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The Coulomb Scattering of Relativistic Electrons and Positrons by Nuclei*

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The Mott series for the Coulomb scattering of electrons and positrons of high energy is summed numerically for angles varying from 30° to 150° and for nuclear charge varying from 13 to 80.

INTRODUCTION

MEASUREMENT of the deviations from the Coulomb scattering of electrons by point nuclei is of great physical interest. In the low energy range, where the electron wavelength is much greater than the nuclear radius, such deviations measure the radiative correction to scattering and thus form the basis of another test of quantum electrodynamics. Recent measurements by both Paul and Reich[1] and Kinzinger and Bothe[2] indicate that for scattering angles greater than 60° a discrepancy exists between theory and experiment. In the higher energy range, where the electron wavelength is of the same order or smaller than the nuclear radius, scattering experiments should furnish information on nuclear charge and current distributions. Such experiments have been recently performed by Lyman, Hanson, and Scott[3] at 15.7 Mev. At the much higher energies available with synchrotrons, the large angle scattering will be sensitive to the charge and current distribution of the nucleons forming the nucleus.[4]

The theoretical cross sections for Coulomb scattering, which are clearly essential for the above studies, have been given by Mott[5] in the form of a conditionally convergent series. Bartlett and Watson[6] have summed the series numerically for Hg($Z=80$) for a range of energies up to $(v/c)\cong 1$; Massey[7] has employed their results to evaluate the corresponding cross section for the scattering of positrons. McKinley and Feshbach[8] have expanded the Mott series in a power series in α and α/β where

$$\alpha=(Ze^2/\hbar c),\quad \beta=v/c, \tag{1}$$

the coefficients depending only on ϑ, the angle of scattering. The series is accurate up to middle Z elements. For light Z they obtained the following formula, since verified by Dalitz,[9] by the second Born approximation:

$$\sigma=\sigma_R[1-\beta^2\sin^2\tfrac{1}{2}\vartheta+\pi\alpha\beta\sin\tfrac{1}{2}\vartheta(1-\sin\tfrac{1}{2}\vartheta)], \tag{2}$$

where σ_R is the Rutherford cross section,

$$\sigma_R=(Ze^2/2m_0c^2)^2(1-\beta^2)/\beta^4\csc^4\tfrac{1}{2}\vartheta. \tag{3}$$

In the present paper, the Mott series is evaluated for $\beta\simeq 1$, for $Z=13$, 29, 47, 62, 80 for $\vartheta=30°$, 45°, 60°, 80°, 90°, 100°, 135°, 150° for both electron and positron scattering. These results should be adequate for electron energies greater than 4 Mev.

THEORY

Mott's result for the differential scattering cross section is

$$\sigma=\lambda\!\!\!^-{}^2\{q^2(1-\beta^2)|F|^2\csc^2\tfrac{1}{2}\vartheta+|G|^2\sec^2\tfrac{1}{2}\vartheta\}, \tag{4}$$

where $q=(\alpha/\beta)$. The functions F and G are defined as follows:

$$\begin{aligned}
&F=F_0+F_1,\quad G=G_0+G_1,\\
&F_0=(i/2)\exp[iq\ln\sin^2\tfrac{1}{2}\vartheta]\Gamma(1-iq)/\Gamma(1+iq),\\
&G_0=-iqF_0\cot^2\tfrac{1}{2}\vartheta,\\
&F_1=i/2\sum_{k=0}^{\infty}[kD_k+(k+1)D_{k+1}](-)^kP_k(\cos\vartheta),\\
&G_1=i/2\sum_{k=0}^{\infty}[k^2D_k-(k+1)^2D_{k+1}](-)^kP_k(\cos\vartheta),
\end{aligned} \tag{5}$$

where Γ is the gamma-function and P_k the Legendre

TABLE I. Comparison of values for G_1.

	Re G_1		Im G_1	
ϑ	B and W[a]	Euler transformation	B and W	Euler transformation
30°	1.349	1.349	−0.026	−0.027
45°	0.851	0.851	0.292	0.290
60°	0.538	0.538	0.331	0.330
90°	0.213	0.213	0.227	0.228
130°	0.0734	0.0735	0.107	0.107
150°	0.0159	0.0158	0.0272	0.0272

[a] J. H. Bartlett and R. E. Watson (see reference 6).

