tainties in the assumed potential. Measurements[17,18] of the ultraviolet absorption of diamond indicate an energy gap of at least 5.5 ev. The calculated width of the valence band 22 ev is consistent with interpretations[19,20] of soft x-ray emission spectra from diamond.

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Bound States and the Interaction Representation

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IN a recent article[1] the temporal development of the state vectors of the free and bound states has been derived from the time-independent formulation of quantum theory. According to this mathematical treatment the transformation operator $W_+(t)$ is non-unitary. On the other hand, it has been observed in Sec. IV of reference 1 that application of an iteration process leads to the conclusion that $W_+(t)$ is unitary. In the present note we shall clarify this apparent inconsistency.

From Eqs. (53) and (36) of reference 1 we see that the integral equations,

$$X(t)=1-i\int_{-\infty}^{t}[H_1(t'),X(t')]dt', \quad (1)$$

$$Y(t)=-i\int_{-\infty}^{t}[H_1(t'),Y(t')]dt', \quad (2)$$

are satisfied by

$$X(t)=W_+(t)W_+^\dagger(t)=\int\Psi_{+\lambda}(t)\Psi_{+\lambda}^\dagger(t)d\lambda, \quad (3)$$

$$Y(t)=\Sigma_s\Psi_s(t)\Psi_s^\dagger(t). \quad (4)$$

However, Eqs. (1) and (2) have also the solutions

$$X(t)=1, \quad (5)$$

$$Y(t)=0. \quad (6)$$

The iteration process described in Sec. IV of reference 1 leads to Eqs. (5) and (6), but not to Eqs. (3) and (4). One cannot, therefore, draw any conclusion about the product $W_+(t)W_+^\dagger(t)$ by means of the iteration process we have considered.

The fact that iteration of Eqs. (1) and (2) does not lead to the solutions given by Eqs. (3) and (4) indicates that these solutions cannot be expanded into power series that satisfy the convergence requirements for the validity of the iteration process. Consider, for example, the wave functions $\Psi_s(\mathbf{p})$ of the bound states of the hydrogen atom in the momentum representation. These wave functions contain the factors $[(nap/\hbar)^2+1]^{-1}$, where $n=1,2,\cdots$, $a=\hbar^2/me^2$, and $p=|\mathbf{p}|$. Expansion of $\Psi_s(\mathbf{p})$ in powers of e involves the power-series expansion

$$[1+(\hbar/nap)^2]^{-1}=1-(\hbar/nap)^2+\cdots,$$

which diverges when $p<\hbar/na$. Similarly, the power-series expansion of the sum $\Sigma_s\Psi_s(\mathbf{p})\Psi_s^*(\mathbf{p}')$ diverges for small p and p'.

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Charge Independence and Multiple Pion Production*

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SEVERAL authors have recently discussed the role of charge independence in processes involving pions and nucleons.[1] We have explored the consequences of charge independence for multiple pion products in nucleon-nucleon and in pion-nucleon collisions. Before presenting our results,[2] we wish to call attention to a point which has been overlooked in previous treatments of single pion production in nucleon-nucleon collisions and leads to an additional relation among the differential cross sections.

Let us denote by $\sigma(\nu_1\nu_2\rightarrow\nu_1'\nu_2'\pi^i)$ the differential cross section for the production of a pion $\pi^i(i=+,-,0)$ in a collision of two nucleons ν_1 and ν_2 ($\nu=p$ or n), which are transformed into two nucleons ν_1' and $\nu_2'(\nu'=p$ or $n)$, respectively. As the differential cross section refers the direction of motion of the outgoing pion to the direction of motion of the incident nucleons, and as all nucleon charges are specified, it follows that in general $\sigma(\nu_1\nu_2\rightarrow\nu_1'\nu_2'\pi^i)$ is distinct from $\sigma(\nu_2\nu_1\rightarrow\nu_1'\nu_2'\pi^i)$ [or from $\sigma(\nu_1\nu_2\rightarrow\nu_2'\nu_1'\pi^i)$] if the initial (or final) nucleons have different charges. If charge symmetry is taken into account, the number of distinct differential cross sections for single pion production reduces to seven which can be written in the form:

