Propagation in Electron-Ion Streams

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A mathematical theory is given for the propagation of electromagnetic waves in electron-ion streams composed of N discrete beams. The solution, which is fully relativistic, is obtained in vector form by an extension of Hansen's theory and takes explicit account of the initial and boundary conditions. When certain restructions are placed upon the transverse boundary conditions the general solution satisfying arbitrary initial conditions can be expanded in terms of a complete orthogonal set of elementary vector solutions. For this case the necessary and sufficient conditions are found for amplification and instability both in the terminated and the unterminated stream. The correct physical interpretation of the conventional "Ansatz" solutions is found together with the conditions under which they are valid. One is then able to distinguish amplified growing waves from reverse waves attenuated in the reverse direction. Finally the analysis is extended to the continuous velocity distribution. It is shown that the present treatment differs significantly from Landau's theory for the thermal plasma and the consequences of this are discussed.

1. INTRODUCTION

THE development of electronic tubes such as the traveling and space-charge wave amplifiers and the discovery that sunspots and solar flares can produce excess noise radiation has led several writers[1–5] to extend the magneto-ionic theory of Appleton to the more general case where the medium contains moving electron-ion streams. However there is considerable disagreement over the physical interpretation of these theories, particularly as to the conditions under which a medium can amplify a given initial disturbance or radiate nonthermal energy into free space. Furthermore some of this work is open to criticism on purely mathematical grounds, especially in those treatments which assume a continuous velocity distribution in the stream.

It is probable that much of the trouble has arisen because the mathematical analysis upon which these theories are usually based contains no explicit reference to the initial and boundary conditions, a procedure the dangers of which have been recently emphasized by Pierce.[6]

It has not proved possible to carry through, in closed form, a complete solution for the propagation in an electron-ion stream which satisfies arbitrary initial and boundary conditions, but this can be done when certain restrictions are placed upon the latter. The resulting solution covers cases of considerable practical importance, which enable one to form an unambiguous picture of the physical nature of the propagation in a moving ionized medium. In particular it enables one to discriminate between waves that are excited directly and those that are excited only by reflections, to find necessary and sufficient conditions under which amplification or instability can take place, and to compare the energy carried by the radiation field to that carried by the moving space charge.

The first part of this paper is largely mathematical and the nature of the solution is discussed only in general terms. The theory is applied to specific cases to establish the physical conditions which can lead to amplification and instability in nature.

2. SIMPLIFYING ASSUMPTIONS AND INITIAL CONDITIONS

The theory of propagation in moving ionized media has been developed along three main lines to apply respectively to gas discharges, to the ionosphere, and to electron wave tubes, and the simplifying approximations employed have naturally depended on which of these cases is to be studied. In the present paper the analysis has been restricted as little as possible consistent with the over-all aim of getting a complete solution under at least some physically realizable boundary conditions. However when a choice has to be made it has been taken with the last two applications in mind rather than with the case of the gas discharge. In fact the starting point of this paper is the fundamental article by Hahn[7] on electron beams and, as far as possible, the symbolism has been chosen to coincide with his. On this standpoint the limiting assumptions are as follows.

(1) *Linearity.* We consider the small signal case, where the square and cross product terms of the time dependent field and space-charge quantities may be neglected, when the Maxwell-Lorentz equations become linear.

(2) *Rectilinear flow.* We assume that the stream is composed of a system of N superimposed beams of charged particles, both electrons and ions, moving in rectilinear unaccelerated paths. Ideally such a flow can be established as discussed by Hahn.[7] In the text we restrict ourselves to the case where all these paths are parallel and where the external force system consists simply of an axial magnetic field. The extension to the cases where the beams are not all parallel and where the external magnetic field has a transverse component can only be achieved with a considerable increase in complexity and in a Cartesian system of coordinates.

[1] V. A. Bailey, Australian J. Sci. Research A.1, 351 (1948); Phys. Rev. 83, 439 (1951).
[2] E. P. Gross, Phys. Rev. 82, 232 (1951).
[3] D. Bohm and E. P. Gross, Phys. Rev. 75, 1851, 1864 (1949).
[4] A. V. Haeff, Proc. Inst. Radio Engrs. 37, 1 (1949).
[5] J. Feinstein and H. K. Sen, Phys. Rev. 83, 405 (1951).
[6] J. R. Pierce, Bell System Tech. J. XXX, no. 3, 626 (1951).
[7] W. C. Hahn, Gen. Elec. Rev. 42, 258 (1939).

The general case, where the charged particles follow nonrectilinear paths under arbitrary external fields, appears as yet quite intractable in a relativistic theory.

(3) *Uniform flow*. We assume that the beams are uniform in cross section so that the dc velocity and space charge density in the stream are independent of position. The analysis will therefore not be applicable to a slipping stream, but the limitation appears essential if the initial conditions are to appear explicitly in a solution in closed form.

(4) *Collisions*. Owing to the discrete nature of the space charge and the presence of ions and neutral molecules both electron-electron and electron-ion scattering will take place. In this paper we follow the usual procedure and treat the stream as if it were a charged jelly. That is we take e/m equal to its experimental value while letting e and m tend to zero. The effects of scattering are then allowed for by introducing a force term $\nu_s m_s \mathbf{v}_s$ into the into the Lorentz force equation where ν_s is the collision frequency of a particle mass m_s and velocity $\mathbf{v}_s$.

This procedure is taken over from the propagation theory of the ionosphere; its extention to the general electron-ion stream is of dubious validity. Thus in a thermal plasma the velocity distribution of the charged particles will presumably be everywhere Maxwellian, but when nonthermal beams are present scattering will change the velocity distribution along the beam; since this is a nonlinear effect it cannot be allowed for by our theory which will only apply when the scattering frequency ν_s is small. Another objection to this procedure is that it cannot allow for the production of radiation by free-free transitions the cross sections of which are proportional to e^2 so that the total incoherent radiation is proportional to Ne^2 or eI which tends to zero on this approximation. This fact must be borne in mind when considering the escape of radiation from an electron-ion stream. However despite these drawbacks we should get a qualitatively accurate approximation for the effects of scattering on coherent phenomena such as amplification or instability.

(5) *Initial conditions*. We assume that the stream is excited by an arbitrary initial spatial distribution of disturbance concentrated between the planes $z=0$, $z=d$, and by a time dependent disturbance at the plane $z=0$ which is not necessarily a surface of discontinuity in the medium.

The principal symbols used in this paper are as follows:

$\mathbf{a}=(0,0,1)$; a unit vector parallel to the z axis.
$\mathbf{B}_0=(0,0,B_0)$; the dc magnetic flux density.
c; the velocity of light in vacuum.
$-e_s$; the electric charge on particles of the sth beam.
$\mathbf{E}(x^1,x^2,z,t)$, $\mathbf{H}(x^1,x^2,z,t)$; the electric and magnetic field vectors.
$E_l(z,t)$, $H_l(z,t)l=1-3$; coefficients in the vector expansions of $\mathbf{E}(x^1,x^2,z,t)$, $\mathbf{H}(x^1,x^2,z,t)$, respectively.
k; the L_z-transform mate of z.
$K_s=1/(1-u_{0s}^2/c^2)$.
m_{0s}, m_s; the rest and relativistic transverse mass of particles of the sth beam, respectively.
$\mathbf{L}$, $\mathbf{M}$, $\mathbf{N}$; fundamental vectors $\nabla\phi(x^1,x^2)$, $\nabla\phi(x^1,x^2)\times\mathbf{a}$, $\phi(x^1,x^2)\mathbf{a}$, respectively.
N; the number of individual beams of charged particles.
p; the mode parameter; $\nabla^2\phi(x^1,x^2)+p^2\phi(x^1,x^2)=0$.
$r_s(z,t)$; a factor of the space charge density. $\rho_s(x^1,x^2,z,t)=r_s(z,t)\phi(x^1,x^2)$.
$\mathbf{u}_{0s}=(0,0,u_{0s})$, $\mathbf{v}_s$; the dc and ac velocities of particles of the sth beam, respectively.
(x^1,x^2,z); generalized cylindrical co-ordinates.
ϵ_0; the dielectric coefficient of free space$=1/(\mu_0c^2)$.
μ_0; the magnetic permeability of free space$=4\pi\cdot10^{-7}$.
ν_s; collision frequency of particles of sth beam.
ρ_{0s}, $\rho_s(x^1,x^2,z,t)$; dc and ac space charge density of particles of sth beam respectively.
$\phi(x^1,x^2)$; the scalar quantity from which the vector solution is derived.
ω; the L_t-transform mate of t.
$\omega_{H,s}=e_sB_0/m_s$; the cyclotron angular frequency of particles of sth beam.
$\omega_{0s}=(-e_s\rho_{0s}/\epsilon_{0s}m_s)^{\frac{1}{2}}$; the plasma angular frequency of particles of sth beam.
Ω_0; angular frequency of external signal.
$L_t\{f(z,t)\}=f^t(z,\omega)=f^t$; the Laplace transform of $f(z,t)$ with respect to t.
$L_t\{f(0,t)\}=f^t(0,\omega)=f^t(0)$.
$L_z\{f^t(z,\omega)\}=f^{z,t}(k,\omega)=f^{z,t}$; the Laplace transform of $f^t(z,\omega)$ with respect to z.
$L_z\{f(z,0)\}=f^z(k,0)=f^z(0)$.

3. AN ELEMENTARY SOLUTION

We shall now obtain an elementary solution for the Maxwell-Lorentz equations in an electron-ion stream of uniform cross section composed of N discrete superimposed beams moving, with dc velocities, u_{01}, $u_{02}\cdots u_{0N}$ and dc charge densities ρ_{01}, $\rho_{02}\cdots\rho_{0N}$, respectively, under the action of an external axial magnetic field of flux density $\mathbf{B}=(0,0,B_0)$. The conditions under which the general solution can be represented as a linear sum over a set of orthogonal elementary solutions are discussed below.

The electromagnetic fields satisfy the Maxwell equations which in MKS units may be written

$$\left.\begin{aligned}\nabla\times\mathbf{E}&=-\partial\mathbf{B}/\partial t, &\nabla\times\mathbf{H}&=\sum_{s=1}^{N}\rho_s\mathbf{v}_s+\partial\mathbf{D}/\partial t,\\ \nabla\cdot\mathbf{D}&=\sum_{s=1}^{N}\rho_s, &\nabla\cdot\mathbf{B}&=0,\end{aligned}\right\}\quad(3.1)$$

where the summations are taken over the N electron velocity classes. Since the electron flow is in vacuum we have the further relations:

$$\mathbf{B}=\mu_0\mathbf{H},\quad \mathbf{D}=\epsilon_0\mathbf{E},\quad(3.2)$$

where $\mu_0=4\pi\times10^{-7}$ and $\mu_0\epsilon_0=1/c^2$.

Since the charge associated with any one beam is conserved we have the N conservation equations

$$\partial\rho_s/\partial t+\nabla\cdot(\rho_s\mathbf{v}_s)=0,\quad(s=1\cdots N).\quad(3.3)$$

Finally each charged particle moves under the Lorentz force equation which for particles of charge $-e_s$ may be written

$$(\partial/\partial t+\nu_s+\mathbf{v}_s\cdot\nabla)m_s\mathbf{v}_s=-e_s(\mathbf{E}+\mathbf{v}_s\times\mathbf{B})\quad(s=1\cdots N),\quad(3.4)$$

where $m_s=m_{0s}(1-v_s^2/c^2)^{-\frac{1}{2}}$ is the transverse relativistic mass, and ν_s is the collision frequency appropriate for particles of the sth beam. In what follows we shall feel free to omit the subscript s whenever this may be done without ambiguity.

