Radial Focusing in the Linear Accelerator*

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The focusing mechanism described in the previous paper is applied to the linear accelerator. The defocusing forces in the linear accelerator are derived, and the particular example of a 4-million-volt accelerator 10 meters long is discussed. It is shown that the defocusing forces can be compensated by magnetic or electric lenses of the alternating gradient type using easily attainable field gradients. For example, a defocusing field gradient of 1000 volts per centimeter on a 100-kv proton beam can be compensated by a lens 20 cm long with field gradients of the order of 1000 gauss per centimeter.

I. INTRODUCTION

THE method of focusing proposed by Courant, Livingston, and Snyder[1] for use in high energy synchronous accelerators seems to be applicable also to the linear accelerator. The usefulness of the linear accelerator to date has been limited by the defocusing which is associated with the mechanism of acceleration. In this paper a brief discussion of the focusing forces in the linear accelerator is followed by a presentation of the method of application of magnetic or electric focusing to compensate for these forces.

Although the linear accelerator is of considerable interest in its own right, it is particularly interesting to our laboratory as a possible injector for high energy positive ion accelerators. If the radial defocusing can be controlled, a linear accelerator injector would present important advantages associated with its geometry and the accessibility of its components. We shall, accordingly, confine this discussion to accelerators in the energy range low enough that relativistic effects can be neglected. Since, in general, the defocusing fields become weaker as the velocity approaches the velocity of light, and since the magnetic focusing procedures to be discussed are still valid at relativistic velocities, the limitation of the discussion to the nonrelativistic range is legitimate.

II. THEORY OF THE LINEAR ACCELERATOR

The z axis of a cylindrical coordinate system will be taken to be coincident with the axis of the accelerator. MKS units will be used.

Axial Electric Field Pattern

The separation between accelerating gaps, L, will be the distance traveled by a particle during one-half cycle of the accelerating signal

$$L=\pi v/\omega, \tag{1}$$

where v is particle velocity and $\omega/2\pi$ is the frequency of the accelerating signal. Now if w is the energy (measured in volts) gained per gap by a particle at the equilibrium phase of the accelerating signal, the rate of change of energy W of the equilibrium particle with distance will be

$$dW/dz=\omega w/\pi v. \tag{2}$$

This relation can be integrated if we assume a specific form for $w(z)$.

For our present purpose, we shall retain sufficient generality if we set

$$w=w_f(z/z_f)^n, \tag{3}$$

where w_f is the energy gain at the final gap and z_f is the length of the accelerator. It should be noted here that distances and times are to be measured on scales representing an accelerator whose injection is at zero volts.

We now substitute $\frac{1}{2}mv^2$ for W in Eq. (2) and integrate to obtain

$$v=v_f(z/z_f)^{(1+n)/3}, \tag{4}$$

$$(z/z_f)^{(2-n)/3}=(2-n)v_ft/3z_f, \tag{5}$$

and

$$W=W_f(z/z_f)^{(2+2n)/3}. \tag{6}$$

The accelerating field will be a complicated function of position along the tube. It can, however, be analyzed into component traveling waves, and it is evident that the only traveling wave component whose effects will average to a value different from zero is the component which travels with the particles at their average velocity. This component will have the form

$$E_z=E_0f(z)\sin\omega\left[t-\int(dz/v)\right]. \tag{7}$$

If the equilibrium particle maintains a constant phase on this wave, $f(z)$ will depend on z in the same fashion as does dW/dz, which we can derive from (6). To retain E_0 as a quantity having the dimensions of an electric field, we write

$$f(z)=\frac{dW}{dz}\bigg/\left(\frac{dW}{dz}\right)_f .$$

In terms of the assumption of Eq. (3) we can rewrite

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[1] Courant, Livingston, and Snyder, preceding article [Phys. Rev. 88, 1190 (1952)].

(7) in the form

$$E_z=E_0(z/z_f)^{(2n-1)/3}\sin\omega\left[t-\frac{3(z/z_f)^{(2-n)/3}z_f}{(2-n)v_f}\right]. \quad (8)$$

The form of E_0 is discussed below and appears in Eq. (13).

