expression for $N(R)$ is used in this equation, one finds

$$_0^{\mathrm{H}}\langle Q_e\rangle_1^{\mathrm{theor}}=a_0^2[{}_0^{\mathrm{H}}\langle(R/R_e)^2\rangle_1(R_e/a_0)^2-0.884$$
$$-0.614\ {}_0^{\mathrm{H}}\langle(R-R_e)/R_e\rangle_1(R_e/a_0)$$
$$+0.410\ {}_0^{\mathrm{H}}\langle(R-R_e)^2/R_e^2\rangle_1(R_e/a_0)^2]. \quad (13)$$

The various averages can be obtained from Ramsey's[14] paper on the effects of zero-point vibration and when these are used, the above equation gives

$$_0^{\mathrm{H}}\langle Q_e\rangle_1^{\mathrm{theor}}=(0.344\pm0.010)\times10^{-16}\ \mathrm{cm}^2. \quad (14)$$

The agreement between the experimental value of Eq. (11) and the theoretical value of Eq. (14) is within the overlap of the errors.

The results of the present measurements are all summarized in Table VI. These are in good agreement with values previously quoted.[4] There are large improvements in accuracy with the present results. $(1-\sigma_{J1})b/H$ is improved by a factor of 100 over previous measurements.[7] The best existing values for γ_p limits the accuracy of the rotational magnetic moment as calculated from the present results, but, there still is an improvement of about 100 over previous measurements. The dependence of the diamagnetic susceptibility on the orientation of the molecules' rotational angular momentum, $\xi_{\pm1}-\xi_0$, is improved from a 65 percent uncertainty to only 5 percent; and the quadrupole moment of the electron distribution, Q_e, from 40 percent to 4 percent uncertainty.

The experimental value of Q_e compares favorably to the theoretically calculated value. The usefulness of the experimental value of Q_e lies in a check of the wave functions used in the theoretical calculation of Q_e and the wave function used to calculate the gradient of the electric field at the nucleus resulting from the electrons.

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Multiple Scattering of Neutrons. III. Scattering by Spin-Dependent Forces and Polarization Phenomena

OTTO HALPERN
University of Southern California, Los Angeles, California
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The treatment presented in two preceding papers is extended to cover the case of spin-dependent forces. A general theorem is derived which reduces the problem of scattering by spin-dependent forces to the problem of scattering by spin-independent forces, which has previously been solved analytically. It is shown that observations on the neutrons returning from the scatterer can be used in many practical cases to determine the ratio of the two scattering amplitudes. Similarly, for very small probability of depolarization per single collision, the depolarization of the neutrons transmitted through the scatterer offers information about the spin dependence of the scattering amplitudes. The necessary material for the evaluation of the observation is presented in table form. A comparison is made with a paper by Borowitz and Hamermesh in which the same subject has been treated by a different method.

I. INTRODUCTION

IN two preceding papers[1] the multiple scattering of slow neutrons in a plane parallel sheet of infinite extension has been discussed. The physical assumptions underlying the treatment were the following: The scattering probability is spherically symmetrical; the scattering is purely elastic; capture processes are admitted. It was also assumed that no correlation exists for the scattering from the various centers. The present paper, as announced at the end of II, extends the treatment to the case of spin-dependent nuclear forces.[2]

The physical interest of the problem here discussed can perhaps best be explained as follows. It has been shown in an earlier paper[3] that the probability Q for the inversion of the spin of the incident neutron during a scattering process is given by

$$Q=\frac{2i(i+1)}{3(2i+1)}\frac{(a_1-a_0)^2}{ia_0^2+(i+1)a_1^2}. \quad (1)$$

Here i denotes the spin of the nucleus in units of $h/2\pi$; a_0 and a_1 stand for the scattering amplitude of the system nucleus plus neutron if its total angular momentum is $i-\frac{1}{2}$ and $i+\frac{1}{2}$, respectively.

One can see by a simple analysis of (1) that Q has a maximum value of $\frac{2}{3}$; this maximum value is taken on for the case of

$$a_0+3a_1=0. \quad (2)$$

The relation (2) is approximately satisfied, for example, in the case of H. One can see in general that sizeable values of Q will only occur if the two amplitudes have opposite sign.

