space along the **k** vector parallel to **E**. Using (1), we obtain

$$(\partial f_1/\partial t)_c = +\int_{S'} f_1(k, E)(1-\cos\theta)P(\theta)dS', \quad (4)$$

$$(\partial f_1/\partial t)_c = +[f_1(k, E)]/[\tau(k)], \quad (5)$$

where τ is the zero field relaxation time. Equation (2) then takes the form

$$(eE/\hbar)(df/dk)+(f_1/\tau)=0, \quad (6)$$

for high as well as low fields.[3]

Since the isotropic part of the distribution will not contribute to the current, the current density is given by

$$j=\int\int\int e\frac{\hbar k}{m}\cos\theta(-f_1\cos\theta)\frac{2}{(2\pi)^3}k^2\sin\theta d\theta d\phi dk, \quad (7)$$

where the integration is over all k space. Using (6) to eliminate f_1, we obtain

$$j=\frac{2}{(2\pi)^3}\frac{e^2}{m}E\int\int\int\tau\frac{df}{dk}k^3\cos^2\theta\sin\theta d\theta d\phi dk. \quad (8)$$

We now make the assumption that the energy surfaces are spherical. The integration can then be carried out, giving

$$j=\frac{1}{3\pi^2}\frac{e^2}{m}E\left[(k^3\tau f)\Big|_0^\infty-\int_0^\infty f\frac{d}{dk}(k^3\tau)dk\right]. \quad (9)$$

For the mechanisms of scattering under consideration, $k^3\tau$ vanishes at the lower limit, and the first term makes no contribution to j. The expression can then be written

$$j=-\frac{e^2}{3m}E\int_0^\infty\frac{1}{k^2}\frac{d}{dk}(k^3\tau)f(k)\frac{1}{\pi^2}k^2dk$$

$$=\frac{e^2}{3m}En\left\langle\frac{1}{k^2}\frac{d}{dk}(k^3\tau)\right\rangle, \quad (10)$$

where n is the number of electrons per unit volume, and the quantity in brackets is to be averaged over all electrons. In terms of the mobility, this result is

$$\mu=\frac{e}{3m}\left\langle\frac{1}{k^2}\frac{d}{dk}(k^3\tau)\right\rangle. \quad (11)$$

In this form the result is valid for low fields and high within the limits that the scattering is nearly elastic. It yields a field-dependent mobility at high fields because the electron distribution is field-dependent.

It is easily checked that this gives correct results in all familiar low field cases. For example, in the case of lattice or impurity scattering in nondegenerate semiconductors at low fields it gives the same mobility as $(e/m)\langle v^2\tau\rangle/\langle v^2\rangle$, which is valid for a Maxwell-Boltzmann distribution.[4] At high fields it gives the correct drift velocity for electrons in a gas, for which the high field distribution is known.[5] It has been pointed out by Wannier that this expression for the mobility is derivable in the high field case from the equation system (17) of his paper.[6]

[3] Essentially this equation is derived for electrons in a gas in high fields in S. Chapman and T. G. Cowling, *Mathematical Theory of Non-Uniform Gases* (Cambridge University Press, Cambridge, 1939), p. 346.

[4] W. Shockley, *Electrons and Holes in Semiconductors* (D. Van Nostrand Company, Inc., New York, 1950), p. 276.

[5] See reference 3, p. 351.

[6] G. Wannier, Phys. Rev. 83, 281 (1951).

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The Comparison between the Longitudinal Correlation and the Time Correlation in a Turbulent Flow*

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Some preliminary applications of high speed computing to the analysis of experimental data on turbulence are reported. A time correlation curve is determined from a recording of velocity fluctuations measured with a single hot-wire anemometer. The shape of the time correlation curve differs from the shape of the longitudinal correlation curve measured in the same fluid flow. One must, therefore, be cautious in reaching conclusions from experiments based on the assumption that time and space spectra of turbulence are identical. A more extensive study of the relation between time and space characteristics of turbulence is in process.

When the relation between a spectrum of turbulence and a correlation coefficient was first found by Taylor,[1] an assumption was made that the turbulence pattern moves in the fluid stream without changing. This assumption was necessary, since Taylor's spectrum described a statistical characteristic of velocity fluctuations at a fixed point of the fluid flow while his correlation coefficient referred to simultaneous velocity fluctuations along the mean velocity direction. Taylor's

* This work was supported by the U. S. Navy Bureau of Ordnance.

