The Effect of Final State Interactions on Reaction Cross Sections

KENNETH M. WATSON
Physics Department, Indiana University, Bloomington, Indiana
(Received August 18, 1952)

Particles produced in a reaction often interact strongly with each other before getting outside the range of their mutual forces. Formal effects of such interactions are discussed, and in particular it is shown that the effect for very strong attractive interactions can be calculated explicitly without having detailed knowledge of the properties of the reaction. Application is made to some meson phenomena.

I. INTRODUCTION

IT frequently happens that, when particles are produced in a nuclear or elementary particle reaction, some of these interact among themselves so strongly that they influence appreciably the properties of the reaction cross section. These interactions we shall call "final state interactions." Their influence is well known in nuclear physics where the Coulomb attraction plays an important role in such phenomena as β-decay.

Recently, strong short-range nuclear interactions have been found to play a striking part in certain reactions involving mesons. It is with such interactions as these that we shall be primarily concerned. The importance of recognizing and analyzing the role played by these interactions lies partly in the observation that they may greatly modify the angular distributions and energy spectra of the produced particles and yet play no important part in the primary mechanism by which the reaction takes place. That is, it is as if the reaction first takes place as if there were no final state interaction and then is distorted by the short-range nuclear interactions before the particles can get out of the range of their mutual forces. It is to be expected that an analysis of the reaction will be considerably simplified by separating the effect of these two mechanisms, which we shall frequently refer to as the "primary mechanism" of the reaction and the "final state interaction," respectively. We shall find that under certain conditions these terms can be made meaningful by explicitly exhibiting their individual contributions to the reaction cross section. Conversely, this separation, when possible, will provide an indirect means of measuring the scattering interaction between two particles produced in a reaction.

Perhaps the most striking example of final state interactions in meson phenomena is that occurring in meson production by nucleon-nucleon collisions,[1,2] where the final state nuclear force influences considerably the spectrum of the produced mesons.[3,4] The capture of π^- mesons by deuterons leads, in a certain fraction of the events, to a single high energy γ-ray and to two neutrons.[5] The nuclear force between the two neutrons modifies the γ-ray spectrum and leads to a means of measuring the neutron-neutron (n-n) force.[6] Present experimental results[7,8] are in agreement with other known characteristics of the n-n force. Finally, there is a slight suggestion from the work of Danysz, Lock, and Yekuteli[9] that a meson-meson interaction may modify multiple meson production cross sections.[10]

We shall be most interested in those reaction processes for which three general conditions are met. These conditions will imply that the final state interaction seriously modifies the reaction cross section and also that the effect of this is calculable. The first of these is that the mechanism of the reaction cross section be a short-range interaction, i.e., that the primary reaction (irrespective of the final state interactions) be confined to a certain volume V of order a^3, where a is the range of the interaction. It is not unlikely that this condition will be met by most of the nonelectromagnetic elementary particle reactions. Second, the effect of the final state interaction is to be considered only for low relative energies of the particles whose interaction is being considered. For high relative energies, this interaction will be expected to be of relatively little importance. Finally, the final state interaction must be strong and attractive, a restriction which will be made more precise in the following paragraphs.

To see this, let us suppose the reaction to be proceeding backwards. That is, we send the produced particles back into the region of their mutual interaction in such a manner that the initial state will result. We suppose that the two particles whose interaction we are studying have a small relative energy and that the scattering cross section of these two particles is large compared to the effective cross-sectional area of the volume of primary interaction. In this case the probability of the two particles "finding" each other in the vicinity in which the reaction is to proceed is proportional to their scattering cross section. That is, the two particles start to scatter and, if the interaction between them is attractive, they will tend to stick together

[1] Cartwright, Richman, Whitehead, and Wilcox, Phys. Rev. **78**, 823 (1950).
[2] Peterson, Iloff, and Sherman, Phys. Rev. **81**, 674 (1951).
[3] K. A. Brueckner, Phys. Rev. **82**, 598 (1951).
[4] K. Watson and K. Brueckner, Phys. Rev. **83**, 1 (1951).
[5] Panofsky, Aamodt, and Hadley, Phys. Rev. **81**, 565 (1950).
[6] K. M. Watson and R. N. Stuart, Phys. Rev. **82**, 738 (1951).
[7] Aamodt, Panofsky, and Phillips, Phys. Rev. **83**, 1057 (1951).
[8] R. Phillips, University of California Radiation Laboratory Report UCRL 1845 (to be published).
[9] Danysz, Lock, and Yekuteli, Nature **169**, 364 (1952).
[10] K. Brueckner and K. Watson, Phys. Rev. **87**, 621 (1952).