In the small signal theory we express the field and charge variables as the sum of a time independent and a time dependent quantity, and neglect second-order time dependent terms. In the present case

$$\mathbf{v}_{0s}=(0,0,u_{0s}),\quad \mathbf{E}_0=(0,0,0),\quad \mathbf{B}_0=(0,0,B_0),$$

where the z-axis is taken parallel to the direction of flow.

Hence, to the first order, we may write

$$\rho_s\mathbf{v}_s=\rho_{0s}u_{0s}\mathbf{a}+\rho_{1s}u_{0s}\mathbf{a}+\rho_{0s}\mathbf{v}_{1s}, \tag{3.5}$$

$$m_s\mathbf{v}_s=m_{0s}(\mathbf{v}_{0s}+\mathbf{v}_{1s})(1-u_{0s}^2/c^2)^{\frac{1}{2}} - m_{0s}(1-u_{0s}^2/c^2)^{-\frac{3}{2}}(u_{0s}^2/c^2)(\mathbf{v}_{1s}\cdot\mathbf{a})\mathbf{a}, \tag{3.6}$$

where $\mathbf{a}=(0,0,1)$ is a unit vector parallel to the direction of flow. Zero-order quantities are denoted by a zero subscript and first-order quantities by a subscript 1 which will in future be omitted.

The time dependent part of Eqs. (3.1), (3.3), and (3.4) may therefore be written

$$\left.\begin{aligned}&\nabla\times\mathbf{E}=-\mu_0\partial\mathbf{H}/\partial t,\\ &\nabla\times\mathbf{H}=\textstyle\sum(\rho_s u_{0s}\mathbf{a}+\rho_{0s}v_s)+\epsilon_0\partial E/\partial t\\ &\nabla\cdot\mathbf{E}=\textstyle\sum\rho_s/\epsilon_0,\quad \nabla\cdot\mathbf{H}=0\end{aligned}\right\} \tag{3.7}$$

$$\partial\rho_s/\partial t+\nabla\rho_s\cdot u_{0s}\mathbf{a}+\rho_{0s}\nabla\mathbf{v}_s=0\quad (s=1\cdots N) \tag{3.8}$$

$$(\partial/\partial t+\nu_s+u_{0s}\mathbf{a}\cdot\nabla)[m_s\mathbf{v}_s+m_s(u_{0s}^2/c^2)\times(1-u_{0s}^2/c^2)^{-1}(\mathbf{v}_s\cdot\mathbf{a})\mathbf{a}] = -e_s[\mathbf{E}+\mathbf{v}_s\times B_0\mathbf{a}+\mu_0u_{0s}\mathbf{a}\times\mathbf{H}]\quad (s=1\cdots N). \tag{3.9}$$

These are the equations to be solved subject to the initial conditions stated above. The conventional procedure used by Bailey[1] and other writers,[2–5] is to look for plane wave solutions where all quantities have a time-spatial dependence of the form $\exp i(k\cdot x+\omega t)$, and thus obtain a characteristic equation relating k and ω. Despite its simplicity this method has many drawbacks. In particular it breaks down when the electron velocity distribution is continuous and has to be modified in electron tube theory where the cross section of the stream is normally circular-cylindrical rather than rectangular. The principal objection however arises from the difficulty of interpreting the characteristic equation without knowing the actual boundary and initial conditions of which this "Ansatz" procedure takes no explicit account.

A possible line of attack is to make use of the stream function introduced by Bunemann[8] into the theory of the magnetron. In principle, with this method, one does not have to restrict oneself to the linear approximation, but the differential equations determining the stream function are normally too complex to be solved except for the small signal theory. This removes much of the attraction of this method. Furthermore the scalar stream function, defined by

$$\nabla\psi_s=(m_sv_s-e_s\mathbf{A}), \tag{3.10}$$

only exists when

$$\nabla\times(m_s\mathbf{v}_s-e_s\mathbf{A})=0, \tag{3.11}$$

where $\mathbf{A}$ is the vector potential. Equation (3.11) is not a consequence of the Maxwell-Lorentz equations but an additional restriction imposed upon them which is not satisfied when, as in the present case, the charged particles have a dc component of velocity in the direction of the external magnetic field. The vector stream function of Gabor,[9] which may be used under the more general conditions when

$$\mathbf{v}_s\times[\nabla\times(m_s\mathbf{v}_s-e_s\mathbf{A})]=0, \tag{3.12}$$

suffers from similar limitations.

None of these alternatives will be employed here; instead we shall employ a method similar to that developed by Hansen[10] in the theory of antenna radiation in which a set of independent vector solutions are derived from a scalar quantity.

In a source free medium the field vectors all satisfy the dissipative wave equation

$$\nabla(\nabla\cdot\mathbf{C})-\nabla\times\nabla\times\mathbf{C}+h^2\mathbf{C}=0, \tag{3.13}$$

where $h^2=\epsilon\mu\omega^2-i\sigma\mu\omega$, and where all quantities are assumed to vary with time as $\exp(-i\omega t)$.

If ψ is a solution of the scalar wave equation,

$$\nabla^2\psi+h^2\psi=0, \tag{3.14}$$

and if $\mathbf{a}$ is a constant unit vector, then three independent vector solutions of Eq. (3.13) are

$$\mathbf{L}'=\nabla\psi(x^1x^2,z),\quad \mathbf{M}'=\nabla\psi(x^1,x^2,z)\times\mathbf{a},\quad \mathbf{N}'=1/h\nabla\times\mathbf{M}', \tag{3.15}$$

where (x^1, x^2, z) are generalized cylindrical coordinates.

Hansen showed that a complete general solution can be constructed of elementary solutions of the type of Eq. (3.15) when $\psi(x^1, x^2, z)$ is any function satisfying Eq. (3.14) and the boundary conditions.

In charge free space such a solution can also be carried through in spherical coordinates, though not in more complex coordinate systems, but this is not possible when moving charge is present.

For our present purposes it is more convenient to select as independent vectors the time independent orthogonal triad

$$\mathbf{L}=\nabla\phi(x^1,x^2),\ \mathbf{M}=\nabla\phi(x^1,x^2)\times\mathbf{a},\ N=\phi(x^1,x^2)\mathbf{a}, \tag{3.16}$$

where $\mathbf{a}$ is the unit constant vector (0, 0, 1) and $\phi(x^1, x^2)$ is a scalar function.

By analogy with the charge free case, we are led to

[8] O. Bunemann, C.V.D. Report, Mag. **37** (1944) (unpublished); also L. R. Walker in *Microwave Magnetrons*, MIT Radiation Laboratory series No. 6 (McGraw-Hill Book Company, Inc., New York, 1948).

[9] D. Gabor, Proc. Inst. Radio Engrs. **33**, 792 (1945).

[10] W. W. Hansen, Phys. Rev. **47**, 139 (1935).

look for a solution of the field variables of the form

$$\mathbf{E}=E_1(z,t)\mathbf{L}+E_2(z,t)\mathbf{M}+E_3(z,t)\mathbf{N}, \quad (3.17)$$

and

$$\mathbf{H}=H_1(z,t)\mathbf{L}+H_2(z,t)\mathbf{M}+H_3(z,t)\mathbf{N}. \quad (3.18)$$

Similarly for the velocity variables we shall look for a solution

$$\mathbf{v}_s=v_{1s}(z,t)\mathbf{L}+v_{2s}(z,t)\mathbf{M}+v_{3s}(z,t)\mathbf{N} \quad (s=1\cdots N), \quad (3.19)$$

and shall show that Eqs. (3.17)–(3.19) are consistent with the Maxwell-Lorentz equations (3.7)–(3.9) provided that $\phi(x^1, x^2)$ satisfies a two-dimensional wave equation of the form

$$\nabla^2\phi(x^1,x^2)+p^2\phi(x^1,x^2)=0. \quad (3.20)$$

$E_1(z, t)$, etc., should not be confused with (E_{x^1}, E_{x^2}, E_z), the components of $\mathbf{E}(x^1, x^2, z, t)$, which are given by

$$\left.\begin{aligned} E_{x^1}(x^1,x^2,z,t)&=E_1(z,t)[\partial\phi/\partial x^1]/h_1 \\ &\quad +E_2(z,t)[\partial\phi/\partial x^2]/h_2, \\ E_{x^2}(x^1,x^2,z,t)&=E_1(z,t)[\partial\phi/\partial x^2]/h_2 \\ &\quad -E_2(z,t)[\partial\phi/\partial x^1]/h_1, \\ E_z(x^1,x^2,z,t)&=E_3(z,t)\phi. \end{aligned}\right\} \quad (3.21)$$

The fundamental vectors **L**, **M**, **N** form a mutually orthogonal triad so that a linear vector equation involving those three vectors alone will be everywhere satisfied if and only if the coefficient of these vectors on each side of the equation be everywhere equal.

Now if we inspect Eqs. (3.7) to (3.9) which are to be solved we see that the vector operations are of two kinds; namely taking the vector product of a field or charge variable with either ∇ or **a**. If the latter operator be applied to a vector of the form

$$\mathbf{F}=f_1(z,t)\mathbf{L}+f_2(z,t)\mathbf{M}+f_3(z,t)\mathbf{N}, \quad (3.22)$$

we get

$$\mathbf{F}\times\mathbf{a}=-f_2\mathbf{L}+f_1\mathbf{M}, \quad (3.23)$$

which leaves the general form of **F** invariant. However,

$$\nabla\times\mathbf{F}=\partial f_2/\partial z\mathbf{L}+(f_3-\partial f_1/\partial z)\mathbf{M}-f_2\nabla^2\phi\mathbf{a}. \quad (3.24)$$

If we apply this last result to the Maxwell equation,

$$\nabla\times\mathbf{E}=-\mu_0\partial\mathbf{H}/\partial t,$$

we see that Eqs. (3.18) and (3.19) are only self-consistent for all (x^1, x^2) if ϕ and $\nabla^2\phi$ are linearly related, that is if ϕ satisfies the two-dimensional wave equation (3.20), where p^2 is a scalar quantity independent of (x^1, x^2, z, t).

Accordingly we get that

$$\nabla\times\mathbf{F}=\partial f_2/\partial z\mathbf{L}+(f_3-\partial f_1/\partial z)\mathbf{M}+p^2f_2\mathbf{N}. \quad (3.25)$$

The scalar operations in Eqs. (3.7) to (3.9) on the other hand either involve taking the scalar product of a field or space charge vector with ∇ or with **a**, or operating on a vector with $\partial/\partial t$ or $\mathbf{a}\cdot\nabla\equiv\partial/\partial z$. The last two operations clearly leave the form of **F** invariant, while the scalar product of **a** or ∇ with **F** is a scalar linearly proportional to $\phi(x^1, x^2)$. Thus

$$\mathbf{a}\cdot\mathbf{F}=f_3\phi, \quad (3.26)$$

while

$$\nabla\cdot\mathbf{F}=[-p^2f_1+\partial f_3/\partial z]\phi. \quad (3.27)$$

From this it follows that the vector equations of Eqs. (3.7) to (3.9) reduce to equations of the general form

$$\alpha(z,t)\mathbf{L}+\beta(z,t)\mathbf{M}+\gamma(z,t)\mathbf{N}=0, \quad (3.28)$$

while the scalar equations are all of the form

$$a(z,t)\phi=0, \quad (3.29)$$

when **E**, **H**, and $\mathbf{v}_s$ are all of the form of Eq. (3.22) and ϕ satisfies Eq. (3.20).

If these equations are everywhere to be consistent we must have

$$\alpha(z,t)=\beta(z,t)=\gamma(z,t)=a(z,t)=0, \quad (3.30)$$

except in the trivial case when $\phi(x^1, x^2)\equiv 0$.