[1] Halpern, Luneburg, and Clark, Phys. Rev. **53**, 173 (1938) (I); O. Halpern and R. K. Luneburg, Phys. Rev. **76**, 1811 (1949) (II).
[2] O. Halpern, Phys. Rev. **75**, 1633 (1949).
[3] O. Halpern and M. H. Johnson, Phys. Rev. **55**, 898 (1939).

If, on the other hand, a_0 and a_1 differ only by a small percentage, then the value of Q diminishes very rapidly. If, e.g., a_0 and a_1 differ by 20 percent, then Q is of the order of 1/150.

Direct experiments[4] of a single scattering type have successfully demonstrated the spin-dependence of the scattering amplitudes of some elements. The observed values were necessarily somewhat inaccurate; the evaluation of the elaborate experiments led, for example, to values of Q exceeding $\frac{2}{3}$. It can scarcely be hoped that the accuracy of this type of experiment can be very much improved and most obviously an arrangement of this kind will be useful only if Q has sizeable values. If, on the other hand, the two amplitudes differ by a small percentage only, then an arrangement involving multiple scattering becomes necessary so that a sufficiently large depolarization effect be obtained.

The result of a large number of collisions during a diffusion process cannot be foreseen by simple arguments, since the number of collisions undergone by the re-emerging neutrons will show large fluctuations. A theory is therefore required which gives directly the polarization state of the emerging neutrons if that of the incident particles is known. Observation on the amount of depolarization will then allow us, with the aid of the theoretical formulas, to determine the value of Q and thereby the ratio of a_0 and a_1.

The usefulness of the study of multiple scattering for the determination of Q has already been recognized by other authors[5] in an earlier paper, which we shall discuss at the end of the present note.

II. THE INTEGRO-DIFFERENTIAL EQUATION FOR SPIN-DEPENDENT FORCES

We shall now show that the problem of diffusion under the influence of spin-dependent forces can be, without difficulty, reduced to the problem already solved, of diffusion under the influence of spin-independent forces.[1]

Following I and II we write again the specialized Boltzmann transport equation for the diffusion of neutrons in the one-dimensional case,

$$\xi\partial w(x,\xi)/\partial x+Aw(x,\xi)=B\bar{w}(x),\quad \xi=\cos\vartheta. \tag{3}$$

Here, w denotes, as before, the probability of finding neutrons at the distance x from the boundary, the velocity of which makes an angle ϑ with the x direction; $\bar{w}$ is given by

$$\bar{w}(x)=\int_{-1}^{+1} w(x,\xi)d\xi. \tag{4}$$

A and B determine the probability of scattering and capture and of scattering alone, respectively. The ratio B/A is again called σ. In the absence of capture, σ takes on the limiting value $\frac{1}{2}$. As before, the treatment is limited to values of of σ in the neighborhood of $\frac{1}{2}$.

Equation (3) holds true in the presence of polarized neutrons and spin-dependent forces as long as we understand w to represent the distribution of all neutrons independent of their spin. To discuss the polarization, we need a second equation which can be obtained easily. Denoting by Π the function analogous to $w(x,\xi)$ but referring to the polarized part only, we observe that always $\Pi\leqq w$ and that for a completely unpolarized condition Π becomes zero. The ratio Π/w gives the percentage of polarization.

The transport equation for Π reads as follows:

$$\xi\partial\Pi(x,\xi)/\partial x+A\Pi(x,\xi)=B(1-2Q)\bar{\Pi}. \tag{5}$$

In fact, the left side of the transport equation, giving the amount of polarization, leaving a certain x, ξ domain is unchanged as compared with (3). The right side, on the other hand, exhibits two changes: The gain in polarization through the entrance of particles into the specified domain is now given by

$$B\bar{\Pi}(1-Q),$$

since only those particles which retain their polarization can be added. Particles, on the other hand, which change their polarization during collision give a negative contribution, their fraction being

$$B\bar{\Pi}Q.$$

The addition of these contributions gives the right side of (5).