momentarily. This helps to bring them together into the region of space from which the primary inverse reaction can proceed—and thus the transition rate for the inverse reaction is expected to be proportional to the scattering cross section of the two particles. On the other hand, if this scattering cross section were less than the cross-sectional area of the primary reaction volume, the final state interactions would be of little help in "drawing the particles into this latter volume."

By detailed balancing we can then obtain the cross section for the reaction itself to be

$$d\sigma \simeq d^3q\sigma_q{}^0 \text{ times other factors,} \tag{1}$$

where q is the small relative momentum of the two particles and $\sigma_q{}^0$ is their scattering cross section for relative momentum q. The d^3q is, of course, just a factor in the volume of phase space into which the reaction proceeds. We suppose q to be small enough that $\sigma_q{}^0$ arises only from S-wave scattering. As long as $q \simeq 0$, one can neglect the q-dependence of the other factors in Eq. (1), keeping only that of

$$d^3q\sigma_q{}^0 \simeq dq \sin^2\delta_q, \tag{2}$$

where δ_q is the S-wave phase shift for the scattering at relative momentum q.

When $\sin^2\delta_q$ becomes of the order of unity for values of q such that $\hbar/q$ is still large compared to a, the range of the primary interaction, a strong correlation in the emission of the interacting particles will result which will be described by Eq. (2). This equation then gives a means of measuring δ_q from an observation of the reaction cross section.

II. BOUNDARY CONDITIONS

In the present section we shall express the transition rate for a reaction process in terms of the wave function which describes the final state interaction. We characterize the initial state of the reaction, containing the colliding particles, by the state function χ_a. χ_a describes the plane wave motion of the center of mass of each of the two incoming particles plus any internal coordinates and satisfies the Schrödinger equation

$$H_0\chi_a = E_a\chi_a, \tag{3}$$

where H_0 is the appropriate Hamiltonian and E_a is the energy of the system.

Among the possible alternate "channels" by which the reaction can proceed, we shall be interested in that one to be designated by the set of states "B." The set "B" refers to a unique set of particle occupation numbers, while a single member of the set "B" specifies also spins, momenta, etc., of each of the particles. We suppose H_0 to be so defined that the state functions χ_B also satisfy

$$H_0\chi_B = E_B\chi_B = E_a\chi_B. \tag{4}$$

The Hamiltonian H of the system will be given by adding to H_0 the interaction $\mathcal{U}$, which is responsible for the transition $a \to B$:

$$H = H_0 + \mathcal{U}. \tag{5}$$

The final state interaction v is introduced by decomposing $\mathcal{U}$:

$$\mathcal{U} = V + v, \tag{6}$$

where V is defined as $\mathcal{U} - v$. v is that specific interaction between pairs of particles in the final state which leads to the correct scattering cross section for any pair of these particles (i.e., we neglect a possible modification of v due to many body forces—this assumption is discussed further in the final section). For simplicity in the following analysis, we can define v to vanish except when operating on that particular set of particle occupation numbers which we have designated by the set "B" above—i.e., we suppose the actual interaction to be multiplied by a projection operator which picks out the set "B." This is permissible since we define V as $\mathcal{U} - v$.

Following the notation of Lippmann and Schwinger,[11] we can write the integral equation for the state function of the system as

$$\psi_a{}^{(\pm)} = \chi_a + \frac{1}{E_a \pm i\epsilon - H_0}(v+V)\psi_a{}^{(\pm)}. \tag{7}$$

Here ϵ is a positive real parameter which goes to zero after the implied integrations are done. We have similarly the wave function describing the scattering of the final state particles among themselves:

$$\phi_B{}^{(\pm)} = \chi_B + \frac{1}{E_a \pm i\epsilon - H_0}v\phi_B{}^{(\pm)}, \tag{8}$$

since $E_a = E_B$.