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[1] W. Paul and H. Reich, Z. Physik **131**, 326 (1952).
[2] E. Kinzinger and W. Bothe (to be published).
[3] Lyman, Hanson, and Scott, Phys. Rev. **84**, 626 (1951).
[4] Amaldi, Dedecaro, and Mariani, Nuovo cimento **1**, 757 (1950).
[5] N. F. Mott, Proc. Roy. Soc. (London) **A124**, 426 (1929); N. F. Mott, Proc. Roy. Soc. (London) **A135**, 429 (1932).
[6] J. H. Bartlett and R. E. Watson, Proc. Am. Acad. Arts Sci. **74**, 53 (1940).
[7] H. S. W. Massey, Proc. Roy. Soc. (London) **A181**, 14 (1942).
[8] W. A. McKinley, Jr., and H. Feshbach, Phys. Rev. **74**, 1759 (1948).
[9] R. H. Dalitz, Proc. Roy. Soc. (London) **A206**, 509 (1951).

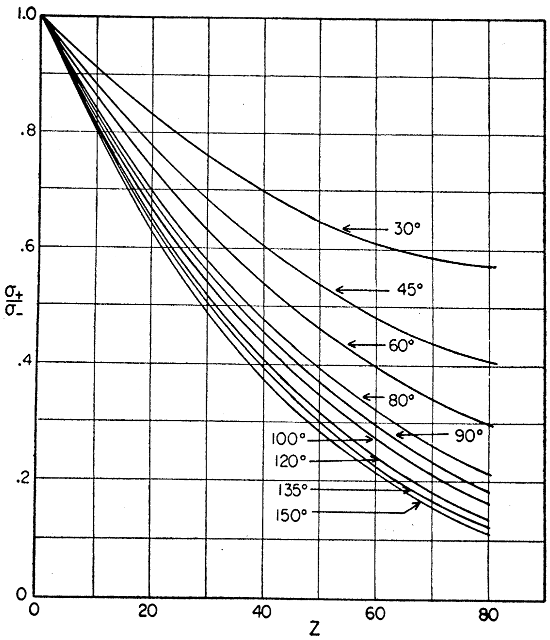

FIG. 1. The ratio of positron to electron scattering (note that β is taken to be one).

polynomial of order k. The coefficients D_k are

$$D_k=\frac{e^{-\pi i k}}{k+iq}\frac{\Gamma(k-iq)}{\Gamma(k+iq)}-\frac{e^{-\pi i\rho_k}}{\rho_k+iq}\frac{\Gamma(\rho_k-iq)}{\Gamma(\rho_k+iq)},\qquad(6)$$

where

$$\rho_k=(k^2-\alpha^2)^{\frac{1}{2}}.$$

The D_k coefficient may be expressed in terms of phases and amplitudes of the complex numbers in (6) as follows:

$$\begin{aligned}(-)^k D_k=(k^2+q^2)^{-\frac{1}{2}}\exp\{-i[\tan^{-1}(q|k)+2\eta_k]\}\\ -(\rho_k{}^2+q^2)^{-\frac{1}{2}}\exp\{-i[\pi(\rho_k-k)\\ +\tan^{-1}(q/\rho_k)+2\sigma_k]\},\qquad(7)\end{aligned}$$

where

$$\eta_k=\text{phase}\Gamma(k+iq),\quad \sigma_k=\text{phase}\Gamma(\rho_k+iq).$$

In evaluating D_k we have placed $\beta=1$, that is, $q=\alpha$, from which it follows that

$$\rho_k{}^2+q^2=k^2,\quad \tan^{-1}(q/\rho_k)=\sin^{-1}(\alpha/k),\quad \text{for}\quad \beta=1.$$

The phases η_k were evaluated for nearly all k by Stirling's approximation which gives the following

TABLE II. Values of the function G defined in Eq. (5).