$$\sigma(np\rightarrow nn\pi^+)=|F_0/\sqrt{6}-F_1'/2|^2,$$
$$\sigma(np\rightarrow np\pi^0)=|F_0/2\sqrt{3}+F_1/2|^2,$$
$$\sigma(np\rightarrow pn\pi^0)=|F_0/2\sqrt{3}-F_1/2|^2,$$
$$\sigma(pn\rightarrow nn\pi^+)=|F_0/\sqrt{6}+F_1'/2|^2,$$
$$\sigma(pp\rightarrow np\pi^+)=|F_1/\sqrt{2}+F_1'/2|^2,$$
$$\sigma(pp\rightarrow pn\pi^+)=|-F_1/\sqrt{2}+F_1'/2|^2,$$
$$\sigma(pp\rightarrow pp\pi^0)=|F_1'/\sqrt{2}|^2,$$

where F_0 is the isotopic singlet amplitude and F_1, F_1' are the two isotopic triplet amplitudes. Thus, the seven cross sections depend on the three absolute values and the two relative phases of these amplitudes. Hence, there exist two relations among the σ's as a consequence of charge independence. One of these is linear:

$$\sigma_1^+ +\sigma_3^+ =2(\sigma_2^+ +\sigma_4); \quad (1)$$

the other is essentially a phase relationship:

$$\cos^{-1}\{\sigma_3^-/2[(\sigma_3^+-\sigma_4)\sigma_4]^{\frac{1}{2}}\}+\cos^{-1}\{\sigma_1^-/2[(\sigma_1^+-\sigma_4)\sigma_4]^{\frac{1}{2}}\}$$
$$=\cos^{-1}\{\sigma_2^-/[(\sigma_1^+-\sigma_4)(\sigma_3^+-\sigma_4)]^{\frac{1}{2}}\}. \quad (2)$$

Here we have set

$$\sigma_1^\pm=\sigma(np\rightarrow nn\pi^+)\pm\sigma(pn\rightarrow nn\pi^+),$$
$$\sigma_2^\pm=\sigma(np\rightarrow np\pi^0)\pm\sigma(np\rightarrow pn\pi^0),$$
$$\sigma_3^\pm=\sigma(pp\rightarrow pn\pi^+)\pm\sigma(pp\rightarrow np\pi^+),$$
$$\sigma_4=\sigma(pp\rightarrow pp\pi^0).$$

Relation (2) can only be obtained by distinguishing between $\sigma(np\rightarrow nn\pi^+)$ and $\sigma(pn\rightarrow nn\pi^+)$, etc. Messiah[1] [see his Eq. (10)] and Luttinger[1] [see his Eq. (2)] do not make this distinction, so that their relation is only valid in so far as *total* cross sections are concerned.

The above considerations can be generalized to multiple pion production in nucleon-nucleon collisions; as the number k of pions increases, both the number S_k of distinct cross sections (in the above sense) and the minimum number R_k of relations resulting from charge independence increase rapidly. It can be shown that

$$S_k = \tfrac{1}{2}I_k + I_{k+1} + \tfrac{1}{2}I_{k+2},$$
$$R_k = I_{k+3} - \tfrac{1}{2}I_{k+2} - 4I_{k+1} - (5/2)I_k + 1,$$
$$I_k = \pi^{-1}\int_0^\pi (1+2\cos\varphi)^k d\varphi.$$

For large k, $S_k \sim R_k \sim k^{-\frac{1}{2}}3^k$. Thus, for $k=2$, $R_2=7$; one relation is particularly interesting:

$$\begin{aligned}4[&\sigma(pp\rightarrow pp\pi^0\pi^0)+\sigma(np\rightarrow np\pi^0\pi^0)+\sigma(np\rightarrow pn\pi^0\pi^0)]\\ &+[\sigma(pp\rightarrow np\pi^+\pi^0)+\sigma(pp\rightarrow np\pi^0\pi^+)+\sigma(pp\rightarrow pn\pi^+\pi^0)\\ &+\sigma(pp\rightarrow pn\pi^0\pi^+)+\sigma(np\rightarrow pp\pi^-\pi^0)+\sigma(np\rightarrow pp\pi^0\pi^-)\\ &+\sigma(np\rightarrow nn\pi^+\pi^0)+\sigma(np\rightarrow nn\pi^0\pi^+)]-2[\sigma(pp\rightarrow pp\pi^+\pi^-)\\ &+\sigma(pp\rightarrow pp\pi^-\pi^+)+\sigma(pp\rightarrow nn\pi^+\pi^+)+\sigma(np\rightarrow np\pi^+\pi^-)\\ &+\sigma(np\rightarrow np\pi^-\pi^+)+\sigma(np\rightarrow pn\pi^+\pi^-)+\sigma(np\rightarrow pn\pi^-\pi^+)]=0, \quad (3)\end{aligned}$$