The probability amplitude for the transition $a \to b$ is[11]

$$T_{Ba} = (\chi_B, (v+V)\psi_a{}^{(+)}). \tag{9}$$

Following the notation of Chew and Goldberger,[12] we express $\psi_a{}^{(+)}$ in terms of the matrix operator $\Omega^{(+)}$:

$$\psi_a{}^{(+)} = \Omega^{(+)}\chi_a, \tag{10}$$

where

$$\Omega^{(+)} = 1 + \frac{1}{E_a + i\epsilon - H_0 - V - v}(V+v). \tag{11}$$

From Eqs. (9) and (10), we write the transition operator T as[12]

$$T = (v+V)\Omega^{(+)}, \tag{12}$$

with

$$T_{Ba} = (\chi_B, T\chi_a). \tag{13}$$

[11] B. Lippmann and J. Schwinger, Phys. Rev. **79**, 469 (1950).

[12] Relations (10) and (12) have been given by C. Moller, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **22**, No. 19 (1946). G. F. Chew and M. L. Goldberger [Phys. Rev. **87**, 778 (1952)] have introduced the expression (11). I am indebted to Dr. Chew and Dr. Goldberger for being informed of this and related algebraic relations in advance of publication.

Remembering that from its definition v vanishes when operating on the state χ_a and using Eqs. (11) and (12) and the operator relation,[13]

$$\frac{1}{A}-\frac{1}{B}=\frac{1}{A}(B-A)\frac{1}{B}, \tag{14}$$

we easily obtain

$$T=\omega^{(-)\dagger}V\Omega^{(+)}. \tag{15}$$

Here

$$\omega^{(-)\dagger}=1+v\frac{1}{E_a+i\epsilon-H_0-v} \tag{16}$$

and $\phi_B{}^{(-)}=\omega^{(-)}\chi_B$ [see Eq. (8)]. Inserting Eq. (15) into Eq. (13), we obtain

$$T_{Ba}=(\phi_B{}^{(-)}, V\psi_a{}^{(+)}). \tag{17}$$

The boundary condition $\phi_B{}^{(-)}$ in Eq. (17) has been derived previously for several special cases.[14–16] The physical basis rests on the requirement that the final state contain outgoing waves only and is formally expressed by the use of "$+i\epsilon$" in Eq. (16).

Equations (15) and (16) do not actually lead as directly to Eq. (17) as we have indicated. The point is that the parameter ϵ in $\omega^{(-)\dagger}$ [Eq. (16)] is the same as the "ϵ" in $\Omega^{(+)}$. This would mean taking the limit $\epsilon\to 0$ for $\phi_B{}^{(-)}$ and $\psi_a{}^{(+)}$ simultaneously for use in Eq. (17). Doing this is clearly not so convenient as being able to take independent limits for $\phi_B{}^{(-)}$ and $\psi_a{}^{(+)}$; i.e., using different ϵ's in $\omega^{(-)\dagger}$ and in $\Omega^{(+)}$. In Appendix (A) it is shown that this can indeed be done, it being only necessary to keep both ϵ's positive as the limit $\epsilon\to 0$ is being approached.

It is interesting that the boundary condition (17) follows also from the requirements of time reversal invariance. It is shown in Appendix (B) that Eq. (16) follows from the general detailed reversibility requirement

$$U\tilde{T}U^{-1}=T, \tag{18}$$

where $\tilde{T}$ is the transpose of the operator T and U is the unitary matrix introduced by Wigner[17] in connection with time reversal. It is not surprising that time reversal invariance should determine the boundary condition on ϕ_B in Eq. (17). We could not expect the reversed system to return to the (time reversed) initial state unless we start it in the correct (time reversed) final state.[18]