Z	Real G	Imag G	$\|G\|^2\sec^2\frac{1}{2}\vartheta$	Real G	Imag G	$\|G\|^2\sec^2\frac{1}{2}\vartheta$	Real G	Imag G	$\|G\|^2\sec^2\frac{1}{2}\vartheta$
		$\vartheta=30°$			$\vartheta=45°$			$\vartheta=60°$	
−80	−1.222	−3.512	14.82	−0.980	−1.146	2.664	−0.614	−0.444	0.765
−62	−1.785	−2.281	8.99	−0.965	−0.678	1.628	−0.541	−0.244	0.470
−47	−1.759	−1.352	5.27(5)	−0.824	−0.376	0.961	−0.439	−0.127	0.278
−29	−1.296	−0.521	2.091	−0.558	−0.136	0.386	−0.288	−0.042	0.113
−13	−0.634	−0.104	0.442	−0.265	−0.026	0.0831	−0.136	−0.007	0.0247
13	0.675	−0.102	0.499	0.288	−0.023	0.0978	0.150	−0.005	0.0303
29	1.514	−0.504	2.729	0.680	−0.110	0.556	0.364	−0.019	0.177
47	2.395	−1.303	7.97	1.183	−0.270	1.725	0.657	−0.028	0.576
62	2.996	−2.223	14.92	1.656	−0.439	3.439	0.957	−0.008	1.221
80	3.405	−3.537	25.83	2.270	−0.649	6.530	1.382	0.095	2.559
		$\vartheta=80°$			$\vartheta=90°$			$\vartheta=100°$	
−80	−0.320	−0.142	0.209	−0.230	−0.082(5)	0.119	−0.164	−0.048	0.0707
−62	−0.266	−0.070	0.129	−0.188	−0.038	0.0736	−0.133	−0.020	0.0438
−47	−0.210	−0.032	0.0769	−0.148	−0.016	0.0443	−0.104	−0.007	0.0263
−29	−0.136	−0.007	0.0316	−0.095	−0.002	0.0181	−0.067	0.001	0.0109
−13	−0.064	−0.001	0.0070	−0.045	0.000	0.0040	−0.031	0.000(4)	0.0023
13	0.072	0.000(5)	−0.0088	0.051	0.001	0.0052	0.036	0.001	0.0031
29	0.177	0.008	0.0535	0.126	0.010	0.0320	0.089	0.010	0.0194
47	0.327	0.040	0.185	0.233	0.043	0.112	0.166	0.039	0.0704
62	0.484	0.107	0.419	0.346	0.107	0.262	0.246	0.095	0.168
80	0.706	0.279(5)	0.982	0.502	0.266	0.646	0.354	0.230	0.431
		$\vartheta=120°$			$\vartheta=135°$			$\vartheta=150°$	
−80	−0.078	−0.015	0.025	−0.040	−0.006	0.011	−0.017	−0.0023	0.0044
−62	−0.063	−0.005	0.016	−0.032	−0.001	0.0070	−0.013(5)	−0.000	0.0026
−47	−0.049	−0.001	0.0096	−0.025	0.000	0.0043	−0.010(3)	0.000	0.0015
−29	−0.031	0.001	0.0038	−0.016	0.000	0.0017	−0.007	−0.001	0.0007
−13	−0.015	0.000(4)	0.0009	−0.008	0.000	0.0004	−0.003	0.0001	0.0001
13	0.017	0.001	0.0012	0.009	0.001	0.0006	0.004	0.0003	0.0002
29	0.043	0.007	0.0076	0.022	0.004	0.0034	0.009	0.002	0.0013
47	0.080	0.025	0.0281	0.041	0.015	0.0130	0.017	0.007	0.0050
62	0.117	0.060	0.0692	0.060	0.035−	0.0329	0.025	0.016	0.0132
80	0.164	0.142	0.188	0.083	0.082(5)	0.0935	0.034	0.037	0.0377

TABLE III. Values of the function F defined in Eq. (5).