The boundary conditions, as discussed in II, take the same form for w and Π. They read

$$w(0,\xi)=f(\xi),\quad \xi>0, \tag{6a}$$

$$w(l,\xi)=0,\quad \xi<0, \tag{6b}$$

$$\Pi(0,\xi)=g(\xi),\quad \xi>0, \tag{7a}$$

$$\Pi(l,\xi)=0,\quad \xi<0. \tag{7b}$$

Here $f(\xi)$ and $g(\xi)$ are known functions.

Equation (3) has been solved in II. The resultant solution for $w(0,\xi)$ and $w(l,\xi)$ as well as for $w(x,\xi)$ inside of the slab was there given in terms of σ and a new transcendental function $P_\sigma(\xi)$, the values of which in the neighborhood of $\sigma\sim\frac{1}{2}$ and for $\xi>0$ can be found in II in the form of a power series and in tables.

By a simple transformation (5) can now be reduced to (3). Substitute in (5) the following:

$$x'=x(1-2Q), \tag{8a}$$

$$A'=A/(1-2Q). \tag{8b}$$

We then obtain

$$\xi\partial\Pi/\partial x'+A'\Pi=B\bar{\Pi}. \tag{9}$$

(3) and (9) are equivalent if one replaces the old by

[4] W. E. Meyerhof and D. B. Nicodemus, Phys. Rev. 82, 5 (1951).
[5] S. Borowitz and M. Hamermesh, Phys. Rev. 74, 1285 (1948).

TABLE I. Values of $P_\sigma^{-1}(z)$.

$\sigma \diagdown z$	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.500	1.00	1.24	1.45	1.64	1.83	2.01	2.19	2.37	2.55	2.73	2.91
0.499	1.00	1.23	1.42	1.60	1.77	1.94	2.10	2.25	2.40	2.55	2.70
0.498	1.00	1.23	1.41	1.59	1.75	1.90	2.06	2.20	2.35	2.49	2.62
0.496	1.00	1.22	1.40	1.57	1.72	1.86	2.00	2.13	2.27	2.39	2.52
0.494	1.00	1.22	1.39	1.55	1.69	1.84	1.96	2.09	2.21	2.32	2.44
0.492	1.00	1.21	1.38	1.54	1.67	1.81	1.93	2.05	2.17	2.27	2.38
0.490	1.00	1.21	1.37	1.53	1.66	1.79	1.90	2.01	2.13	2.22	2.32
0.485	1.00	1.20	1.36	1.50	1.62	1.74	1.85	1.95	2.05	2.14	2.23
0.480	1.00	1.19	1.35	1.48	1.59	1.70	1.80	1.89	1.98	2.06	2.14
0.470	1.00	1.18	1.32	1.44	1.55	1.64	1.73	1.81	1.88	1.96	2.02
0.460	1.00	1.17	1.30	1.41	1.51	1.60	1.67	1.74	1.81	1.87	1.92
0.450	1.00	1.17	1.29	1.39	1.48	1.55	1.63	1.69	1.74	1.80	1.85
0.440	1.00	1.16	1.27	1.37	1.45	1.52	1.58	1.64	1.70	1.75	1.79
0.430	1.00	1.15	1.26	1.35	1.42	1.49	1.55	1.60	1.65	1.69	1.73
0.420	1.00	1.15	1.25	1.33	1.40	1.46	1.52	1.57	1.61	1.65	1.68
0.410	1.00	1.14	1.24	1.31	1.38	1.44	1.49	1.53	1.58	1.61	1.64
0.400	1.00	1.13	1.23	1.30	1.36	1.41	1.46	1.50	1.54	1.57	1.60

the new variable and also changes the coefficients in the manner indicated. The substitution (8) is legitimate as long as $Q<\frac{1}{2}$. Since we are only interested in the problem of small values of Q, the substitution (8) is always valid in our case.

The presence of spin-dependent forces thus affects the transport equation by changing the unit of length and increasing the "capture cross section." The percentage of polarization can immediately be determined if the ratio of the two solutions of (3) and (9) is formed.

III. DISCUSSION OF THE SOLUTION

The effect of depolarization can be studied observing the returning as well as the transmitted neutrons; the magnitude of Q and perhaps experimental convenience will decide the method to be used. We want to illustrate here the evaluation for some special cases of interest. The discussion will be carried out under the assumption that $A=2B$. This means that there is no direct capture present. A' is, of course, larger than $2B$.