In Eq. (17) the final state interaction contributes most explicitly to $\phi_B{}^{(-)}$, but is also contained in $\psi_a{}^{(+)}$. We shall see in the next section that Eq. (17) is already in a useful form for the applications of that section. It is possible, however, to exhibit more explicitly the occurrence of v in Eq. (17). For this purpose we define $\Omega_0{}^{(+)}$ to be $\Omega^{(+)}$ [Eq. (11)] with $v=0$,

$$\Omega_0{}^{(+)}=1+\frac{1}{E_a+i\epsilon-H_0-V}V, \tag{19}$$

and

$$\psi_a{}^{0(+)}=\Omega_0{}^{(+)}\chi_a. \tag{20}$$

Using Eq. (14), we have

$$\Omega^{(+)}=\Omega_0{}^{(+)}+\frac{1}{E_a+i\epsilon-H_0-V-v}v\Omega_0{}^{(+)}. \tag{21}$$

From this and Eqs. (17), (19), and (20), we obtain

$$T_{Ba}=(\phi_B{}^{(-)}, V\psi_a{}^{0(+)})+(\psi_B{}^{(-)}-\phi_B{}^{(-)}, v\psi_a{}^{0(+)}), \tag{22}$$

where $\psi_B{}^{(-)}$ satisfies Eq. (7) with χ_a replaced by χ_B. The first term in Eq. (22) is similar to Eq. (17), except now v occurs only in $\phi_B{}^{(-)}$. Following the methods of Chew and Goldberger,[12] the second term can readily be developed in the form

$$(\phi_B{}^{(-)}, Q\psi_a{}^{0(+)}),$$

where Q is a power series in v.

The second term in Eq. (22) can be expected to be small compared to the first in two limiting cases. Remembering that v was defined in such a manner that it vanished except when operating on the set "B" of particle occupation numbers, we see that $v\psi_a{}^{0(+)}$ describes the production of a final state "B" followed by a scattering of the particles in this state. Since $\psi_B{}^{(-)}-\phi_B{}^{(-)}$ vanishes when $V\to 0$, this term describes the repeated interaction of the particles through the primary interaction V to produce again a state of the type "B." It is thus clear that when V can be treated as a small perturbation the second term in Eq. (22) is of higher order in V than is the first and can be neglected. On the other hand, if V is a strong interaction leading to many virtual states (many sets of particle occupation numbers) we note that the second term in Eq. (22) picks one very restricted set of intermediate states (i.e., of type "B") and may consequently be small to the extent that these intermediate states contribute relatively little to the reaction.

III. THE DEPENDENCE OF THE REACTION CROSS SECTION ON THE FINAL STATE INTERACTION

We return to the expression (17) for the transition amplitude T_{Ba}. We suppose that n particles are pro-

[13] G. F. Chew and G. C. Wick, Phys. Rev. **85**, 636 (1952).
[14] N. F. Mott and H. S. W. Massey, *The Theory of Atomic Collisions* (Oxford University Press, London, 1933).
[15] W. Rarita and J. Schwinger, Phys. Rev. **59**, 556 (1941).
[16] Rose, Biedenharn, and Arfken, Phys. Rev. **85**, 5 (1952).
[17] E. P. Wigner, Gottingen Nachr. **31**, 546 (1932).
[18] It seems likely that the boundary condition on ϕ_B expressed by Eq. (17) should probably also follow from "causality" requirements, as discussed by W. Schützer and J. Tiomno [Phys. Rev. **83**, 249 (1951)]. The work of these authors suggests that the matrix T has no singularities (except on the imaginary axis) in the positive-imaginary half-plane, when presented in diagonal form. $\Omega^{(+)}$ and $\omega^{(+)}$ contain the scattering matrix, and a related property is expected for these. On the other hand, $\omega^{(-)}$ is simply related to the complex conjugate of $\omega^{(+)}$, so $\omega^{(-)\dagger}$ should also have this causal property and thus the T of Eq. (15). This would not seem to be the case, however, had we used, for example, $\omega^{(+)\dagger}$ in Eq. (15).

duced in the state B with rest masses M_i, momenta k_i, and energies W_i. The differential cross section for the reaction in the center-of-mass system is then (we set $\hbar=c=1$)

$$d\sigma=\frac{(2\pi)^4}{\mu_r}\int\delta(\sum_{i=1}^{n}W_i-E_a)\delta(\sum_{i=1}^{n}\mathbf{k}i)\times\prod_{i=1}^{n}d^3k_i\sum_{\text{spins}}|T_{Ba}|^2. \quad (23)$$

Here μ_r is the relative velocity of the colliding particles. The integration is to be carried out over four of the $3n$ momentum variables in such a way as to satisfy the conditions imposed by the δ-functions. The expression $\sum_{\text{spins}}$ is an appropriate average and sum over initial and final spin substates, respectively.