On substituting for **E**, **H**, $\mathbf{v}_s$ from Eqs. (3.17)–(3.19) in the continuity equations (3.8), we get

$$(\partial/\partial t+u_{0s}\partial/\partial z)\rho_s=-\rho_{0s}[-p^2v_{1s}+\partial v_{3s}/\partial z]\phi \quad (s=1\cdots N), \quad (3.31)$$

using Eq. (3.27), so that we may write

$$\rho_s(x^1,x^2,z,t)=r_s(z,t)\phi(x^1,x^2) \quad (s=1\cdots N), \quad (3.32)$$

where

$$(\partial/\partial t+u_{0s}\partial/\partial z)r_s=-\rho_{0s}[-p^2v_{1s}+\partial v_{3s}/\partial z] \quad (s=1\cdots N). \quad (3.33)$$

From the scalar Maxwell equations of (3.7) we get

$$-p^2E_1+\partial E_3/\partial z=\sum r_s/\epsilon_0, \quad (3.34)$$

$$-p^2H_1+\partial H_3/\partial z=0. \quad (3.35)$$

From the vector Maxwell equations of (3.7) we get

$$\left.\begin{aligned} \partial H_2/\partial z&=\sum\rho_{0s}v_{1s}+\epsilon_0\partial E_1/\partial t, \\ H_3-\partial H_1/\partial z&=\sum\rho_{0s}v_{2s}+\epsilon_0\partial E_2/\partial t, \\ p^2H_2&=\sum(u_{0s}r_s+\rho_{0s}v_{3s})+\epsilon_0\partial E_3/\partial t, \end{aligned}\right\} \quad (3.36)$$

and

$$\left.\begin{aligned} \mu_0\partial H_1/\partial t&=-\partial E_2/\partial z; \\ \mu_0\partial H_2/\partial t&=(E_3-\partial E_1/\partial z); \\ \mu_0\partial H_3/\partial t&=-p^2E_2. \end{aligned}\right\} \quad (3.37)$$

Finally, from the Lorentz force, Eq. (3.9), we get

$$\left.\begin{aligned} &(\partial/\partial t+\nu_s+u_{0s}\partial/\partial z)m_sv_{1s} \\ &\qquad =-e_s[E_1+u_{0s}\mu_0H_2]+m_s\omega_{Hs}v_{2s}, \\ &(\partial/\partial t+\nu_s+u_{0s}\partial/\partial z)m_sv_{2s} \\ &\qquad =-e_s[E_2-u_{0s}\mu_0H_1]-m_s\omega_{Hs}v_{1s}, \\ &(\partial/\partial t+\nu_s+u_{0s}\partial/\partial z)m_sK_sv_{3s}=-e_sE_3, \end{aligned}\right\} \quad (3.38)$$

where

$$\omega_{Hs}=e_sB_0/m_s \quad (3.39)$$

is the cyclotron angular frequency associated with particles of the sth beam, and where $K_s=(1-u_{0s}^2/c^2)^{-1}$.

These equations are just sufficient to determine the set of unknown quantities in terms of the initial conditions. It follows that a particular solution of Eqs. (3.7) to (3.9) can be found in the form of Eqs. (3.17) to (3.19).

To this stage the solution is valid even when the dc quantities are arbitrary functions of z. We now assume that these quantities are all independent of position, and take the Laplace transforms of Eqs. (3.33) to (3.38) first with respect to t and then with respect to z, where

$$\left.\begin{aligned} L_t\{f(z,t)\}&=f^t(z,\omega)\\ &=\int_0^\infty f(z,t)\exp(-i\omega t)dt,\\ L_t^{-1}\{f^t(z,\omega)\}&=f(z,t)\\ &=\frac{1}{2\pi}\int_{-\infty-i\gamma_1}^{\infty-i\gamma_1}\exp(i\omega t)f^t(z,\omega)d\omega,\end{aligned}\right\} \tag{3.40}$$

and

$$\left.\begin{aligned} L_z\{f^t(z,\omega)\}&=f^{z,t}(k,\omega)\\ &=\int_0^\infty f^t(z,\omega)\exp(-ikz)dz,\\ L_z^{-1}\{f^{z,t}(k,\omega)\}&=f^t(z,\omega)\\ &=\frac{1}{2\pi}\int_{-\infty-i\gamma_2}^{\infty-i\gamma_2}f^{z,t}(k,\omega)\exp(ikz)dk.\end{aligned}\right\} \tag{3.41}$$

γ_1, γ_2 are positive real numbers such that all the singularities of $f^t(z,\omega)$ lie above the line $\mathrm{Im}(\omega)+\gamma_1=0$ and all the singularities of $f^{z,t}(k,\omega)$ lie above the line $\mathrm{Im}(k)+\gamma_2=0$ in the complex ω and k-planes, respectively.[11]

There is no agreed nomenclature for double Laplace or Fourier transforms of the Maxwell equations. The present choice has the virtue of being self-explanatory and seems as good a compromise as any. We shall normally write $f^{z,t}(k,\omega)$ as $f^{z,t}$ while $f^t(0,\omega)$, which defines conditions over the initial surface $z=0$, will be written $f^t(0)$. Similarly $f^z(k,0)$ will be written $f^z(0)$ whenever this may be done without ambiguity.

Applying these transformations to Eq. (3.33) we get, for $r_s^{z,t}$, the equation

$$\begin{aligned} i(\omega+u_{0s}k)r_s^{z,t}&=r_s^z(0)+u_{0s}r_s^t(0)\\ &\quad-\rho_{0s}[-p^2v_{1s}^{z,t}+ikv_{3s}^{z,t}-v_{3s}^t(0)].\end{aligned} \tag{3.42}$$

From Eq. (3.37) we get for $\mathbf{H}^{z,t}$ the equations

$$\left.\begin{aligned} i\omega\mu_0H_1^{z,t}&=-ikE_2^{z,t}+\mu_0H_1^z(0)+E_2^t(0),\\ i\omega\mu_0H_2^{z,t}&=-[E_3^{z,t}-ikE_1^{z,t}+E_1^t(0)]\\ &\quad+\mu_0H_2^z(0),\\ i\omega\mu_0H_3^{z,t}&=-p^2E_2^{z,t}+\mu_0H_2^z(0).\end{aligned}\right\} \tag{3.43}$$

From the transform of Eq. (3.35), $H_3^z(0)$ is given in terms of $H_1^z(0)$ by the equation

$$ikH_3^z(0)=p^2H_1^z(0)+H_3(0,0), \tag{3.44}$$

while, from Eq. (3.34), $E_3^z(0)$ is given in terms of $E_1^z(0)$ and $r_s^z(0)$ by the equation

$$-p^2E_1^z(0)+ikE_3^z(0)-E_3(0,0)=\sum r_s^z(0)/\epsilon_0. \tag{3.45}$$

Taking transforms of Eq. (3.38) and eliminating the magnetic field components by Eq. (3.43) we get three equations for $v_{ls}^{z,t}$ which may be solved in terms of $E_l^{z,t}$ and the initial conditions. Finally, taking transforms of Eq. (3.36) and eliminating $r_s^{z,t}$, $H_l^{z,t}$, and $v_{ls}^{z,t}$, we get three linear nonhomogeneous equations for $E_l^{z,t}$ which, in matrix notation, may be written

$$\mathfrak{E}(k,\omega)\mathfrak{A}(k,\omega)=\mathfrak{C}^*(k,\omega)+\mathfrak{C}^\dagger(k,\omega), \tag{3.46}$$

where $\mathfrak{E}(k,\omega)$ is the row matrix $(E_1^{z,t},E_2^{z,t},E_3^{z,t})$.

$\mathfrak{A}(k,\omega)$ is the 3×3 matrix with element $a_{lm}(k,\omega)$ defined by

$$\mathfrak{A}(k,\omega)=\begin{bmatrix} \left[k^2-\frac{\omega^2}{c^2}+\frac{1}{c^2}\sum\frac{\omega_0^2\omega_b\omega_{b\nu}}{\omega_{b\nu}^2-\omega_H^2}\right], & -\frac{1}{c^2}\sum\frac{i\omega_0^2\omega_b\omega_H}{\omega_{b\nu}^2-\omega_H^2}, & \left[ik+\frac{i}{c^2}\sum\frac{u_0\omega_0^2\omega_b}{\omega_{b\nu}^2-\omega_H^2}\right],\\ \frac{1}{c^2}\sum\frac{i\omega_0^2\omega_b\omega_H}{\omega_{b\nu}^2-\omega_H^2}, & \left[k^2+p^2-\frac{\omega^2}{c^2}+\frac{1}{c^2}\sum\frac{\omega_0^2\omega_b\omega_{b\nu}}{\omega_{b\nu}^2-\omega_H^2}\right], & -\frac{1}{c^2}\sum\frac{u_0\omega_H\omega_0^2}{\omega_{b\nu}^2-\omega_H^2},\\ p^2\left[ik+\frac{i}{c^2}\sum\frac{u_0\omega_0^2\omega_{b\nu}}{\omega_{b\nu}^2-\omega_H^2}\right], & \frac{p^2}{c^2}\sum\frac{u_0\omega_H\omega_0^2}{\omega_{b\nu}^2-\omega_H^2}, & \frac{\omega^2}{c^2}\left[1-\sum\frac{\omega_0^2}{K\omega_b\omega_{b\nu}}\right]-p^2\left[1+\frac{1}{c^2}\sum\frac{u_0^2\omega_{b\nu}\omega_0^2}{\omega_b(\omega_{b\nu}^2-\omega_H^2)}\right] \end{bmatrix}, \tag{3.47}$$

where

$$\omega_b=\omega+u_{0s}k\quad\text{and}\quad\omega_{b\nu}=\omega+u_{0s}k-i\nu_s, \tag{3.48}$$

and $\omega_{0s}^2=(-e_s\rho_{0s}/\epsilon_0m_s)^{\frac{1}{2}}$ is the plasma angular frequency associated with the sth beam.

It may be noted that ω_0^2, ω_b, $\omega_{b\nu}$, K, and u_0 all depend on s and therefore must be included under the summation sign.

$\mathfrak{C}^*(k,\omega)$ is the column matrix with elements $C_l^*(k,\omega)$

[11] It will be noted that these transforms are rotated through an angle $\pi/2$ in the complex k and ω-planes from the conventional definitions for the Laplace transforms, so that a direct comparison may be made between the present analysis and the "Ansatz" procedure.