[1] G. I. Taylor, Proc. Roy. Soc. (London) A164, 476 (1938).

assumption is, of course, meant as an approximation and in many cases its use is fully justified. In a field of decaying turbulence the time-spectrum[1] cannot be identical with the one-dimensional longitudinal spectrum[2]; if it were, there would not be any decay whatsoever. One must, therefore, be rather cautious in reaching conclusions concerning the decay of turbulence when these conclusions are based on the assumption that the time-spectrum and the longitudinal spectrum are identical. The relation between certain time and space characteristics has been discussed elsewhere.[2,3] It is difficult to establish a rigorous theoretical relation between these characteristics without other assumptions. A direct experimental comparison between them may, therefore, be valuable. The present note describes some preliminary data obtained during the preparation of an extensive investigation in this field.

Oscillographic recordings of the fluctuations of the turbulent velocity made with a hot-wire anemometer were used for this investigation. The measurements (made available by Dr. G. B. Schubauer of the National Bureau of Standards) were obtained in a wind-tunnel at a point 10 feet downstream of a $3\frac{1}{2}$-inch mesh grid at a wind speed of 20 feet per second. The velocity was determined as a function of the time by reading, on the oscillographic recording, the velocity at 1779 instants, t_1, $t_2, \cdots$ spaced at intervals equal to approximately 1/3100 second.

Various statistical characteristics of turbulence, including frequency distributions and second- and third-order correlation curves, were then determined using punch-card computing machines.[4] The second-order time correlation coefficient for the turbulent velocities u_i' and u_{ith}' at two instants t_i and t_{ith} separated by an

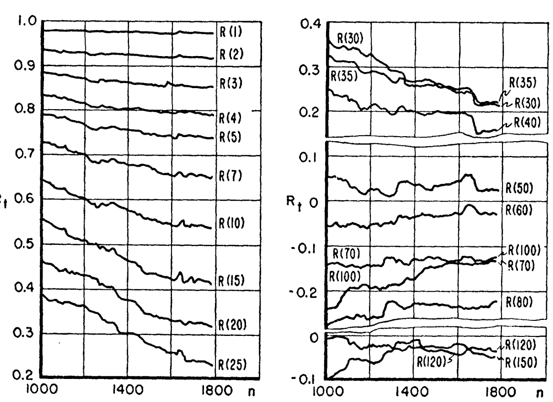

FIG. 1. Time-correlation coefficient $R_t(h)$ as a function of the length of the oscillographic recording. The interval of time h and the length of the recording n are expressed in multiples of 1/3100 second.

[2] F. N. Frenkiel, Compt. rend. **222**, 367 (1946).
[3] F. N. Frenkiel, Proc. 7th Int. Congr. Appl. Mech. **2**, 112 (1948).
[4] Frenkiel, Edelson, and Rawling, Nav. Ord. Lab. Memo 10815 (1950) (unpublished).

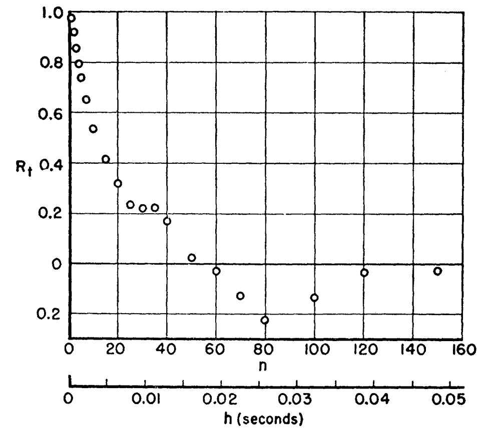

FIG. 2. Time-correlation curve calculated using the total length of the oscillographic recording ($n=1779$ corresponding to about 0.57 second).

interval of time h was calculated with the formula

$$R_t(h)=\sum_{i=1}^{i=n-h} u_i'u_{i+h}'\left[\sum_{i=1}^{i=n-h}(u_{i+h}')^2\right]^{-1},$$

with $n=1, 2, \cdots, 1779$ and h the interval of time measured in multiples of 1/3100 second. Figure 1 represents the time-correlation coefficient for several values of the interval of time h as a function of the number of points used for its computation. The curves show a general trend indicating the limits toward which the correlation coefficient values appear to tend. Figure 2, representing the correlation coefficient R_t as a function of h, is given only as an illustration of the general shape of the time-correlation curve. One notices immediately the large negative values of the correlation coefficient for h in the region of 0.02 to 0.04 second. None of the available measurements of longitudinal correlation coefficients made under the same experimental conditions present such large negative values.[5] While the length of the oscillographic recording was too small to give very accurate numerical values for the correlation coefficients, one can conclude from Fig. 1 that the negative values of $R_t(h)$ are of the order of magnitude of those represented on Fig. 2. From these preliminary investigations, it appears that the time-correlation curve has a sensibly different shape than the longitudinal correlation curve.