Particle Phase Oscillations

In order to evaluate E_0 and to determine the possible limits of n, it is necessary to investigate the phase oscillations of the accelerated particles. Thus far, we have discussed only particles which travel at the equilibrium phase ϕ_0 and at the velocity v of the electric field wave. For a general particle, we now define a quantity δ in terms of which we shall discuss the particle velocity $\dot{z}$. δ will be defined by

$$\int dz/v=t-\phi_0/\omega-\delta/\omega, \quad (9)$$

whence we derive the relations

$$\dot{z}=v(1-\dot{\delta}/\omega), \quad (10)$$

$$\ddot{z}=-v\ddot{\delta}/\omega+v(dv/dz)(1-\dot{\delta}/\omega)^2. \quad (11)$$

We now substitute from (7), (9), and (11) in the relation $m\ddot{z}=eE_z$ to obtain

$$-mv\ddot{\delta}/\omega+mv(dv/dz)(1-\dot{\delta}/\omega)^2=eE_0f(z)\sin(\phi_0+\delta). \quad (12)$$

If δ is assumed to be small, (12) becomes approximately

$$\ddot{\delta}/\omega-(dv/dz)(1-2\dot{\delta}/\omega)+(eE_0/mv)f(z)(\sin\phi_0+\delta\cos\phi_0)=0. \quad (12a)$$

For δ and its derivatives equal to zero, (12a) defines E_0. Making the appropriate substitutions from (4), (5), (6), and (8) we find

$$E_0\sin\phi_0=2(1+n)W_f/3z_f, \quad (13)$$

and, from (4), (5), (12a), and (13) the equation describing the phase oscillation is

$$\frac{\ddot{\delta}}{\omega}+\frac{(1+n)}{(2-n)}\frac{2}{t}\frac{\dot{\delta}}{\omega}+\frac{(1+n)\omega\cot\phi_0}{(2-n)t}\frac{\delta}{\omega}=0. \quad (14)$$

The asymptotic solution of (14) for t large is

$$\frac{\delta}{\omega}=At^{-(2+5n)/(8-4n)}\sin\left(\frac{4(1+n)\omega t\cot\phi_0}{2-n}\right)^{\frac{1}{2}}, \quad (15)$$

where A is a constant. This represents a damped oscillation provided n lies between -0.4 and 2 and $\cot\phi_0$ is positive. Physically, n cannot be greater than 2; for n greater than 2, it is evident from (1) and (4) that L must increase at a rate faster than z, a situation which is not realizable in practice.

Radial Electric Field and Azimuthal Magnetic Field

The radial motion of the particles in a linear accelerator will be influenced by two field components, E_r and B_θ. To a first approximation, these components are given by

$$E_r=-\tfrac{1}{2}r\partial E_z/\partial z, \quad (16)$$

$$B_\theta=-\tfrac{1}{2}(r/c^2)\partial E_z/\partial t. \quad (17)$$

The form of these components can be derived from (8) and (13). (We neglect the small term which includes the derivative of $f(z)$ and those time varying fields which follow from other field components than (7).)

$$E_r=\frac{r(1+n)\omega W_f}{3\sin\phi_0 v_f z_f}(z/z_f)^{(n-2)/3}\cos(\phi_0+\delta), \quad (18)$$

$$B_\theta=\frac{r(1+n)W_f\omega}{3c^2\sin\phi_0 z_f}(z/z_f)^{(2n-1)/3}\cos(\phi_0+\delta). \quad (19)$$

Radial Forces on the Accelerated Particles

The radial force equation $m\ddot{r}=eE_r-evB_\theta$ becomes, upon substitution from (18), (19), and (4),

$$m\ddot{r}=\frac{r(1+n)\omega W_f}{3\sin\phi_0 z_f v_f}(z/z_f)^{(n-2)/3}(1-v^2/c^2)\cos(\phi_0+\delta). \quad (20)$$

Since we are neglecting relativistic effects, the second term in the bracket, which arose from the effect of B_θ, is negligible. Also, the average radial force on a particle will not differ materially from the radial force on the equilibrium particle. Thus, the radial deflections are those which result from the radial electric field

$$E_r=\frac{r\omega(1+n)W_f}{3z_f v_f}(z/z_f)^{(n-2)/3}\cot\phi_0. \quad (21)$$

Evidently, the closer n approaches its limiting value of 2, the more nearly constant becomes the defocusing field as a function of distance along the accelerator.

Numerical Example

As a numerical example, we consider a 4-million-volt linear accelerator 10 meters long. Protons will be injected into this accelerator at 100 kv and will be accelerated by a 30-megacycle signal. We assume, further, that the equilibrium phase is 45°. Evaluation of other parameters depends on the choice of n. We shall consider three cases: $n=0$ (voltage equal on all gaps), $n=\frac{1}{2}$ (constant amplitude axial electric field), and $n=1$ (gap voltage increasing linearly with distance along the accelerator).