proportional to the values of the field and space charge variables at $z=0$, defined by

$$\left.\begin{aligned}
C_1^*(k,\omega)&=-ik_1E_1^t(0)-i\omega\mu_0H_2^t(0)-\sum\left\{\frac{u_0\omega_0^2}{c^2}\Delta_1[E^t(0)]-i\omega\mu_0\rho_0u_0\Delta_1[v_s^t(0)]\right\},\\
C_2^*(k,\omega)&=-ikE_2^t(0)+i\omega\mu_0H_1^t(0)-\sum\left\{\frac{u_0\omega_0^2}{c^2}\Delta_2[E^t(0)]-i\omega\mu_0\rho_0u_0\Delta_2[v_s^t(0)]\right\},\\
C_3^*(k,\omega)&=p^2E_1^t(0)+\sum\left\{\frac{u_0\omega\mu_0}{\omega_b}\left[u_0r_s^t(0)+\rho_0v_{3s}^t(0)\left(1+\frac{\omega}{\omega_{b\nu}}\right)\right]-\frac{p^2u_0^2}{\omega_b}\left(\frac{i\omega_0^2}{c^2}\Delta_1[E^t(0)]+\omega\mu_0\rho_0\Delta_1[v_s^t(0)]\right)\right\},
\end{aligned}\right\}\quad(3.49)$$

and $\mathfrak{C}^\dagger(k,\omega)$ is the column matrix with elements $C_l^\dagger(k,\omega)$, proportional to the values of the field and space charge variables at $t=0$, defined by

$$\left.\begin{aligned}
C_1^\dagger(k,\omega)&=\frac{i\omega}{c^2}E_1^z(0)+ik\mu_0H_2^z(0)+\mu_0\sum\left\{\frac{u_0\omega_0^2}{c^2}\Delta_2[H^z(0)]+i\omega\rho_0\Delta_1[v_s^z(0)]\right\},\\
C_2^\dagger(k,\omega)&=\frac{i\omega}{c^2}E_2^z(0)-\frac{i\mu_0(p^2+k^2)}{k}H_1^z(0)-\mu_0\sum\left\{\frac{u_0\omega_0^2}{c^2}\Delta_1[H^z(0)]-i\omega\rho_0\Delta_2[v_s^z(0)]-\frac{iH_3(0,0)}{k}\right\},\\
C_3^\dagger(k,\omega)&=-\frac{\omega p^2}{c^2k}E_1^z(0)-\mu_0\sum\left\{\frac{\omega^2}{\omega_b}\left[\frac{r_s^z(0)}{k}-\frac{\rho_{0s}v_{3s}^z(0)}{\omega_{b\nu}}\right]-\frac{\omega}{c^2k}E_3(0,0)-p^2\mu_0H_2^z(0)\right.\\
&\qquad\left.+p^2u_0\left(\frac{\omega_0^2}{c^2}\Delta_2[H^z(0)]-\frac{i\omega\rho_0}{\omega_b}\Delta_2[v_s^z(0)]\right)\right\},
\end{aligned}\right\}\quad(3.50)$$

where

$$\left.\begin{aligned}
\Delta_1[F]&=(i\omega_{b\nu}F_1+\omega_HF_2)/(\omega_{b\nu}^2-\omega_H^2),\\
\Delta_2[F]&=(-\omega_HF_1+i\omega_{b\nu}F_2)/(\omega_{b\nu}^2-\omega_H^2),
\end{aligned}\right\}\quad(3.51)$$

and F is any one of $E_l^z(0)$, $H_l^t(0)$, etc.

By Cramer's rule the solution of Eq. (3.46) may be written

$$E_l^{z,t}(k,\omega)=\sum_{m=1}^{3}[C_m^*(k,\omega)+C_m^\dagger(k,\omega)]A_{ml}(k,\omega)/\det\mathfrak{A}(k,\omega),\quad(3.52)$$

where $A_{ml}(k,\omega)$ is the co-factor of $a_{ml}(k,\omega)$ in the matrix $\mathfrak{A}(k,\omega)$ defined by Eq. (3.47).

We have thus obtained an elementary solution of the Maxwell-Lorentz equations of the form of Eqs. (3.17) to (3.19), where

$$E_l(z,t)=\left(\frac{1}{2\pi}\right)^2\int_{-\infty-i\gamma_1}^{\infty-i\gamma_1}\exp(i\omega t)d\omega\int_{-\infty-i\gamma_2}^{\infty-i\gamma_2}\exp(ikz)dk\times\sum_{m=1}^{3}[C_m^*(k,\omega)+C_m^\dagger(k,\omega)]\frac{A_{ml}(k,\omega)}{\det\mathfrak{A}(k,\omega)},\quad(3.53)$$

together with similar expressions for $H_l(z,t)$, $v_{ls}(z,t)$ and $r_s(z,t)$, which is valid inside the electron-ion stream provided that $\phi(x^1,x^2)$ satisfies the two-dimensional wave equation (3.20).

It still remains to find the transverse boundary conditions satisfied by the elementary solution. For the stream of infinite cross section the transverse boundary conditions require only that the solution be well behaved at infinity. We show below that this is so if we restrict our choice of p, in Eq. (3.17), to the real numbers $-\infty<p<\infty$, when the general solution satisfying arbitrary initial conditions can be expressed in terms of a continuous orthogonal set of elementary solutions.

For the stream of finite cross section the transverse boundary conditions are only satisfied by a single elementary solution under the limited conditions discussed in Sec. 4.

4. THE GENERAL SOLUTION FOR THE STREAM OF INFINITE CROSS SECTION

The elementary solution of Eq. (3.53) does not satisfy arbitrary initial conditions. On the contrary it was assumed that the initial conditions depended upon the transverse coordinates in a manner determined uniquely by the particular choice of the scalar $\phi(x^1,x^2)$ from which the solution was derived. Specifically we assumed that the initial spatial distribution of velocity modulation on the sth beam at time $t=0$ could be written in the form

$$\mathbf{v}_s(x^1,x^2,z,0)=v_{1s}(z,0)\nabla\phi-v_{2s}(z,0)\mathbf{a}\times\nabla\phi+v_{3s}(z,0)\phi\mathbf{a},\quad(4.1)$$

where ϕ satisfies Eq. (3.20) and $\mathbf{a}=(0,0,1)$ is a unit axial vector, together with similar assumptions as to the initial values of the transverse components of the electromagnetic field and the velocity and density modulations at the surface $z=0$.

It follows that the general solution can only be expressed by an integral representation over the set of elementary solutions if an arbitrary vector field can be expressed as an integral over a set of terms of the general form of Eq. (4.1).

The proof that this is possible involves the classical theory of orthogonal integral representation which is amply covered in the literature. The only new feature of the present case arises because of the vector nature of the expansion. To illustrate this let us consider the solution in the special case of a Cartesian coordinate system.

In Cartesian coordinates (x, y, z, t) a typical solution of Eq. (3.20) is

$$\phi(x, y)=\exp[i(mx+ny)], \tag{4.2}$$

where $m^2+n^2=p^2$, when the general solution can be given by an integral representation over the set of $\phi(x, y)$ obtained by letting m, n take all real values in the range $-\infty<m<\infty$, $-\infty<n<\infty$ as long as an arbitrary vector field can be expressed in the form

$$\mathbf{F}(x, y, z, t)=\left(\frac{1}{2\pi}\right)^2\int_{-\infty}^{\infty}dm\int_{-\infty}^{\infty}dn\{f_1(m, n, z, t)\nabla - f_2(m, n, z, t)\mathbf{a}\times\nabla+f_3(m, n, z, t)\mathbf{a}\} \times\exp[i(mx+ny)]. \tag{4.3}$$

To prove that this is possible we first obtain a Fourier integral expansion of the three scalars $\mathbf{F}\cdot\mathbf{a}$, $(\nabla-\mathbf{a}\partial/\partial z)\cdot\mathbf{F}$ and $\mathbf{a}\cdot[\nabla\times\mathbf{F}]$, and then apply the Fourier integral theorem to show that

$$f_3(m, n, z, t)=\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}dy(\mathbf{a}\cdot\mathbf{F}) \times\exp[-i(mx+ny)], \tag{4.4}$$

$$f_1(m, n, z, t)=\frac{-1}{(m^2+n^2)}\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}dy(\nabla-\mathbf{a}\partial/\partial z)\cdot\mathbf{F} \times\exp[-i(mx+ny)], \tag{4.5}$$

$$f_2(m, n, z, t)=\frac{1}{(m^2+n^2)}\int_{\infty}^{\infty}dx\int_{-\infty}^{\infty}\mathbf{a}\cdot[\nabla\times\mathbf{F}] \times\exp[-i(mx+ny)], \tag{4.6}$$

provided that $\mathbf{F}$ and its first derivatives belong to $L^2(-\infty, \infty)$.

Let us suppose that an expansion of the form of Eq. (4.1) has been carried through for all the given initial vector fields $\mathbf{v}_s(x^1, x^2, 0, t)$, $E(x^1, x^2, z, 0)$, etc. In each expansion we then select the component appropriate to a particular choice of m, n and obtain the elementary solution of Eq. (4.1) in the form

$$E_l(m, n, z, t)=\left(\frac{1}{2\pi}\right)^2\int_{-i\gamma_1-\infty}^{-i\gamma_1+\infty}d\omega\int_{-i\gamma_2-\infty}^{-i\gamma_2+\infty}dk \times E_l^{z,t}(m, n, k, \omega)\exp[i(kz+\omega t)]. \tag{4.7}$$

By the superposition principle the general solution may then be expressed by the fourfold integral

$$\mathbf{E}(x^1, x^2, z, t)=\left(\frac{1}{2\pi}\right)^4\int_{-\infty}^{\infty}dm\int_{-\infty}^{\infty}dn\int_{-i\gamma_1-\infty}^{-i\gamma_1+\infty}dk \times\int_{-i\gamma_2-\infty}^{-i\gamma_2+\infty}dk\exp[i(kz+\omega t)]\{E_1^{z,t}(m, n, k, \omega)\nabla -E_2^{z,t}(m, n, k, \omega)\mathbf{a}\times\nabla+E_3^{z,t}(m, n, k, \omega)\mathbf{a}\} \times\exp[i(mx+ny)]. \tag{4.8}$$

That the set of elementary solution is orthogonal follows immediately from the Fourier integral theory. A similar analysis can be carried through in any other orthogonal cylindrical coordinate system.

5. THE STREAM OF FINITE CROSS SECTION

For the stream of finite cross section the transverse boundary conditions are satisfied by a single elementary solution only in the very restricted case when

(i) The boundaries of the stream coincide with one or other of the orthogonal cylindrical coordinate surfaces.
(ii) The tangential components of the electric vector are identically zero at the boundary of the stream.
(iii) The electromagnetic fields decompose into independent TE and TM modes for which $E_1(z, t)\equiv 0 \equiv E_3(z, t)$ and $E_2(z, t)\equiv 0$, respectively. As in the single-beam theory of Hahn,[7] this is only possible when $\omega_H=0$, or $\omega_H=\infty$ or $\omega_0^2=0$.

It is implicit in Hahn's analysis[7] that these conditions are necessary even for the simple case of the single beam stream excited in the symmetrical mode. To show that they are sufficient for the general case let us assume that the boundary of the stream coincides with the coordinate surfaces

$$x^1=a_1, a_2;\quad x^2=b_1, b_2, \tag{5.1}$$

and that $\phi(x^1, x^2)$ is separable in (x^1, x^2) so that we may write

$$\phi(x^1, x^2)=X(x^1)Y(x^2). \tag{5.2}$$

When condition (ii) is satisfied the transverse boundary condition may be written

$$\mathbf{E}\times\mathbf{n}=0 \tag{5.3}$$

where $\mathbf{n}$ is a unit vector normal to the boundary surfaces of Eq. (5.1). From Eq. (3.21) we have on substituting in Eq. (5.3), that

$$\left.\begin{array}{ll} E_3(z,t)X(a)Y(x^2)=0 & a=a_1, a_2, \quad b_1<x^2<b_2, \\ E_3(z,t)X(x^1)Y(b)=0 & a_1<x^1<a_2, \quad b=b_1, b_2, \end{array}\right\} \quad (5.4)$$

together with

$$E_1(z,t)Y(b)(\partial X/\partial x^1)/h_1 - E_2(z,t)(X(x^1)/h_2)[\partial Y/\partial x^2]_{x^2=b}=0, \quad (5.5)$$
$$b=b_1, b_2 \quad \text{and} \quad a_1<x^1<a_2,$$

$$E_1(z,t)(X(a)/h_2)(\partial Y/\partial x^2) + E_2(z,t)(Y(x^2)/h_1)[\partial X/\partial x^1]_{x^1=a}=0, \quad (5.6)$$
$$a=a_1, a_2 \quad \text{and} \quad b_1<x^2<b_2.$$

If these conditions are to be satisfied for all $0<z<d$ and all $t>0$ we must in general have either

$$\left.\begin{array}{l} \text{(i)}\ X(a)=Y(b)=0; \quad a=a_1, a_2; \quad b=b_1, b_2 \\ \text{and} \qquad E_2(z,t)\equiv 0, \end{array}\right\} \quad (5.6)$$

or

$$\left.\begin{array}{l} \text{(ii)}\ [\partial X/\partial x^1]_{x^1=a}=[\partial Y/\partial x^2]_{x^2=b}=0; \\ \qquad\qquad a=a_1, a_2; \quad b=b_1, b_2 \\ \text{and} \qquad E_1(z,t)\equiv 0\equiv E_3(z,t), \end{array}\right\} \quad (5.7)$$

which is possible if and only if condition (iii) is satisfied.