This restriction is only introduced for illustrative purposes, since the formulas in II include the general case, with true capture present.

If observations are made on the returning beam, then it is convenient to assume that the slab is infinitely thick; this restriction again is insignificant, since II, formulas (5.5) and (6.3), are valid also for large but finite thickness of the slab.

The velocity distribution of the returning beam $w(0,\xi)$ then reads

$$w(0,\xi)=\frac{\xi_0}{2(\xi_0-\xi)}\frac{1}{P_0(\xi_0)}\frac{1}{P_0(-\xi)}. \tag{10}$$

In (10) P_0 denotes the new transcendental function $P_\sigma(\xi)$ discussed before for the value of $\sigma=\frac{1}{2}$; ξ is, of course, always negative, while ξ_0 denotes the direction cosine of the velocity of the incident beam. For most practical cases $\xi_0=1$; but, as shown in I and II, the general case of an arbitrary velocity distribution can be obtained by multiplying (10) with the distribution function and integrating over ξ_0.

Similarly, $\Pi(0,\xi)$ is given by

$$\Pi(0,\xi)=\frac{\sigma\xi_0}{(\xi_0-\xi)}\frac{1}{P_\sigma(-\xi)}\frac{1}{P_\sigma(\xi_0)}. \tag{11}$$

The main difference between (10) and (11) is the appearance of σ in place of the factor $\frac{1}{2}$ and the replacement of P_0 by P_σ. We repeat the definition of σ in the present case

$$\sigma=B/A', \tag{12}$$

The percentage of polarization Π/w is now given by

$$\chi(\xi)=\frac{\Pi(0,\xi)}{w(0,\xi)}=2\sigma\frac{P_\sigma(-\xi)P_\sigma(\xi_0)}{P_0(-\xi)P_0(\xi_0)}. \tag{13}$$

It depends, as we see, quite markedly on the direction cosine of the velocity of the returning beam.

Table I gives the values of $P_\sigma(\xi)$ in the range of greatest practical interest. The polarization ratio can be determined with its aid by two simple divisions. We notice that the percentage of polarization of the returning beam increases monotonously with decreasing $-\xi$. Depolarization is by far strongest for neutrons which emerge perpendicular to the boundary.

Numerically, one can see that the method of measuring the depolarization of returning neutrons will require a value of Q which is not too small. Assuming for the sake of discussion that a depolarization effect of about 30 percent is required for sufficiently accurate measurements, then we see from Table I that observation of neutrons returning almost perpendicularly would give such an effect if 2σ is about 0.99. This would mean a depolarization probability Q per collision of approximately 1/200 or a difference in the scattering amplitudes of about 20 percent. Smaller differences between the two scattering amplitudes could then not be measured with the aid of observations on returning beams.

The expressions for transmitted beams are given in II (6.4) and (5.6) with the aid of II (5.13), (6.28), and (4.23). They read as follows:

$$w(l,\xi)=\frac{\xi_0}{2(m+Al)}\frac{1}{P_0(\xi_0)}\frac{1}{P_0(\xi)}. \tag{14}$$

$$\Pi(l,\xi)=\frac{2\sigma\xi_0 e^{-A'l'/\alpha}\alpha\mathcal{S}(\alpha)}{1-\mathcal{S}^2(\alpha)e^{-2A'l'/\alpha}}\times\frac{1}{(\alpha-\xi_0)(\alpha-\xi)}\frac{1}{P_\sigma(\xi_0)}\frac{1}{P_\sigma(\xi)}. \tag{15}$$

The quantities α, $\mathcal{S}(\alpha)$, and m are defined by the relations

$$\sigma=\left(\alpha\log\frac{\alpha+1}{\alpha-1}\right)^{-1}, \tag{16}$$

$$\mathcal{S}(\alpha)=(1/2\alpha)[P_\sigma^2(\alpha)/\rho'(\alpha)], \tag{17}$$

$$\rho(z)=P_\sigma(z)P_\sigma(-z), \tag{18}$$

$$m=1.43. \tag{19}$$

If $\alpha \gg 1$, we have approximately

$$\mathcal{S}(\alpha)=1-(m/\alpha). \tag{19a}$$

(14) together with the values of Table I shows that the transmitted intensity is approximately given by

$$\mathcal{J}=\int_0^1 \xi w(l,\xi)d\xi=\frac{\xi_0}{2(m+Al)}\frac{1}{P_0(\xi_0)}\int_0^1\frac{\xi d\xi}{P_0(\xi)}$$

$$\cong\frac{1.68}{1.43+\mathrm{Al}} \text{ for } \xi_0=1. \tag{20}$$

If the thickness of the slab is about ten mean free paths (Al=10), then the transmitted current (in the absence of capture) amounts to about 15 percent of the incident or 17 percent of the returning current.