Let us suppose that there are at least three particles in the final state and that the two whose interaction we are studying have momentum vectors $\mathbf{k}_1$ and $\mathbf{k}_2$. Let us introduce their relative momentum, $\mathbf{q}=\frac{1}{2}(\mathbf{k}_1-\mathbf{k}_2)$, and their total momentum $\mathbf{p}=\mathbf{k}_1+\mathbf{k}_2$. $d\sigma$ will now contain a factor $d^3q=q^2dqd\Omega_q$, where $d\Omega_q$ is an element of solid angle in the direction of $\mathbf{q}$. The values of $\mathbf{q}$, a relative momentum, will not be affected by the δ-function describing momentum conservation. When q is sufficiently small compared to E_a, we can neglect the effect of energy conservation on the values of q. We then have the q-dependence of the phase space factor in Eq. (23) as just

$$q^2dqd\Omega_q. \quad (24)$$

In any actual problem it is, of course, straightforward to calculate exactly the phase space factor, but for purposes of a general discussion it is convenient to use the simple form (24).

It now remains to find the dependence of T_{Ba} on q, subject to the restrictions mentioned in the introduction. Returning to the expression (17) for T_{Ba} we define

$$(\xi|R|a)=V\psi_a^{(+)}, \quad (25)$$

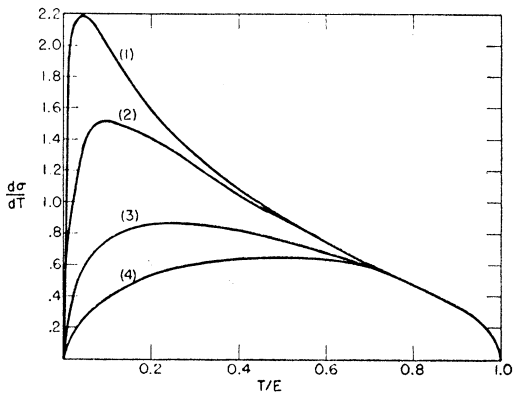

FIG. 1. The spectrum of relative energies of two interacting particles emitted in a reaction. The total energy of the two particles is considered to be constant. Curves (1) to (4) are calculated for the following respective values of $2\alpha^2/ME$ [see Eq. (35)]: 0.1, 0.25, 1.0, ∞. A much sharper peak could, of course, have been obtained by taking even smaller values of this parameter.

where ξ is an appropriate set of coordinates. R is just the transition operator in a mixed representation and has been used previously[4] in the study of meson production in nucleon-nucleon collisions. Let $\mathbf{r}$ be that member of the coordinate set ξ which describes the relative position of particles "1" and "2." In accordance with the assumption that the reaction takes place within a certain volume, we suppose R to vanish as r becomes appreciably larger than a, the radius of this volume. Neglecting the interaction of particles "1" and "2" with others of the final state particles (phase space arguments lead one to expect a small probability that more than two particles will have small relative momenta) we can factor out of $\phi_B^{(-)}$ that part, $g_q^{(-)}(\mathbf{r})$, which describes the relative motion of particles "1" and "2." Writing $\phi_B^{(-)}=h_{B'}{}^{(-)}(\xi')g_q^{(-)}(\mathbf{r})$, where B' and ξ' do not contain q or r as variables, we next define

$$(r|\bar{R}|a)\equiv(h_{B'}{}^{(-)},R), \quad (26)$$

and have

$$T_{Ba}=(g_q{}^{(-)},\bar{R}). \quad (27)$$

Now it is important to note that $\bar{R}$ does not depend explicitly upon the variable q, so that the only dependence of T_{Ba} on q comes from $g_q{}^{(-)}$. Energy conservation will, of course, impose a q-dependence on $\bar{R}$, but we have seen above that this is a small effect when q is small.