In this case the boundary conditions define two independent sets of orthogonal eigenfunctions in terms of which the general *TE* and the general *TM* solution can be expanded, respectively.

In the special case where the boundary surfaces reduce to a pair of coordinate surfaces $x^1=a_1, a_2$, and when $\phi(x^1, x^2)$ is independent of x^2 a certain simplification results when conditions (i) and (iii) are met as then general transverse boundary conditions can be satisfied independently by either the *TE* or the *TM* modes.

Even in this case, however, the transverse boundary conditions cannot in general be satisfied by a single elementary solution. The only exceptions occur when the moving stream is replaced by a stationary plasma or when $\omega_{Hs}=\infty$ and only the *TE* waves are excited. The solution is then effectively the same as that for a dielectric filled wave guide which has been extensively analyzed in the literature.[12]

When the limiting conditions (i)–(iii) are not satisfied the general solution cannot be represented as an orthogonal sum over a set of the elementary solutions, and the initial conditions cannot be included explicitly in the solution as developed here. Essentially this is because we have taken too simple a form for the elementary solutions. However elementary solutions of a form complex enough to satisfy arbitrary transverse boundary conditions have not yet been found. It may be that complete orthogonal sets of such solutions do not exist and it is certainly open to question as to whether they can be derived from a single scalar quantity.

At present our only resource is to fall back upon an "Ansatz" and develop the method given by Hahn[7] in a form appropriate to the vector solution of this paper. In this procedure one represents the general solution as the sum over an infinite set of partial waves the propagation constants of which are determined by the angular frequency ω. The amplitudes of these waves can then be found in terms of the initial and terminal boundary conditions by matching across the terminal surfaces of discontinuity at $z=0$ and $z=d$. Unfortunately the transverse distribution functions associated with the allowable propagation modes in the stream form a nonorthogonal and indeed overcomplete set. One there-therefore has to introduce an auxiliary set of orthogonal functions which do form a complete set in the region over which the matching has to be carried out. Both the initial conditions and the solution in the stream have then to be expanded in terms of this auxiliary set, thus giving an infinite sequence of infinite equations to determine the amplitudes of the various partial waves, the exact solutions of which involve the ratio of two infinite determinants.

This procedure is straightforward and can, in principle, be used to yield an approximation of arbitrarily high order as we have shown elsewhere,[13] but even in the simplest cases the algebra is very heavy and the solution too complicated to throw light on the physical nature of the propagation. Accordingly it will not be considered further here. For the rest of this paper we shall confine attention to the elementary solution of Eq. (3.53) and assume that the transverse boundary conditions can be satisfied for suitable real values of p.

6. THE SOLUTION UNDER PARTICULAR INITIAL CONDITIONS

In earlier treatments of the unbounded plasma it has been customary to assume an initial distribution of disturbance which depends upon z as $\exp(ik_0z)$, where k_0 is an arbitrary real number.

As pointed out by Landau[14] this choice involves no loss of generality in a small signal theory since any initial disturbance that is physically realizable can be expanded as a Fourier integral of the form

$$F(z,0)=\frac{1}{2\pi}\int_{-\infty}^{\infty} f(k_0)\exp(ik_0z)dk_0. \quad (6.1)$$

Alternatively we can assume that the initial disturbance is concentrated at a single point in the medium when formally it may be expressed by the Dirac δ-function. This procedure is also quite general since an arbitrary initial disturbance may be expressed by

[12] J. A. Stratton, *Electromagnetic Theory* (McGraw-Hill Book Company, Inc., New York, 1941).

[13] R. Q. Twiss, Services Electronics Research Lab. Tech. J. 1 (1951).

[14] L. Landau, J. Phys. (U.S.S.R.) 10, 25 (1946).

the integral

$$F(z, 0)=\int_{-\infty}^{\infty} f(z_0)\delta(z-z_0)dz_0. \quad (6.2)$$

Both these choices have advantages which can be combined by assuming that

$$E_l(z, 0)=E_l(k_0)\exp(ik_0z)+E_l(z_0)\delta(z-z_0), \quad (6.3)$$

together with similar expressions for the other field and space charge variables. In this case the Laplace-Fourier transform of $E_l(z, 0)$ with respect to z is given by

$$E_l^z(k, 0)=E_l^z(0)=E_l(k_0)/i(k-k_0) +E_l(z_0)\exp(-ikz_0). \quad (6.4)$$

Substituting from Eq. (6.4) in Eq. (3.52), we have that

$$E_l^{z,t}(k, \omega)=\sum_{m=1}^{3}[C_m^*(k, \omega)+C_m^{z_0}(k, \omega)\exp(-ikz_0) +C_m^{k_0}(k, \omega)/i(k-k_0)]A_{ml}(k, \omega)/\det\mathfrak{A}(k, \omega) \quad (6.5)$$

where $C_m^{z_0}(k, \omega)$, $C_m^{k_0}(k, \omega)$ are given by Eq. (3.50) with $E_l^z(0)$ replaced by $E_l(z_0)$ and $E_l(k_0)$, respectively, and where $C_m^*(k, \omega)$ depends only upon the time-dependent fields at $z=0$.

Now $E_l^{z,t}(k, \omega)$ as given by Eq. (6.5) is a meromorphic function of k which is $O(k^{-1})$ as $k\rightarrow\infty$ along the line $\mathrm{Im}(k)+\gamma_2=0$, while it can be shown by direct expansion of Eq. (3.47) that all the poles of $E_l^{z,t}(k, \omega)$ except that at $k-k_0=0$ occur at the zeros of the equation

$$\prod_{s=1}^{N}[(\omega_{b\nu,s}^2-\omega_{H,s}^2)(\omega_{b\nu,s}\omega_{H,s})]\det\mathfrak{A}(k, \omega)=0. \quad (6.6)$$

A rational equation for k of the $(4N+4)$th order.[15]

If we had used an "Ansatz," and assumed that all ac quantities were proportional to $\exp[i(kz+\omega t)]$ one would have obtained Eq. (6.6) as the characteristic equation relating k, ω though further discussion would now be needed to determine whether the zeros of the first two factors ever gave nontrivial roots or whether all the solutions could be derived from the zeros of the characteristic determinant,

$$\det\mathfrak{A}(k, \omega)=0. \quad (6.7)$$

This point is clearly particularly relevant for the continuous velocity distribution when $N\rightarrow\infty$, and we show below that Eq. (6.7) does not yield all the solutions in this case.

If $k_n(\omega)$ be any root of Eq. (6.6), then $E_l^t(z, \omega)$, defined by

$$E_l^t(z, \omega)=\frac{1}{2\pi}\int_{-\infty-i\gamma_2}^{\infty-i\gamma_2}\exp(ikz)E_l^{z,t}(k, \omega)dk, \quad (6.8)$$

may be expressed as a sum of partial waves by the equation

$$E_l^t(z, \omega)=\sum_{m=1}^{3}[C_m^{k_0}(k_0, \omega)A_{ml}(k_0, \omega)/\det\mathfrak{A}(k_0, \omega)] \times\exp(ik_0z)+i\sum_{m=1}^{3}\sum_{n=1}^{4N+4}\alpha_{ml}(k_n\omega)[C_m^*(k_n, \omega) +C_m^{z_0}(k_n, \omega)\exp(-ik_nz_0)U(z-z_0) +C_m^{k_0}(k_n, \omega)/i(k_n-k_0)]\exp(ik_nz), \quad (6.9)$$

where $U(z)$ is the Heaviside unit step function, and where $\alpha_{ml}(k_n, \omega)$ is the residue of $A_{ml}(k, \omega)/\det\mathfrak{A}(k, \omega)$ at the pole $k=k_n$.

This result is derived on the assumption that all the roots of the characteristic equation are distinct, which will be the case except at a finite number of points in the complex ω plane. As ω tends to the value at which $k_r(\omega)=k_s(\omega)$, say, the terms proportional to $\exp(ik_rz)$ and $\exp(ik_sz)$ tend to infinity. However their sum remains finite and tends uniformly to a limit proportional to

$$\exp(ik_rz)\lim_{k_s\rightarrow k_r}\frac{\sin(k_r-k_s)z}{k_r-k_s}\sim z\exp(ik_rz). \quad (6.10)$$

Therefore $E_l^t(z, \omega)$ may be expressed by Eq. (6.9) even when some of the $k_n(\omega)$ are equal as long as the infinite terms are grouped so that their sum is finite. This will always be possible unless $E_l^t(z, \omega)$ has a singularity at a multiple root in the ω-plane. In this case however the point lies by definition above the line $\mathrm{Im}(\omega)+\gamma_2=0$ along which $E_l^t(z, \omega)$ is to be integrated.

Instead of discussing the solution in the general case we shall consider separately the three special cases where the medium is excited (i) by an external signal incident on $z=0$ at $t=0$, (ii) by an initial disturbance concentrated at the point $z=z_0$, (iii) by an initial sinusoidal distribution of disturbance. Between them these cover most of the cases of physical interest. The physical situations to which they apply have been discussed by Feinstein and Sen.[5]

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In this case where we take $C_m^{z_0}\equiv C_m^{k_0}\equiv 0$, the solution of Eq. (6.9) is not expressed entirely in terms of known quantities since $C_m^*(k_n, \omega)$ depends upon $E_s^t(0)$, $H_s^t(0)$ $(s=1, 2)$, which are determined by the boundary conditions at $z=0$, $z=d$, on the transverse components of the electromagnetic field. These conditions yield four independent linear nonhomogeneous equations relating $E_s^t(0)$, $H_s^t(0)$ to the external fields and the initial density and velocity modulations on the stream which are just sufficient to determine the former quantities uniquely.

In the general case, the elimination of these unknown quantities is very cumbersome, and not essentially

[15] Using Eqs. (3.44) and (3.45) it can be shown that there is no pole at $k=0$.

different from that given in an earlier paper,[16] referred to hereafter as I, for the special case of pure transverse propagation. The principal conclusions will therefore be stated without the mathematical proofs which are straightforward extensions of those in I.

To begin with, let us assume that all the beams composing the stream are moving in the positive direction, and let us consider the contribution to the solution at (z, t) from a partial wave with propagation constant $k_r(\omega)$. As shown in I, this depends upon the asymptotic nature of $k_r(\omega)$ as $\omega \rightarrow \infty - i\gamma_2$.

From Eq. (3.47) it can be shown that two of the roots of Eq. (6.6) $\sim +\omega/c$ as $\omega \rightarrow \infty$ and therefore are to be associated with the reverse field waves. As in I it can be shown that these waves can only be excited by reflection, and do not contribute to the solution at z until a time $t > (2d-z)/c$.

Two other roots such that $k \sim -\omega/c$ as $\omega \rightarrow \infty$ correspond to the forward field waves. The remaining space charge waves have propagation constants for which

$$k \sim (-\omega/u_r)(1+\epsilon_r) \quad r = 1 \cdots N,$$

where ϵ_r is a small quantity $\rightarrow 0$ as $\omega \rightarrow \infty$ which can take one of four different values for given r.

It can be shown as in I, that a space charge or forward field wave will not contribute to the disturbance at a point z until a time

$$t > z \lim_{\omega \rightarrow \infty - i\gamma_1} (-k_r/\omega).$$

Furthermore the expressions for $E_s^t(0)$, $H_s^t(0)$ involving the terminal boundary conditions can be expanded as a sum of partial waves which can be interpreted as arising from successive reflections at the terminal boundaries. If $E_s^t(0)$, $H_s^t(0)$ have any poles in the complex ω-plane below the real axis, instability will result due to the presence of reflected waves.