where the notation is an obvious generalization of the single pion notation. If one now introduces the following numbers for the 2π production reactions:

$$N_{+,-,0} = \langle N_{+,-,0}\rangle_{pp} + \langle N_{+,-,0}\rangle_{np},$$

where, e.g., $\langle N_+\rangle_{pp}$ is the average number of positive pions produced in a pp collision in which two pions are emitted with given direction (the other quantities are likewise defined). One easily finds from (3)

$$N_+ + N_- = 2N_0, \quad (4)$$

which is a special case of Watson's[1] general relation. One can start with Watson's relation which holds for an arbitrary number of pions and reverse the argument to obtain the generalization of Eq. (3) for an arbitrary pion multiplicity.[3]

The consequences of charge independence for multiple pion production in pion-nucleon collisions have also been examined;[2] unfortunately, some of the relations cannot be tested experimentally because they involve cross sections referring to an incident π^0. An illustration of the latter type of relationship which is independent of the pion multiplicity is

$$\sigma(\pi^+p)+\sigma(\pi^-p)=2\sigma(\pi^0p), \quad (5)$$

where $\sigma(\pi^ip)$ denotes the sum of all cross sections corresponding to a specified multiplicity produced by a pion π^i incident on a proton. Secondly, considering $\pi-p$ collisions of given multiplicity, relation (4) again holds true provided one defines $N_+ = \langle N_+\rangle_{p\pi^+} + \langle N_+\rangle_{p\pi^-} + \langle N_+\rangle_{p\pi^0}$, etc.

When there is only one final pion (i.e., pion scattering by a proton), (4) and (5) are the only relations which follow from charge independence and actually reduce to one, namely Heitler's[1] relation, because of charge symmetry and detailed balancing.

For k final pions, $R_k = I_{k+3} - \frac{3}{2}I_{k+2} - (5/2)I_{k+1} + 1$, R_k^{ch} (referring to an incoming $\pi^\pm$ only) $= I_{k+3} - \frac{3}{2}I_{k+2} - 3I_{k+1} - \frac{1}{2}I_k + 1$. Thus, $R_2=6$, $R_2^{\mathrm{ch}}=1$, whereas asymptotically $R_k \sim R_k^{\mathrm{ch}} \sim k^{-\frac{1}{2}}3^k$.

Of course, if the further assumption is made that the pion-nucleon interaction is restricted to the state of total isotopic spin $\frac{3}{2}$,[4] the number of relations involving only incident charged pions is greatly increased; for example, Eq. (5) is replaced in this approximation by

$$\tfrac{1}{3}\sigma(\pi^+p)=\tfrac{1}{2}\sigma(\pi^0p)=\sigma(\pi^-p), \quad (6)$$

while additional relations hold for specific multiplicity, such as:

$$\begin{aligned}2[&\sigma(p\pi^-\rightarrow p\pi^-\pi^0)+\sigma(p\pi^-\rightarrow p\pi^0\pi^-)]\\ &=\sigma(p\pi^-\rightarrow n\pi^+\pi^-)+\sigma(p\pi^-\rightarrow n\pi^-\pi^+)+4\sigma(p\pi^-\rightarrow n\pi^0\pi^0).\end{aligned}$$

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[1] See K. M. Watson, Phys. Rev. **85**, 852 (1952); R. L. Garwin, Phys. Rev. **85**, 1045 (1952); A. M. L. Messiah, Phys. Rev. **86**, 430 (1952); J. M. Luttinger, Phys. Rev. **86**, 571 (1952), in addition to the original reference of W. Heitler, Proc. Roy. Irish Acad. **51**, 33 (1946). One of us (A.P.) is indebted to J. M. Luttinger for several discussions on these topics.

[2] Details of the method will be found in L. Van Hove, Rochester Report NYO-3074, unpublished.

[3] It may be pointed out that additional linear relations are obtained for pion production in nucleon-nucleon collisions when the nucleons emerge as a deuteron. See reference 2.