Since q is considered to be very small, we shall for the moment suppose that only the S-wave part of $g_q{}^{(-)}$ is important. That is, we write

$$g_q{}^{(-)}=(e^{-i\delta}/qr)\sin(qr+\delta), \quad (28)$$

where δ is the S-wave phase shift for the scattering of particles "1" and "2." Equation (28) is valid outside the range of the interaction of these two particles. We next express δ in terms of the low energy scattering parameters:

$$q\cot\delta=\alpha+\tfrac{1}{2}r_0q^2. \quad (29)$$

We are interested in $r\simeq a$ or less, so when q is small enough that qr is small, we have

$$\sin(qr+\delta)\simeq\sin\delta[1+\alpha r+\tfrac{1}{2}q^2r(r_0-r)]. \quad (30)$$

Neglecting the q^2 term above, we have for the pertinent values of r,

$$g_q{}^{(-)}(r)\simeq e^{-i\delta}(\sin\delta/q)f(r), \quad (31)$$

where $f(r)$ is independent of q. Equation (31) is also valid inside the region of interaction of particles "1" and "2" since then $f(r)$ is the integral of the Schrödinger equation, which is to a good approximation independent of q for small q-values. (For more detail see the paper by Bethe[19] from which the present argument is taken.) Inserting Eq. (31) into Eq. (27), we have

$$T_{Ba}=e^{i\delta}(\sin\delta/q)(f(r),\bar{R}). \quad (32)$$

[19] H. A. Bethe, Phys. Rev. **76**, 38 (1949).

The factor $(f, \bar{R})$ is now essentially independent of q for small q-values (the only dependence coming from energy conservation, as remarked previously).

Inserting this expression into Eq. (23) and using (24) for the phase space factor, the cross section can be written as

$$d\sigma = \sin^2\delta \, dq \text{ times a factor independent of } q. \quad (33)$$

If we keep only the α-term in Eq. (29), this becomes

$$d\sigma \simeq q^2 dq/(\alpha^2+q^2). \quad (34)$$

In Eq. (30) we have neglected the term of order $\frac{1}{2}q^2a(r_0-a)$ compared to unity. Thus (34) is expected to be valid for q-values satisfying this condition or for a wavelength $\hbar/q$ large compared to the radius of the region of primary interaction. If α is larger than the maximum q-values for which Eq. (33) remains valid, Eq. (34) gives just the usual phase space dependence for the cross section which is expected to hold near threshold.[20] In this case we would not find so pronounced an effect from the final state interaction—and indeed might expect to be able to neglect it entirely.

We note that Eq. (33) agrees exactly with the estimate made in the introduction [Eq. (27)]. The criterion here found, that $|\alpha a|$ be appreciably less than unity, is just the condition described in the introduction, i.e., that the low energy scattering cross section of particles "1" and "2" be large compared to the cross section of the volume of primary interaction.

To illustrate Eq. (34), we suppose that particles "1" and "2" have a fixed total energy (we introduce the phase space factor for the center-of-mass motion of these two particles). The resulting energy spectrum of relative motion is plotted in Fig. 1. In Fig. 2 we plot the relative angle of emission of the two particles in the coordinate system in which the entire scattering system is at rest. For angles less than 15° to 20° there is a sharp peak, which for larger angles merges roughly with that dependence to be expected from phase space arguments. The shape of this correlation curve is approximately

$$\frac{d\sigma}{d\Omega_{12}} \simeq \left[\frac{(2\alpha)^2}{ME}+\theta^2\right]^{-\frac{1}{2}}, \quad (35)$$

where M is the mass of the particles, E is their total energy, and θ is the angle between them.

We have assumed that we need consider only S-state relative motion of particles "1" and "2," which is probably the case of most practical importance. On the other hand, gradient couplings do seem to occur in nature. If to a gradient coupling there should also correspond a strong P-state scattering at low energies of two emitted particles, again correlation can occur. Suppose that in a partial wave analysis of $\bar{R}$ it is found that only an orbital angular momentum L is important.

[20] E. P. Wigner, Phys. Rev. **73**, 1002 (1948).