When the stream consists in part of reverse beams of charged particles, the solution for the terminated stream becomes prohibitively complicated. The density and velocity modulations on the reverse beams at $z=0$ can no longer be given *a priori* but depend on the manner in which these beams enter the interaction space. However when the terminal surface at $z=d$ is infinitely remote, things become much simpler since, as is proved below, the space charge waves associated with the reverse beams can only be excited by reflection. Accordingly, the density and velocity modulations on these reverse beams at $z=0$ can be determined immediately from the requirement that the amplitudes of the reverse space charge waves, for which

$$\lim_{\omega \rightarrow \infty - i\gamma_1} (k_r/\omega) > 0, \tag{6.11}$$

be identically zero.

Even when $d \rightarrow \infty$, it is not possible to evaluate explicitly the transient solution,

$$E_l(z, t) = \frac{1}{2\pi} \int_{-\infty - i\gamma_1}^{\infty - i\gamma_1} \exp(i\omega t) E_l^t(z, \omega) d\omega, \tag{6.12}$$

except in the very simplest cases. The difficulties arise largely because the $k_n(\omega)$, and therefore $E_l^t(z, \omega)$, are multivalued functions of ω which cannot in general be found explicitly, since this would involve the algebraic solution of an equation of the $(4N+4)$th degree.

It may be however that the only contribution to $E_l(z, t)$ which does not decay with time is that proportional to $\exp(i\Omega_0 t)$, where Ω_0 is the angular frequency of the external signal. In this case we have the steady-state solution

$$E_l(z, t) = i \sum_{n=M}^{4N+4} \alpha_{ml}(k_n, \Omega_0) C_m^*(k_n \Omega_0) \exp[i(k_n z + \Omega_0 t)],$$

where M is chosen to exclude all the partial waves for which the inequality of Eq. (6.11) is satisfied. However, the condition that the transient part of the solution decays with time must be established by a separate investigation. This steady-state solution is the same as that obtained by an "Ansatz" in which the characteristic equation is solved for k with $\omega = \Omega_0$.

One important result of this discussion is to settle the ambiguity between amplified growing waves and growing waves attenuated in the negative direction, since the latter satisfy the inequality (6.11).

It is usually a simple matter to identify the reverse field waves, except perhaps in the neighborhood of the cyclotron frequency, but more care is needed to discriminate between the forward and reverse space charge waves.

The Medium Excited at a Single Internal Point

In this case $C_m^{k_0}(k, \omega) \equiv 0$, but $C_m^*(k_n, \omega)$ must be found in terms of $E_l(z_0)$, etc., from the terminal boundary conditions at $z=0$ and $z=d$.

In a frame of reference in which all the beams in the stream are moving in the positive direction one can extend the analysis of I to show that the reverse field waves do not exist in the region $z > z_0$ until a time $t > (d-z)/c$, while in the region $z < z_0$ they provide the only terms that contribute to the disturbance as long as $t < z/c$. As in the case of the externally excited medium, the space charge waves can be grouped in sets of four such that the front edge of their contribution to the disturbance propagates at the velocity u_r of the associated beam. If we now transform to a new frame of reference in which some of the beams are moving in the reverse direction, it follows that none of the space charge waves associated with these beams contribute to the disturbance in the region $z > z_0$ in the new frame. More generally we conclude that no partial wave can be excited directly in the region $z > z_0$ by an internal disturbance originally localized in the region $z < z_0$ or

[16] R. Q. Twiss, Phys. Rev. 84, 448 (1951).

by an external disturbance incident at $z=z_0$, if the associated propagation constant obeys the inequality (6.11).

For the infinite unterminated medium it can be shown that $E_l^t(z,\omega)$ is given by choosing $C_m^*(k_n,\omega)$ so that

$$C_m^*(k_n,\omega)+C_m^{z_0}(k_n,\omega)\exp(ik_nz_0)=0 \quad (6.13)$$

for all $k_n(\omega)$ for which the inequality (6.11) holds good, and

$$C_m^*(k_n,\omega)=0$$

for all other $k_n(\omega)$. These $4N+4$ equations are just sufficient to determine the $4N+4$ unknowns in $C_m^*(k_0,\omega)$.

The Medium Excited Sinusoidally

In this case it can be shown, along the lines employed in I, that the only term of $E_l^t(z,\omega)$ that contributes to $E_l(z,t)$ for $t<z/c$ and $t<(d-z)/c$ is that proportional to $\exp(ik_0z)$, so that

$$E_l^t(z,\omega)=\sum_{m=1}^{3}[C_m^{k_0}(k_0,\omega)A_{ml}(k_0,\omega)/\det\mathfrak{A}(k_0,\omega)] \times\exp(ik_0z) \quad (6.14)$$

for $t<z/c$ and $t<(d-z)/c$. For the infinite unterminated medium where $z=d-z=\infty$, it follows from Eq. (6.9) that

$$C_m^*(k_n,\omega)+C_m^{k_0}(k_n,\omega)/i(k_n-k_0)=0 \quad (n=1\cdots 4N+4).$$

Since $E_l^t(z,\omega)$ is a rational function of ω, $O(\omega^{-1})$ as $\omega\rightarrow\infty$, we have immediately that

$$E_l(z,t)=i\sum_{n=1}^{4N+4}\beta_{nl}\exp[i(k_0z+\omega_nt)], \quad (6.15)$$

where ω_n is any root of the characteristic equation (6.6) with $k=k_0$, and where β_{nl} is the residue of

$$\sum_{m=1}^{3}C_m^{k_0}(k_0,\omega)A_{ml}(k_0,\omega)/\det\mathfrak{A}(k_0,\omega)$$

at the pole $\omega=\omega_n$. The solution of Eq. (6.15) is the same as that given by an "Ansatz" in which one solves the characteristic equation for ω with $k=k_0$.

If some of these characteristic roots lie in the lower half of the complex ω plane for certain ranges of k_0, then an initial spatial disturbance with Fourier components in these ranges will build up exponentially with time since in general the β_{nl} are all nonzero.

However one cannot decide without further discussion whether the stream is amplifying or unstable since the two phenomena are indistinguishable if the initial distribution of disturbance is sinusoidal. To differentiate one must consider the stream excited by a disturbance concentrated initially at $z=z_0$. If the disturbance remains finite within a finite distance of $z=z_0$ for all t then the stream is stable in this particular frame of reference and vice versa. In practice the distinction is only important in a frame of reference in which all the charge is moving in one direction, when only the reverse field waves exist in the region $z<z_0$ of an infinite unterminated medium. Hence in this case one would only expect instability to arise if one of the ω_n that lie below the real ω axis for some real k_0 can be identified as a field wave, that is if

$$\lim_{k_0\rightarrow\infty-i\gamma_2}[\omega_n(k_0)/k_0]=\pm 1/c.$$

This is rarely if ever the case. In general it is amplification rather than instability that can arise in an unterminated stream.

This discussion serves to underline the similarities between the two alternative "Ansatz" solutions. It is misleading to assert that one solution gives conditions for instability, the other conditions for amplification. As has been pointed out by Feinstein and Sen,[5] the physical differences between the two lies only in the different initial conditions, and either solution can be used to find whether amplification is possible in an unterminated stream. If the infinite stream is neither unstable nor amplifying, this can be shown more easily by solving the characteristic equation for ω for arbitrary real k_0, than by examining the steady-state solution, since, in the latter case, one must also prove that any growing waves are also reverse waves. On the other hand the former procedure can only be applied to the terminated stream when the solution can be represented by standing waves as in the charge free resonator with perfectly conducting walls. If such a resonator is filled with moving charge one has to use the steady state analysis to find the resonant frequencies, which are given, in this case, by the values of ω for which $H_s^t(0)$ is a maximum. Admittedly, to the first approximation, the only effect of the moving charge on the field waves is to alter the effective dielectric constant of the medium, and to this order the resonant frequencies may still be found from the roots of the characteristic equation for ω. However this procedure goes badly astray when the energy carried by the space charge waves becomes comparable with that carried by the field waves, as happens in the neighborhood of the cyclotron frequency or when the charge velocity approaches that of light.

7. THE CONTINUOUS VELOCITY DISTRIBUTION

Until now we have assumed that the electron-ion stream consists of a finite number of separate beams. When the number of these beams becomes very large, as in the case of a thermal plasma, the discrete analysis becomes excessively complicated and one is led to approximate the discontinuous velocity distribution of physical reality by an idealized continuous distribution.[17]

The solution of Eqs. (3.47) to (3.52) can formally be

[17] It must be remembered that one is here concerned with the actual velocity distribution at a given time rather than with the ensemble average distribution.

applied both to the discrete and the continuous case if we replace the summations over the N electron-ion beams by Riemann-Stieltjes integrals. To justify this step one has to show that the operations of evaluating the integral of Eq. (6.8) and proceeding to the limit $N\rightarrow\infty$ are commutable. The proof is a straightforward exercise in ϵ-analysis and will not be given here since its validity does not appear to be in dispute. When the stream is composed of two or more kinds of particles, electrons and ions say, one distinguishes between the separate velocity distribution and replaces terms such as

$$\sum_{s=1}^{N}\frac{\omega_{0s}^2\omega_{b\nu,s}\omega_{H,s}}{\omega_{b\nu,s}^2-\omega_{H,s}^2}$$

by a sum of Riemann-Stieltjes integrals of the form

$$\sum_{l=1}^{L}\int_{u_{1l}}^{u_{2l}}du\frac{\omega_{0l}^2(u)\omega_{Hl}(u)\omega_{b\nu,l}(u)}{\omega_{b\nu,l}^2(u)-\omega_{Hl}^2(u)},$$

where L is the number of different particles and

$$\omega_{Hl}(u)=(eB_0/m_{0l})(1-u^2/c^2)^{\frac{1}{2}},\quad \omega_{b\nu,l}(u)=\omega+uk-i\nu_l(u).$$

For the sake of simplicity however we shall assume in this section that there is only one class of charged particles, though the more general case does lead to new results when the axial magnetic field is nonzero.

In the continuous case the "Ansatz" solution breaks down in that it does not yield the complete solution. This was pointed out by Landau in his criticism of Vlasov's theory[18] of the thermal plasma but the former's treatment is also open to objection, since it is only rigorously valid for the physically impossible case where the velocity distribution of the charged particles is analytic up to infinite velocities. It is possible to modify Landau's theory to cover the case where the electron velocities are always finite, but the solution then becomes much more complex, while it is only valid when both the dc and the ac velocity distributions are analytic functions of velocity.

Accordingly we shall develop an alternative theory valid for any form of velocity distribution. In order to compare this procedure with that followed by Landau we shall consider the special case

$$p^2=\omega_H=\nu=0, \tag{7.1}$$

in which the stream is excited by an initial sinusoidal distribution of density modulation of the form

$$\rho_u(z,0)=g(u)\exp(ik_0z). \tag{7.2}$$

The general case is discussed briefly below.

If in Eqs. (3.47) to (3.52) we make the further substitution

$$\omega_0^2(u)K(u)=\omega_0^2f_0(u),$$

we get

$$E_3(z,t)=\frac{1}{2\pi}\int_{-\infty-i\gamma_1}^{\infty-i\gamma_1}\exp(i\omega t)E_3{}^t(z,\omega)d\omega, \tag{7.3}$$

where

$$E_3{}^t(z,\omega)=\frac{\dfrac{-1}{\epsilon_0}\displaystyle\int_{u_1}^{u_2}g(u)du/k_0(\omega+uk_0)}{1-\omega_0^2\displaystyle\int_{u_1}^{u_2}f_0(u)du/(\omega+uk_0)^2}, \tag{7.4}$$

and where u_1u_2 are the limits of the continuous velocity distribution.