Z	Real F	Imag F	$\|F\|^2 \csc^2\frac{1}{2}\vartheta$	Real F	Imag F	$\|F\|^2 \csc^2\frac{1}{2}\vartheta$	Real F	Imag F	$\|F\|^2 \csc^2\frac{1}{2}\vartheta$
		$\vartheta=30°$			$\vartheta=45°$			$\vartheta=60°$	
−80	−0.4841	0.1889	4.031	−0.3825	0.3661	1.914	−0.2821	0.4585	1.1592
−62	−0.3939	0.3290	3.932	−0.2778	0.4392	1.844	−0.1849	0.4920	1.1050
−47	−0.2989	0.4108	3.853	−0.1936	0.4741	1.791	−0.1164	0.5027	1.0650
−29	−0.1789	0.4703	3.780	−0.1050	0.4939	1.741	−0.0536	0.5038	1.0267
−13	−0.0767	0.4947	3.741	−0.0411	0.4994	1.715	−0.0167	0.5012	1.0059
13	0.0703	0.4956	3.740	0.0320	0.5000	1.714	0.0053	0.5015	1.0061
29	0.1484	0.4792	3.757	0.0602	0.5008	1.737	−0.0036	0.5076	1.0307
47	0.2266	0.4468	3.747	0.0795	0.5020	1.764	−0.0336	0.5171	1.0741
62	0.2844	0.4051	3.657	0.0878	0.4988	1.752	−0.0740	0.5202	1.1043
80	0.3436	0.3298	3.386	0.0932	0.4780	1.619	−0.1353	0.5007	1.0760
		$\vartheta=80°$			$\vartheta=90°$			$\vartheta=100°$	
−80	−0.1737	0.5208	0.7295	−0.1303	0.5382	0.6133	−0.0932	0.5503	0.5309
−62	−0.0924	0.5248	0.6873	−0.0565	0.5331	0.5748	−0.0264	0.5384	0.4952
−47	−0.0424	0.5190	0.6563	−0.0142	0.5227	0.5468	0.0094	0.5247	0.4693
−29	−0.0055	0.5088	0.6266	0.0127	0.5097	0.5199	0.0280	0.5100	0.4446
−13	0.0059	0.5019	0.6098	0.0145	0.5020	0.5044	0.0216	0.5020	0.4303
13	−0.0200	0.5018	0.6104	−0.0296	0.5017	0.5052	−0.0378	0.5015	0.4310
29	−0.0653	0.5078	0.6344	−0.0892	0.5064	0.5288	−0.1094	0.5045	0.4541
47	−0.1466	0.5124	0.6875	−0.1908	0.5056	0.5841	−0.2283	0.4978	0.5111
62	−0.2403	0.5022	0.7502	−0.3059	0.4848	0.6572	−0.3615	0.4654	0.5918
80	−0.3773	0.4446	0.8230	−0.4726	0.4005	0.7675	−0.5531	0.3530	0.7337
		$\vartheta=120°$			$\vartheta=135°$			$\vartheta=150°$	
−80	−0.0358	0.5643	0.4263	−0.0052	0.5696	0.3801	0.0160	0.5724	0.3514
−62	0.0198	0.5437	0.3947	0.0441	0.5452	0.3505	0.0608	0.5457	0.3231
−47	0.0454	0.5261	0.3718	0.0643	0.5261	0.3291	0.0773	0.5257	0.3026
−29	0.0512	0.5097	0.3499	0.0633	0.5092	0.3085	0.0716	0.5087	0.2828
−13	0.0326	0.5017	0.3370	0.0384	0.5015	0.2964	0.0424	0.5014	0.2714
13	−0.0503	0.5009	0.3379	−0.0570	0.5005	0.2973	−0.0616	0.5003	0.2723
29	−0.1407	0.5004	0.3603	−0.1572	0.4976	0.3190	−0.1687	0.4955	0.2936
47	−0.2865	0.4816	0.4187	−0.3174	0.4712	0.3781	−0.3387	0.4631	0.3528
62	−0.4474	0.4269	0.5099	−0.4928	0.4023	0.4741	−0.5240	0.3835	0.4519
80	−0.675(3)	0.2608	0.6987	−0.7385	0.2022	0.6869	−0.7812	0.1578	0.6808

asymptotic series:

$$\eta_k \cong (q/2)\ln(k^2+q^2)+(k-\tfrac{1}{2})\tan^{-1}(q/\rho_k) - q[1+(1/12)(k^2+q^2)^{-1} - (1/360)(3k^2-q^2)(k^2-q^2)^{-3}]. \quad (8)$$

To obtain σ_k replace k by ρ_k in Eq. (8). Equation (8) could almost always be employed down to $k=2$ with little error. For $k=1$, the standard formula given by Jahnke-Ende[10] or the gamma-function recurrence relation was employed.

The values of D_k so obtained were inserted into the series for F_1 and G_1 which were then summed numerically. Since the series are only conditionally convergent, a summation method appropriate to a series being evaluated on its circle of convergence was employed. This is the Euler transformation,[11] which was found to be accurate in reference 8. Another check is provided by comparison with Bartlett and Watson's results for $Z=80$, as given in Table I.

The B and W values will differ somewhat from the values they quote because we have included several more terms in the infinite series they needed to sum. It was also necessary to extrapolate in β to $\beta=1$.

The results of these calculations are given in Tables II and III. The negative Z refers to positron scattering, the positive Z to electron scattering. In Fig. 1 we plot the ratio of the cross section for positron scattering to that for electron scattering. We note that this ratio is less than one, decreasing as either ϑ or Z increases. Effects of this kind have been observed by Lipkin and White.[12]

[10] E. Jahnke and F. Ende, *Funktrontafeln* (B. G. Teubner, Leipzig, 1933).

[11] See, for example, T. J. I. 'A, Bowmwich, *Introduction to the Theory of Infinite Series* (The MacMillan Company, New York, 1947).

[12] H. J. Lipkin and M. G. White, Phys. Rev. **79**, 892 (1950).