It is rather easy to find the values of $\Pi(l,\xi)$ for various practically occurring values of σ. A simple inspection of the structure of (15) shows that one has it in one's hand by moderately increasing the thickness to obtain sizeable depolarization effects even if σ lies very close to $\frac{1}{2}$; i.e., Q is very small. It is fortunate that the decrease of the transmitted current with thickness is so slow that even for thicknesses which are very large compared to the mean free path, the transmitted intensity forms a substantial fraction of the returning intensity. The experimenter should find no difficulty in choosing the proper arrangement with the aid of the formulas here presented.

The paper by Borowitz and Hamermesh[5] has several points in common with the present investigation; there exist, on the other hand, fundamental differences[2] in the treatment of the problem, the numerical results obtained, and the interpretation of the solution.

The authors named start with the same transport equation and equally use our expression for Q.[3] Since they do not find themselves in the possession of our analytic solution of the diffusion problem, they use an approximate numerical method due to Wick to solve the problem for the special case where the thickness of the slab equals ten mean free paths. The determination of the ratio of polarization is done in the same manner in both papers.

The data given by Borowitz and Hamermesh as a result of their numerical calculations of one single case differ considerably from those obtained by our general analytic solution as specialized for the condition discussed by the authors. Since we have at present no reason to doubt the correctness of our analytic solution, we are forced to conclude that the numerical method used is unsatisfactory. To illustrate the differences occurring, we refer to their value for the transmitted density of 0.3, while our value (in the absence of any capture) is only 0.25. Similarly, they give for the transmitted current, which constitutes probably the best quantity observable, a value of about 8×10^{-2}, while our analytical value leads to 15×10^{-2}. The reflected current is given by those authors, in the absence of capture, as 1, while we have a value of 0.85. Similar differences exist in the values in the presence of capture.[6] We may conclude that these numerical differences are sufficiently big to affect very seriously any attempts to evaluate Q or the ratio of the scattering amplitudes with the aid of these formulas.

The authors did not make calculations for the use of, nor did they discuss, the method of observing the depolarization of returning neutrons; it seems that their calculations would not indicate any sizeable depolarization in this case. The authors give, furthermore, a discussion of the intensity transmitted and the choice of the thickness most advisable for various values of Q, which seems to us to be due to a mistake. They mention in the text on pages 1291 and 1292 as well as in their Table II that the intensity of the emerging beam in case of a thickness of the slab of ten mean free paths is about 10^{-4} of that of the incident beam. Therefore, they advise the experimenter to choose carefully between a very big loss in intensity with increased thickness and a larger depolarization obtained thereby. This statement must be due to a misunderstanding of the formulas. The transmitted current for the case discussed is, as we have shown, not 10^{-4} of the incident or returning current but rather 15 percent or, respectively, 17 percent of it. We have pointed out at the end of the preceding paragraph that the decrease in the transmitted intensity, even for large thicknesses, is so slow that, in the case of very small Q's, thick slabs may be chosen with great advantage.

[6] W. E. Meyerhof and D. B. Nicodemus, Phys. Rev. **82**, 5 (1951), express the opinion (see their footnote 6) that the differences between our results, as announced previously (see reference 2), and those of Borowitz and Hamermesh are due to a wrong labeling of their Fig. 3. We are unable to follow this opinion. In particular, the terminal value for the density of 0.3, as given in Fig. 3 of Borowitz and Hamermesh, is confirmed by their value given in the text, namely 299×10^{-3}. Furthermore, as shown in the text of the present note, there are (even larger) differences which arise from comparison with the other diagrams of Borowitz and Hamermesh.