If $f_0(u)$ has a first derivative throughout the range $u_1<u<u_2$ the denominator in Eq. (7.4) may be written in the equivalent form,

$$1-\omega_0^2\int_{u_1}^{u_2}du[df_0(u)/du]/k_0(\omega+uk_0),$$

as long as $f_0(u_1)=0=f_0(u_2)$, when our solution is identical with that of Landau allowing for the change of symbolism

The difference arises at the next stage in the inversion from the ω-plane onto the real t axis and centers on the treatment of the integrals

$$I_1=\int_{u_1}^{u_2}g(u)du/(\omega+uk_0), \tag{7.5}$$

$$I_2=\int_{u_1}^{u_2}f_0(u)du/(\omega+uk_0)^2, \tag{7.6}$$

which are undefined when $\omega+uk_0=0$.

As defined by the Laplace integral,

$$E_3{}^t(z,\omega)=\int_0^{\infty}E_3(z,t)\exp(-i\omega t)dt$$

only converges on and below the line $\mathrm{Im}(\omega)+\gamma_1=0$, and in this region I_1, I_2 and $E_3{}^t(z,\omega)$ are single valued analytic functions of ω.

However to find $E_3(z,t)$ explicitly it is desirable to use Eq. (7.4) to define $E_3{}^t(z,\omega)$ by analytic continuation in the upper half of the complex ω plane. One way of doing this is to insert a cut along the real ω axis between the points

$$\omega=-u_1k_0,\quad \omega=-u_2k_0.$$

This removes the singularities of I_1 and I_2 which are now single valued analytic functions of ω all over the accessible regions of the ω plane, and $E_3{}^t(z,\omega)$ is now a single valued function of ω everywhere analytic except at the isolated poles at the zeros of

$$1-\omega_0^2\int_{u_1}^{u_2}f_0(u)du/(\omega+uk_0)^2=0. \tag{7.7}$$

[18] A. Vlasov, J. Exptl. Theoret. Phys. (U.S.S.R.) 8, 291 (1938); J. Phys. (U.S.S.R.) 9, 25 (1945).

We now form a closed contour from the line

$$\mathrm{Im}(\omega)+\gamma_1=0,$$

and the infinite upper half-circle together with contours around the isolated poles of $E^t(z,\omega)$ and around the cut, within which $E_3{}^t(z,\omega)$ is everywhere analytic and $O(\omega^{-1})$ as $\omega\rightarrow\infty$.

By Cauchy's theorem the integral of Eq. (7.6) for $t>0$ is equal to $2\pi i$ times the sum of the residues of the isolated poles of $E_3{}^t(z,\omega)$ together with the contribution from the contour around the cut taken in a counter clockwise direction. By Jordan's lemma the contribution from the infinite half-circle is zero.

The contribution from around the cut is in general nonzero, as is shown in Appendix II for the special case where $g(u)$ and $f_0(u)$ are analytic functions of u, and can be split into three separate terms. The first term arises when one of the zeros of Eq. (7.7) lies on the cut. As we show in Appendix II this is never the case when $f_0(u)$ is analytic in the neighborhood of the real axis in the complex u plane, but such zeros do occur when $f_0(u)$ is discontinuous, as witness the simple case

$$f_0(u)=f_1 \quad u_1\leqslant u\leqslant \bar{u},$$
$$f_0(u)=f_2 \quad \bar{u}<u\leqslant u_2.$$

The second term in the integral around the cut arises because I_1, of Eq. (7.5) has, in general, a discontinuity equal to $2\pi i g(-\Omega_0/k_0)$ as we cross the cut from the upper to the lower half of the complex ω plane at the point $\omega=\Omega_0$. As a typical example consider the case

$$g(u)=f_0(u)=1 \quad u_1<u<u_2,$$

where the integral around the cut is equal to

$$\frac{i}{\epsilon_0 k_0}\int_{u_1}^{u_2} du \frac{1}{1-\omega_0{}^2(u_2-u_1)/(\omega+u_2k_0)(\omega+u_1k_0)}.$$

This term is nonzero even when $\omega_0{}^2=0$ and corresponds to the so-called "gas modes" discussed by Bohm and Gross.[3] Finally a third term arises due to the fact that the I_2, of Eq. (7.6), has a discontinuity as one crosses the cut equal to

$$2\pi i[df_0(-\Omega_0/k_0)/du]/k_0,$$

when $f_0(u)$ has a derivative for all $u_1<u<u_2$. The presence of such terms is ignored in the usual "Ansatz" solutions which often omit the "gas modes" as well.

Admittedly I_2 becomes continuous across the cut if we represent $f_0(u)$ as the limit of a series of square pulse functions

$$f_0(u)=f(n) \quad u_n<u<u_{n+1}, \quad n=1\cdots M,$$

where

$$u_{n+1}-u_n=\Delta u=(u_2-u_1)/M,$$

which can be done as long as $f_0(u)$ is integrable in the sense of Riemann.

However this simplification only arises in the special case of Eq. (7.1). In general the characteristic determinant is discontinuous across the cut whatever the form of $f_0(u)$, except when $f_0(u)$ can be represented as the sum of a series of δ functions. This last case is trivial since the stream now reduces to a system of discrete beams when the cut is redundant.

The procedure, that we have just outlined, is the normal one in transform theory valid whatever the nature of $f_0(u)$ and $g(u)$ as long as these functions are integrable in some sense. In particular there is no need for $f_0(u)$ and $g(u)$ to be analytic functions of u; they can be represented by a sum of δ functions or of square pulse functions or by a power series, whichever is the more convenient.

However this is certainly not possible in the treatment given by Landau as we show in Appendix III.

When ω_H, ν and p^2 are all nonzero we see by inspection of Eqs. (3.47) to (3.50) that cuts must be made in the ω plane along the lines

$$\omega=\omega_H-uk_0+i\nu(u); \quad \omega=-uk_0+i\nu(u); \quad \omega=-uk_0,$$

which all lie on or above the real ω-axis.

A similar procedure must be followed for the medium excited by an external signal or by an initially localized disturbance before inverting from the complex k plane onto the real ω-axis. In this case also the contributions from the cuts are, in general, nonzero and can be of importance particularly in the noise theory of the terminated stream.

8. DISCUSSION

The vector solution of this paper can be applied in principle to a stream of charged particles with different charge/mass ratios and arbitrary velocity distributions under arbitrary initial and boundary conditions. Unlike the theories that employ a stream function one is not restricted to the case of zero axial magnetic field and since the analysis is fully relativistic there is no need to treat the transverse and longitudinal fields separately.

However the general theory is very complex and in this paper we have discussed only the special case where the solution satisfying arbitrary initial conditions can be expanded as a sum over a complete orthogonal set of independent elementary solutions. Such an expansion can always be carried out for the stream of infinite cross section, but for the finite stream it is only possible under the limited transverse boundary conditions of Sec. 5. Each elementary solution can then be expressed explicitly in terms of the initial conditions by means of a double Laplace transform.

The use of a Laplace transform was introduced by Landau[14] in his theory of the longitudinal oscillations in a thermal plasma. However his treatment is only valid when the ac and dc velocity distribution functions are analytic, and, in its original form, can only be applied to the unphysical case where the electron velocity distribution extends to infinity. The solution of this paper is free from these restrictions and contains terms, which are inevitably omitted in an "Ansatz" solution where one assumes *ab initio* that all ac quantities are proportional to $\exp[i(\mathbf{k}\cdot\mathbf{z}+\omega t)]$. This "Ansatz" procedure, which is used by the majority of writers, only gives the necessary conditions that a solution should satisfy the Maxwell-Lorentz equations. It is therefore much less informative than the more rigorous Laplace transform analysis which gives the complete solution under given initial conditions, and from which one can find both the physical interpretation of the "Ansatz" solutions and the initial conditions to which they correspond.

In one "Ansatz" procedure one solves the characteristic equation for ω in terms of fixed $k=k_0$, where k_0 is a real propagation constant, and obtains $4N+4$ possible modes of oscillation, where N is the number of discrete velocity beams. We have shown that this solution only applies in general to the infinite unterminated stream excited by an initial distribution of disturbance proportional to $\exp(ik_0z)$. For the stream terminated at $z=0$ the solution is only valid for $t<z/c$ except in the very special circumstances discussed in Sec. 6. If any of the roots of the characteristic equation lie in the lower half of the complex ω-plane for certain real k_0, then an associated initial disturbance will build up exponentially with time. However one cannot tell from this whether the stream is amplifying or unstable, a distinction that is important when the stream is excited by a disturbance confined initially to a small region of the stream. We concluded on heuristic rather than rigorous grounds that the stream would only be unstable in a frame of reference in which all the charge is moving in one direction if one of the growing roots was associated with a field wave; that is if $\mathrm{Lim}(\omega_n/k_0)=\pm 1/c$ as $k_0\rightarrow\infty - i\gamma_2$, where $\omega_n(k_0)$ lies below the real ω axis for some real k_0 and where γ_2 is chosen so as to avoid any multiple roots of the characteristic determinant. The distinction between instability and amplification is usually unimportant in a frame of reference moving with the stream as both processes will normally occur together.

The alternative "Ansatz" solution is found by solving the characteristic equation for k in terms of fixed $\omega=\Omega_0$, when one obtains $4N+4$ possible values for the propagation constant. This solution applies to a stream excited at a material boundary by an external signal of frequency Ω_0, and is only valid when the transient part of the solution for the singly terminated stream decays with time. The conditions when this last restriction is satisfied can be found from the first "Ansatz" solution when all the charge is moving in one direction. However when reverse beams are present together with amplified growing waves the stream is often unstable so that no steady-state solution exists.

In order to interpret this solution one must be able to distinguish between waves that are directly excited at the incident surface $z=0$, and those that can only be excited by reflection from some subsequent surface or region of discontinuity. When all the charge is moving in the positive direction we showed in an earlier paper[16] that the asymptotic value of the propagation constant of the reflected waves is given by $\mathrm{Lim}[k_n(\Omega_0)/\Omega_0]=+1/c$ as $\Omega_0\rightarrow\infty - i\gamma_1$. When reverse beams are present we have shown that the associated space charge waves can only be excited by reflection so that we now have the general result that $\mathrm{Lim}[k_n(\Omega_0)/\Omega_0]>0$ as $\Omega_0\rightarrow\infty - i\gamma_1$ for all reflected waves and vice versa.

Although the two "Ansatz" solutions correspond to quite different initial conditions, there are many similarities between them. In particular either procedure can be used to find whether the unterminated stream can amplify. This is contrary to some earlier interpretations where one "Ansatz" is used to find whether the stream is unstable the other to find whether it can amplify. To some extent the confusion is merely a matter of terminology, but it can easily lead to false conclusions. Thus some writers have argued that amplified growing waves can exist in a stream even if all the roots of the characteristic equation lie on or above the real axis in the ω-plane. However in this case any growing waves with propagation constants in the lower half of the complex k plane are always reverse waves attenuated in the reverse direction.

The vector solution of Eqs. (3.17) to (3.19) is valid even when the dc quantities such as u_{0s}, ρ_s, etc., are functions of z. In this case one can integrate Eqs. (3.33), (3.37), and (3.38) to find $r_s^t(z,\omega)$, $H_l^t(z,\omega)$, and $v_{ls}^t(z,\omega)$ explicitly in terms of $E_l^t(z,\omega)$. On substituting in Eq. (3.37) one gets a system of integro-differential equations for $E_l^t(z,\omega)$, but these can no longer be reduced to algebraic equations by a Laplace transform with respect to z and small progress has been made toward their solution.