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Lower Limits for Interaction Times in Photon Scattering Processes

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In a semi-classical treatment of the scattering of light by charged particles a switching-on function was used to avoid the appearance of "run-away" radiation reaction terms.[1] A consistent formulation appeared to be possible only if the duration of the interaction, the "interaction time," exceeded a lower limit. Recently, several authors[2] suggested that it might be necessary to use switching-on functions also in quantum theory. A tentative way of doing this, which leads to a confirmation of the semi-classical conclusions concerning interaction times, is to multiply the matrix elements $H_{0\mathrm{I}}$ for transitions from the initial state in the equations of the time dependent perturbation theory by a switching-on function f. In solving these equations, we retain the terms of lowest order containing time derivatives of f. Apart from this we follow the usual procedure and obtain for second-order processes[3]

$$\dot{b}_0 = (i/4\pi)\int d\Omega(f\Delta + if\Gamma + i\dot{f}\Delta_1 - \dot{f}\Gamma_1)\cdot b_0, \quad (1)$$

where

$$\Delta = \frac{1}{\hbar}\Sigma\frac{H_{0\mathrm{I}}H_{\mathrm{I}0}}{E_{\mathrm{I}}-E_0}, \quad \Delta_1 = \Sigma\frac{H_{0\mathrm{I}}H_{\mathrm{I}0}}{(E_{\mathrm{I}}-E_0)^2},$$
$$\Gamma = \frac{\pi}{\hbar}S_F\rho_F\left\{\Sigma\frac{H_{0\mathrm{I}}H_{\mathrm{I}F}}{E_{\mathrm{I}}-E_0}\right\}^2, \quad (2)$$
$$\Gamma_1 = \pi S_F\rho_F\left\{\Sigma\frac{H_{0\mathrm{I}}H_{\mathrm{I}F}}{E_{\mathrm{I}}-E_F}\Sigma\frac{H_{F\mathrm{II}}H_{\mathrm{II}0}}{(E_{\mathrm{II}}-E_0)^2} + \Sigma\frac{H_{0\mathrm{I}}H_{\mathrm{I}F}}{(E_{\mathrm{I}}-E_0)^2}\Sigma\frac{H_{F\mathrm{II}}H_{\mathrm{II}0}}{E_{\mathrm{II}}-E_0}\right\}.$$

Σ denotes summation over the intermediate states I, II; and S_F summation and averaging over the directions of spin and polarization in the final state. Δ represents a frequency shift, and Γ a damping coefficient; Δ_1, Γ_1 are dimensionless.

The well-known formulas for scattering cross sections and self-energy effects follow from Eq. (1) if

$$(\tau_{1,2}\dot{f})_{\mathrm{Av}} \ll (f)_{\mathrm{Av}} \cong 1, \quad (3)$$

where

$$\tau_1 = \Delta_1/\Gamma, \quad (4a)$$
$$\tau_2 = \Gamma_1/\Delta. \quad (4b)$$

Using $f = 1-\exp(-\alpha t)$, we evaluate these averages for the time interval between $t=0$ and the interaction time t. It appears that condition (3) will be fulfilled if $1/\alpha \ll t$, and $\tau_{1,2} \ll t$. The times τ_1, τ_2 are thus seen to represent lower limits for the interaction time of processes for which the formulas of the quantum theory of radiation are known to be approximately valid.

For scattering of photons by electrons initially at rest, we evaluate Eqs. (2) in the usual way and obtain from Eqs. (4)

$$\tau_1 = \left(\frac{\hbar c}{e^2}\right)^2\frac{r_0}{c}\frac{k_0^2}{k^2}\frac{\mu(k_0+k)}{(k_0+k)^2+k_0k(1+x^2)} \quad (5a)$$

$$\tau_2 = \frac{r_0}{c}\frac{1+x}{2}\frac{k^2}{k_0^2}\frac{(k_0-k)^2+k_0k(1+x)}{\mu(k_0+k)}, \quad (5b)$$

where $r_0 = e^2/mc^2$, and x is the cosine of the scattering angle. Except for $k/\mu = \gamma > 137$ and small scattering angles, $\tau_2 \ll \tau_1$. We shall thus, at present, consider τ_1 only.

In the extreme relativistic case ($\gamma \gg 1$) in which we are primarily interested, τ_1 is largest for $x \cong -1$: $\tau_{1\,\mathrm{max}} \cong 4\gamma\tau_0$, $\tau_0 = \hbar^2/me^2c$. The extreme relativistic average of τ_1 is $2\gamma\tau_0/\ln(2\gamma)$. These results confirm the conclusion arrived at in the semi-classical theory that a lower limit for interaction time of the order of magnitude $\gamma\tau_0$ should be taken into consideration.

The same method can be used to investigate the validity of quantum theoretical formulas for higher order processes. For extreme relativistic photon-photon scattering a rough estimate leads to the order of magnitude $\gamma\tau_0$ with $\tau_0 = (\hbar c/e^2)^4(r_0/c) \cong 3.3 \times 10^{-15}$ sec for the lower limit of the interaction time. The coin-