From Eq. (21) we can now derive the radial field gradient,

$$G=E_r/r=9.1\times10^5(1+n)(0.1z)^{(n-2)/3}(\text{volts/m})/\text{m}. \quad (22)$$

The value of z corresponding to injection follows from (6). The values of G for the three n's are as follows:

	$n=0$	$n=\frac{1}{2}$	$n=1$
G at injection (volts/cm)/cm	3640	870	455
G final (volts/cm)/cm	91	136	182

The $n=1$ case is obviously preferable from the standpoint of radial defocusing during the early stages of acceleration. The practical application of this case presents some difficulty, but the same desirable results can be achieved by accelerating in several stages, the accelerating voltage increasing stage by stage as the particle is accelerated.

III. THE FOCUSING MECHANISM

The technique of magnetic focusing proposed by Courant, Livingston, and Snyder[1] involves the passage of a beam of particles through successive magnetic fields of alternating transverse gradients. As in the preceding section, we choose the axis of the beam as the z axis. We suppose that the beam travels with velocity v, and that from $z=0$ to $z=l$ we apply a magnetic field having $B_x=B'y$ and $B_y=B'x$ where B' is a constant. From $z=l$ to $z=2l$, we apply a reversed field $B_x=-B'y$ and $B_y=-B'x$. Courant, Livingston, and Snyder demonstrate that this system acts like a converging lens whose focal length is given by their Eq. (19).

It is worthy of note that exactly the same technique will work with electric fields of the form $E_x=E'x$, $E_y=E'y$, etc. In this case the properties of the lens will be given by substituting E'/v for B' in the magnetic lens formulas.

IV. APPLICATION OF FOCUSING TO THE LINEAR ACCELERATOR

If the defocusing forces in the linear accelerator are to be compensated by magnetic lenses of the type discussed in the preceding section, we must take into account the fact that the effective electric gradient G given by Eq. (22) is acting throughout, in addition to the forces due to B'. The lens will now be one in which one section has an effective restoring field of $B'-G/v$ and the other has an effective field of $-B'-G/v$. The range of stability is given by Courant, Livingston, and Snyder's Eq. (5) and their Fig. 2 if only we substitute $l^2e(B'\pm G/v)/4\pi^2mv$ for their n_1/N^2 and $-n_2/N^2$.

We now make an arbitrary choice of design parameters satisfying the stability conditions. We shall determine values for B' and l which correspond to the stable combination $n_1/N^2=0.07$, $n_2/N^2=0.05$. This choice yields the following relations:

$$B'=6G/v \text{ (or } E'=6G) \quad \text{and} \quad l^2=0.79W/G, \tag{23}$$

where W is the particle energy.

From (22) and (23) we can determine the lens parameters for our 4-Mev accelerator. The result of these operations for the injection end of the accelerator is summarized in Table I

TABLE I.

	Lens length ($2l$) cm	B' (gauss per cm)	E' (volts per cm per cm)
$n=0$	9.4	5050	21,800
$n=\frac{1}{2}$	19	1200	5200
$n=1$	26	620	2700

As we progress along the accelerator, the permissible lens length increases and the magnetic or electric field gradients decrease. From (4), (22), and (23) we see that the lens length can increase as $z^{(n+4)/6}$. In this case B' will decrease with distance like $1/z$ or, if electric lenses are used, E' will decrease like $z^{(2-n)/3}$. The final lens length will be of the order of meters, and it will probably be desirable to make a new choice of stable parameters which results in shorter lenses near the exit end of the accelerator.

The applications of these principles to practical accelerators will place some limitations on the radio-frequency technique. If magnetic lenses are used, the accelerator tube diameter should be small, and it will be necessary to use lumped constant circuits rather than wave guides. Electric lenses could be combined with the drift tubes so that the same unit is both a lens component and a drift tube.

Lenses of this sort appear to be equally practicable for higher energy accelerators. We have computed possible lens parameters for an accelerator in which (as in the Berkeley linear accelerator) injection is at 4 million volts, energy gain is of the order of a million volts per foot, n is $\frac{1}{2}$, and the accelerating frequency is 200 megacycles. In this case, as usual, the worst defocusing is at the injection end of the accelerator where effective radial gradients exist of about 5000 volts per cm per cm. These could be compensated by lenses of our type 50 cm long having magnetic field gradients of 1100 gauss per cm or electric field gradients of 30,000 volts per cm per cm.