If the time correlation curve $R_t(h)$ is compared with the longitudinal correlation curve $R_x(hU)$ (U is the mean velocity and $hU=x$ is the distance between the two points for which the correlation between the u-velocity components is measured), one may expect that the difference between R_t and R_x will decrease as h decreases. However, even if this difference is negligible for small h, it does not necessarily follow that the

[5] Dryden, Schubauer, Mock, and Skramstad, National Advisory Committee for Aeronautics, Rept. 581 (1937).

difference between time and space velocity gradients is negligible, and there may be, therefore, a non-negligible difference between time and space microscales of turbulence.

An extensive investigation is needed before complete experimental results can be given concerning the relation between time and space characteristics of turbulence. This work is now in process in cooperation with the National Bureau of Standards.[6]

[6] New experimental measurements, now in process, are being made in cooperation with Dr. J. Laufer (National Bureau of Standards) and Mr. I. Katz (Applied Physics Laboratory).

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Statistical Mechanics of Irreversibility*

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The fluctuation-dissipation theorem relating spontaneous equilibrium fluctuations to the conductance in a dissipative thermodynamic system is extended to the case of several variables, using a quantum statistical analysis. The conductance matrix is shown to be subject to certain symmetry relations, providing a generalization of the Onsager reciprocity theorem. The susceptance matrix is also shown to be subject to similar symmetries. The symmetries apply to all frequency components, and hence to arbitrary transient processes.

1. INTRODUCTION

The theory of irreversible processes consists essentially of two types of theorems. The first of these theorems is the reciprocity relations of Onsager, treating of the symmetry of the mutual interference among several simultaneously occurring irreversible processes.[1,2] The second is the fluctuation-dissipation theorem, relating the spontaneous fluctuations in an equilibrium system and the parameter (the conductance) which characterizes the dissipative aspects of an irreversible process.[3–5] The purpose of this and the following paper is to extend the fluctuation-dissipation theorem to several variables and to exhibit its relation to a generalization of the Onsager reciprocity theorem.

Both the Onsager theorem and the fluctuation-dissipation theorem have been investigated by quantum statistical and thermodynamic methods. In the present paper we shall be concerned exclusively with the quantum statistical analysis.

We shall show that the fluctuation-dissipation theorem may be interpreted as a matrix equation when extended to several variables. The mean square fluctuation of a single variable $\langle Q^2\rangle$ is replaced by a matrix, the elements of which are the spontaneous mutual correlation moments of two fluctuating variables $\langle Q_jQ_k\rangle$. The admittance (and hence the conductance) is replaced by an admittance matrix, the element Y_{jk} describing the response of the variable Q_j to the force V_k. Furthermore, we shall show that both the conductance and susceptance matrices are subject to a symmetry relation. The symmetries so established apply to all frequency components and consequently are applicable to arbitrary types of transient processes. An indication will be given of the application of this reciprocity theorem to steady-state processes.

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[1] L. Onsager, Phys. Rev. **37**, 405 (1931); **38**, 2265 (1931).
[2] H. B. G. Casimir, Revs. Modern Phys. **17**, 343 (1945).
[3] H. B. Callen and T. A. Welton, Phys. Rev. **83**, 34 (1951).
[4] J. L. Jackson, Phys. Rev. **87**, 471 (1952).
[5] H. B. Callen and R. F. Greene, Phys. Rev. **86**, 702 (1952).

2. THE ADMITTANCE MATRIX

In this section we shall define the admittance matrix and develop a useful quantum statistical expression for it.

We consider a system whose Hamiltonian in isolation is H_0. Let the system be acted on by a perturbation which induces the irreversible processes of interest. For a single variable this perturbation may be written in the form[3] $V(t)Q(\cdots q_r\cdots p_r\cdots)$, in which $V(t)$ is a time-dependent scalar which measures the instantaneous strength of the applied perturbation, and which therefore plays the role of a driving force, and in which $Q(\cdots q_r\cdots p_r\cdots)$ is a function of the coordinates and momenta of the particles composing the system. For the general case in which several simultaneous perturbations act, we shall write the total perturbation as $\sum_k V_k(t)Q_k(\cdots q_r\cdots p_r\cdots)$.

We shall assume the perturbations to be sufficiently small so that first-order perturbation theory is valid. As we shall see, this assumption linearizes the system in the following sense. Under the influence of the perturbations, the expectation value of Q_j becomes a function of time, and if $\dot{Q}_j(\omega)$ denotes the Fourier component of the time derivative of this expectation value, then $\dot{Q}_j(\omega)$ is a linear function of the Fourier