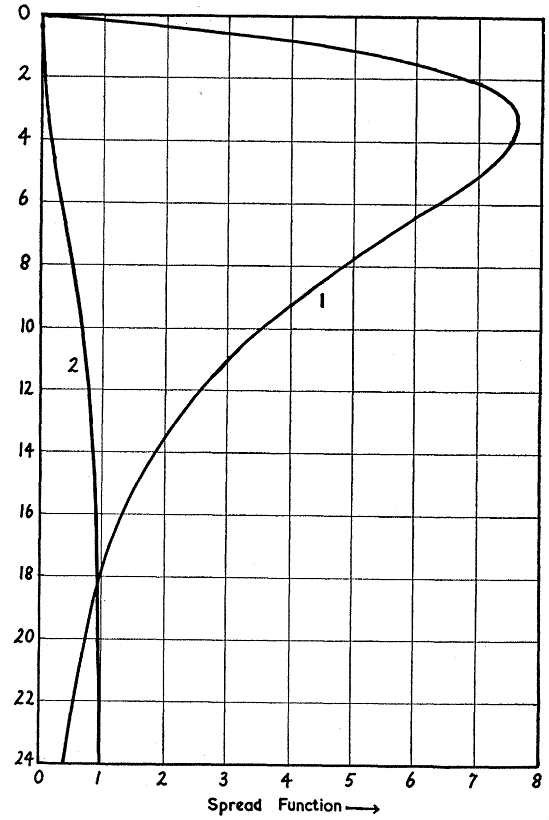

FIG. 1. The spread function normalized in such a way that, multiplied by $(E_s/E)^2$, it gives the mean square distance of electrons of energy E from the shower axis. The curves are calculated for an electron primary power law spectrum with exponent $\gamma=1.5$. The vertical scale gives the atmospheric depth l measured in cascade units. Curve 1 is for an isothermal atmosphere; curve 2 for an atmosphere assuming a constant mean density.

variable density the mean square distance from the shower axis is a linear combination of four terms, namely, the $\chi_{m,n}(4.5, 1)$ given by (35), divided by the average numbers. It can be seen from (36) and (37) that the behavior of the $\chi_{m,n}$ is determined by the *spread function* $J_1(x)$ in the atmosphere, or $J_2(x)$ in a medium of constant density. Since the function $J_1(x)$ describes the spread of all components of the cosmic radiation (see reference 13) in the atmosphere, it has been tabulated for us by Dr. E. A. Cornish of the Commonwealth Scientific and Industrial Research Organization, with results given in the Appendix. After several cascade units, only the term derived from $\chi_{1,1}$ makes an essential contribution to the spread; and we have therefore singled out this term for special consideration.

In order to compare the behavior of the mean square spread of showers initiated by a spectrum of electrons in the atmosphere, assumed to be (a) of constant density and (b) isothermal, we have plotted the J_1 and J_2 as functions of depth, multiplied by the appropriate

TABLE II. The spread function $J_1(x)$.

x	$J_1(x)$	x	$J_1(x)$
0.0	0.0000000	5.8	0.06681354
0.2	0.1679646	6.0	0.06119279
0.4	0.2823759	6.2	0.05613459
0.6	0.3563793	6.4	0.05158116
0.8	0.4002031	6.6	0.04748001
1.0	0.4217734	6.8	0.04378372
1.2	0.4272021	7.0	0.04044957
1.4	0.4211756	7.2	0.03743916
1.6	0.4072634	7.4	0.03471810
1.8	0.3881625	7.6	0.03225562
2.0	0.3658910	7.8	0.03002427
2.2	0.3419402	8.0	0.02799953
2.4	0.3173958	8.2	0.02615959
2.6	0.2930317	8.4	0.02435576
2.8	0.2693846	8.6	0.02295855
3.0	0.2468114	8.8	0.02156481
3.2	0.2255345	9.0	0.02029013
3.4	0.2056768	9.2	0.01912236
3.6	0.1872884	9.4	0.01805071
3.8	0.1703675	9.6	0.01706557
4.0	0.1548762	9.8	0.01615839
4.2	0.1407521	10.0	0.01532155
4.4	0.1279177	10.5	0.01349454
4.6	0.1162866	11.0	0.01197937
4.8	0.1057688	11.5	0.01071006
5.0	0.09627376	12.0	0.009636510
5.2	0.08771300	12.5	0.008720378
5.4	0.08000181	13.0	0.007932085
5.6	0.07306021		

factors, in Fig. 1. We were doubtful as to what constant density δ should be chosen for the "atmosphere" of constant density, but have in fact taken a mean value equal to the actual density at 12 cascade units. In the case of the isothermal atmosphere, δ_0 is the surface density; thus, no such difficulty is encountered.

It is obvious from Fig. 1 that the variation of the shower spread with height depends very sensitively on the law of variation of density which is adopted. The values calculated for constant density would be much more applicable to the actual atmosphere if they were multiplied by the ratio of the assumed density to the actual density at each level. This procedure would, of course, be quite inconsistent, and even if applied would result in a percentage error.

Results have hitherto been calculated, either in the neighborhood of the cascade maximum,[6,7] which varies widely with the energy and initial condition considered, or averaged over all depths.[2–5,10] It is clear from Fig. 1 that neither of these procedures could result in values bearing even an approximate relation to those at sea level, or other given depths. This conclusion holds whether one assumes media of constant or variable density.

It should, of course, be borne in mind that the concept of incident electrons at the top of the atmosphere is probably unphysical. However, since all previous work has adopted this concept, it is interesting to note that the maximum spread of particles with a given energy is attained in the upper half of the atmosphere. No such maximum is attained in a medium of constant density, where the spread tends to a constant value.

To the errors already noted in work by previous authors, should be added those arising from miscellaneous approximations introduced in evaluating the already approximate integrals obtained. In view of the above it is obvious that treatments which have endeavored to account for the low energy showers, by the introduction of ionization losses, are practically meaningless.

APPENDIX

Table II gives the spread function $J_1(x)$, defined in (36). It can be expressed in terms of an indefinite integral:

$$J_1(x)=e^{-x}\int_0^x \{\mathrm{Ei}(x)-\ln x-\gamma\}x^{-1}dx,$$

where γ is Euler's constant and

$$\mathrm{Ei}(x)\equiv\int_{-\infty}^x e^x x^{-1}dx.$$

The values given here were computed by Dr. Cornish by summing the infinite series

$$J_1(x)=e^{-x}\sum_{n=0}^{\infty} x^n/n!n^2,$$

which is the most convenient method for arguments up to 13 or 15. Beyond this, it is easier to use another development

$$J_1(x)=2\sum_{n=2}^{\infty} c_n x^{-n}\Gamma_x(n+1)/\Gamma(n+1),$$

where c_n is the coefficient of x^n in the power series expansion of $\{\ln(1-x)\}^2$. Values for arguments from 13 to 50 are being computed in this way.