The theory can also be extended to the case where the dc magnetic field and the dc velocities have transverse as well as axial components as discussed in Appendix I. However this can only be carried through in Cartesian coordinates and the algebra in the general case is very cumbersome. The best hope is then to look for some approximate solution in which, for example, one considers only the transverse or only the longitudinal part of the solution. The errors involved in this and similar simplifications are discussed.

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APPENDIX I

In the text the dc components of the external magnetic flux density and of space charge velocity were assumed to be of the form $\mathbf{B}_0=(0,0,B_0)$ and $\mathbf{u}_0=(0,0,u_0)$, respectively. We shall now show how the analysis may be extended when these vectors possess transverse components so that $\mathbf{B}_0=\mathbf{B}_{\tau 0}+B_0\mathbf{a}$, $\mathbf{u}_0=\mathbf{u}_{\tau 0}+u_0\mathbf{a}$ where $\mathbf{B}_{\tau 0}\cdot\mathbf{a}=\mathbf{u}_{\tau 0}\cdot\mathbf{a}=0$ and $\mathbf{B}_{\tau 0}$, $\mathbf{u}_{\tau 0}$ are constant vectors independent of position. In order to maintain the steady-state flow it is necessary to introduce additional dc **E**-fields but as these do not affect the small signal ac equations they will be left unspecified.

Assuming, for the moment that $\mathbf{u}_{\tau 0}=0$, the only effect of the transverse components of the dc magnetic field is to add additional terms to the Lorentz force equation which, in its linear approximation, may now be written

$$(\partial/\partial t+\nu_s+u_{0s}\partial/\partial z)(m_s\mathbf{v}_s+m_sK_s(u_{0s}^2/c^2)(\mathbf{v}_s\cdot\mathbf{a})\mathbf{a}) \\ =-e_s[\mathbf{E}+\mathbf{v}_s\times B_0\mathbf{a}+\mu_0u_{0s}\mathbf{a}\times\mathbf{H}+\mathbf{v}_s\times\mathbf{B}_{\tau 0}], \quad \text{(I.1)}$$

differing from Eq. (3.9) in the term $\mathbf{v}_s\times\mathbf{B}_{\tau 0}$.

The elementary solution derived above for the case $\mathbf{B}_{\tau 0}=0$ relied on the fact that all the vector Maxwell-Lorentz equations could be expressed as linear combinations of the three fundamental vectors **L**, **M**, **N** defined in (3.16). These vectors determine an auxiliary coordinate system in which we may express $\mathbf{B}_{\tau 0}$ by the vector equation

$$\mathbf{B}_{\tau 0}=B_{01}\mathbf{L}+B_{02}\mathbf{M}, \quad \text{(I.2)}$$

where B_{01}, B_{02} are scalar coefficients that depend upon (x^1, x^2) in such a manner that $\mathbf{B}_{\tau 0}$ is a constant vector.

Substituting for $\mathbf{B}_{\tau 0}$ in Eq. (I.1) and assuming that $\mathbf{v}_s$ is given by (3.19) the term $\mathbf{v}_s\times\mathbf{B}_{\tau 0}$ may be written

$$\mathbf{v}_s\times\mathbf{B}_{\tau 0}=-B_{02}v_{3s}\phi\mathbf{L}-B_{01}v_{3s}\phi\mathbf{M} \\ +(B_{01}v_{2s}-B_{02}v_{1s})[\nabla\phi]^2/\phi\mathbf{N}. \quad \text{(I.3)}$$

In the general case where ϕ is any solution of the two-dimensional wave equation (3.20) consistency requires that

$$B_{0l}\phi(x^1, x^2) \quad \text{and} \quad B_{0l}[\nabla\phi(x^1, x^2)]^2/\phi(x^1, x^2) \quad l=(1, 2)$$

must both be independent of (x^1, x^2). This is only possible in Cartesian coordinates and then only when the solution of Eq. (3.20) is expressed as a traveling wave

$$\phi(x^1, x^2)=\exp[i(lx+my)],$$

where $l^2+m^2=p^2$.

In this case $B_{0l}\sim\phi^*(x^1, x^2)$, the complex conjugate of $\phi(x^1, x^2)$ when $\mathbf{B}_{\tau 0}$ as defined by Eq. (I.2) is indeed independent of position, so that a consistent solution can be obtained.

A similar discussion shows that the same restriction must be imposed on the coordinate system when the dc velocity has transverse components. In the latter case especially the algebra becomes extremely cumbersome and its importance does not seem sufficiently great to warrant even a summary of the solution.

APPENDIX II

When $g(u)$ the ac velocity distribution function of Sec. 7 is analytic we can immediately find the discontinuity in I_1, defined by Eq. (7.5), as we cross the real ω axis.

Since $g(u)$ is analytic we may expand $g(u)\equiv g(u+\omega/k_0-\omega/k_0)$ as a Taylor series in (ω/k_0+u) to give

$$g(u)=g(-\omega/k_0)+\sum_{n=1}^{\infty}g^n(-\omega/k_0)(u+\omega/k_0)^n/n!.$$

If we substitute in Eq. (7.5) and integrate we get

$$I_1=g(-\omega/k_0)\log\left(\frac{u_2+\omega/k_0}{u_1+\omega/k_0}\right) \\ +\sum_{n=1}^{\infty}\frac{g^n(-\omega/k_0)}{nn!k_0}\left[\left(u_2+\frac{\omega}{k_0}\right)^n-\left(u_1+\frac{\omega}{k_0}\right)^n\right].$$

As we cross the real ω axis at the point $\omega=-\bar{u}k_0\equiv\Omega_0$, $u_1<\bar{u}<u_2$, the logarithmic term in I_1 changes from

$$i\pi g(+\bar{u})\equiv i\pi g(-\Omega_0/k_0)$$

just above the real ω-axis to

$$-i\pi g(-\bar{u})\equiv -i\pi g(-\Omega_0/k_0)$$

just below the real ω-axis.

Similarly if $f_0(u)$ is analytic we may write

$$f_0(u)=f_0(-\omega/k_0)+\sum_{n=1}^{\infty}f_0^n(-\omega/k_0)(u+\omega/k_0)^n/n!,$$

when substituting in Eq. (7.6), and integrating term by term we get

$$I_2=\frac{f_0(-\omega/k_0)}{k_0^2}\cdot\frac{u_2-u_1}{(u_2+\omega/k_0)(u_1+\omega/k_0)} \\ +\frac{f_0'(-\omega/k_0)}{k_0^2}\cdot\log\left(\frac{u_2+\omega/k_0}{u_1+\omega/k_0}\right) \\ +\sum_{n=1}^{\infty}\frac{f^n(-\omega/k_0)}{(n-1)n!k_0^2}[(u_2+\omega/k_0)^{n-1}-(u_1+\omega/k_0)^{n-1}],$$

which has a discontinuity of

$$2\pi i[f_0'(-\Omega_0/k_0)]/k_0^2$$

across the real ω-axis, when $\omega=-\bar{u}k_0\equiv\Omega_0$ and $u_1<\bar{u}<u_2$.

It follows from this that the characteristic equation of Eq. (7.7) has no zeros on or immediately above the cut between the points $-u_2k_0<\omega<-u_1k_0$ if $f_0(u)$ is analytic, except possibly at the zeros of $f_0'(-\Omega_0/k_0)$. However these zeros form a set of measure zero if $f_0(u)$ is an integral function of u as do the zeros of Eq. (7.7) so that this possibility can effectively be ignored.

APPENDIX III

In the solution given in Sec. 7 for the stream with a continuous velocity distribution it was only necessary that $f_0(u)$ and $g(u)$, the dc and ac velocity distribution functions should be integrable.

However in the alternative solution given by Landau[14] it is necessary that $f_0(u)$ and $g(u)$ be integral functions

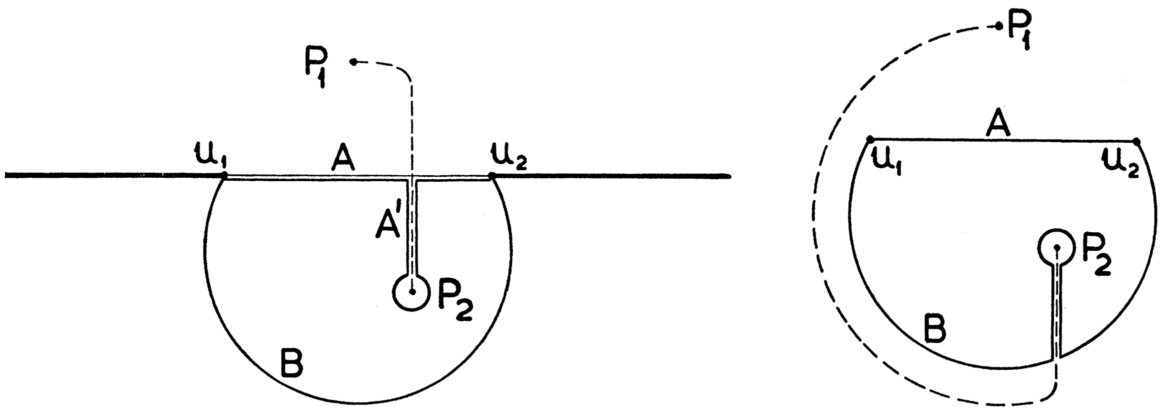

FIG. 1(a), (b). Contours of integration in complex u plane for analytical continuation of I_1, I_2 in complex ω plane.

of u, while this writer's procedure has to be further modified when one allows for the fact that no material particle can have a velocity greater than c.

Instead of inserting a cut in the complex ω-plane, Landau obtains the analytic continuation of $E_3{}^t(z, \omega)$ by deforming the contours of integration of I_1 and I_2, defined by Eqs. (7.5) and (7.6), respectively, in the complex *u plane*. For values of ω lying in the lower half of the complex ω plane the point $u=-\omega/k_0$ lies in the upper half of the complex u plane if k_0 is positive and the contour of integration may be taken along the real u axis, contour A in Fig. 1(a). Provided that $f_0(u)$ and $g(u)$ are integral functions of u this contour may be deformed into contour B without affecting the value of I_1 and I_2. Let us now suppose that the point $u=-\omega/k_0$ is allowed to move into the lower half of the complex *u plane* to the point P_2, crossing the real axis between $u_1<u<u_2$. The contour B may now be deformed into the contour A' giving a different value for I_1I_2 than that obtained by integrating along the contour A. By this means one can obtain an analytic continuation of I_1I_2 and $E_3{}^t(z, \omega)$ into the upper half of the complex ω plane. However this procedure does not lead to a single valued expression for I_1I_2 since the point P_2 could have been approached by a path that crosses the real u axis outside the range $u_1<u<u_2$, as shown in Fig. 1(b), when the contour B can be deformed back into the contour A. Admittedly this difficulty does not arise in the case considered by Landau when $(u_1, u_2)\equiv(-\infty, \infty)$, but when u_1u_2 are finite we must insert cuts into the real u axis in the ranges $-\infty>u>u_1$, $u_2<u<\infty$ to ensure that I_1, I_2 are single valued. In this case I_1, I_2 are undefined at values of $\omega=-\bar{u}k_0$ where $\bar{u}$ lies on these cuts and we must introduce a corresponding system of cuts into the complex ω-plane in the ranges $-\infty<\omega<-u_2k_0$, $-u_1k_0<\omega<\infty$.

Thus Landau's procedure leads to no increase in simplicity over the alternative we have followed in the text, while it is only valid when $f_0(u)$, $g(u)$ are integral functions of u. Furthermore some of his conclusions as to the nature of the solution for the thermal plasma have to be modified appreciably when one takes account of the fact that $|u_1|$, $|u_2|$ are both less than c, since it is not now possible to deform the contour of integration of Eq. (7.3) into the upper half of the complex ω plane outside the range $-u_2k_0<\omega<-u_1k_0$. The physical consequences of this are discussed.