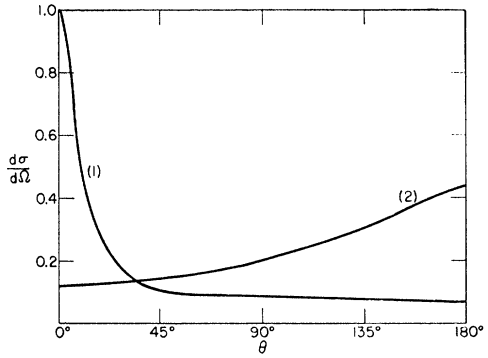

FIG. 2. The differential cross section for the production of a pair of interacting particles with an angular separation θ. Curve (1) is for an attractive interaction for which $\alpha^2/ME=0.01$ [see Eq. (35)]. Curve (2) is the correlation to be expected to a repulsive interaction of about ten times the strength of that for curve (1).

If the low energy scattering is this orbital state in large, an analysis such as that made for S-states leads to

$$T_{Ba} = e^{i\delta_L}\frac{\sin\delta_L}{q^{L+1}} \text{ times a quantity independent of } q. \quad (36)$$

Here δ_L is the phase shift for the Lth partial wave in the scattering of particles "1" and "2." Equation (33) now becomes

$$d\sigma \simeq \frac{\sin^2\delta_L}{q^{2L}}dq. \quad (37)$$

If the formula

$$q^{2L+1}\cot\delta_L = \alpha^{2L+1} \quad (38)$$

holds for a sufficient range of q-values, then

$$d\sigma \simeq q^{2L+2}dq/(\alpha^{4L+2}+q^{4L+2}), \quad (39)$$

and a considerable correlation could result. This would place quite stringent restrictions, however, on the interaction leading to δ_L.

IV. APPLICATION TO MESON PRODUCTION IN NUCLEON-NUCLEON COLLISIONS

There are at present two types of known mesonic phenomena in which the present theory is applicable[1,2,5] and possibly a third (the hypothetical meson-meson interaction[9,10]). These phenomena have been discussed previously,[4,6] so need not be considered in detail here. However, the observation of a bound final state of the neutron and proton in the reaction $p+p\rightarrow\pi^++n+p$ (or d) necessitates a slight extension of the theory of the last section. In reference 4 the meson energy spectrum was calculated by solving the Schrödinger equation explicitly for $g_q^{(-)}(r)$, the neutron-proton wave function, rather than by using the low energy scattering parameters. As argued in reference 4, the assumption that $|\alpha a|$ is small seems to be particularly good for this reaction, since the range a is expected to be (near the energetic threshold) considerably smaller than the range of the nuclear forces. Thus we can immediately

use Eq. (32) with the low energy n-p triplet[4] scattering parameters for $\sin\delta$. This gives the part of the meson spectrum corresponding to an unbound state of the n-p system.

To calculate the fraction of the time that the reaction goes as $p+p\rightarrow\pi^{+}+d$, we need the deuteron wave function $g_D(r)$ to insert into Eq. (27).[21] Actually, we need only the ratio of $g_D(r)$ to $g_q^{(-)}(r)$, for values of r less than the range of the n-p force. To a sufficient approximation, we can write this as

$$g_D(r)=N_D f(r), \tag{40}$$

where $f(r)$ is the same as in Eq. (31) since the binding and kinetic energies are small compared to the well depth [see reference (19)]. N_D is a constant normalization factor.

An approximate evaluation of N_D can be obtained from Heisenberg's theory of the S-matrix.[22] We write $g_D(r)$ outside of the range of the n-p triplet force as

$$g_D(r)=Ce^{-\gamma r}/r, \tag{41}$$

where C is a normalization factor and γ^2 is the product of the nucleon mass by the deuteron binding energy. Let $S(q)$ be the S-matrix for n-p triplet S-wave scattering. Then Heisenberg's relation is

$$\oint dqS(q)=2(2\pi)^2|C|^2, \tag{42}$$

where the integral is a closed contour about the pole of the S-matrix in the positive imaginary half-plane at the point $q=i\gamma$. Expressing

$$S(q)=(\cot\delta+i)/(\cot\delta-i),$$

and using Eq. (29) (with α replaced by $-|\alpha|$) we readily evaluate the integral to find that

$$\gamma=|\alpha|(1+\tfrac{1}{2}r_0|\alpha|)$$

and

$$C=\left[\frac{\gamma}{2\pi(1-r_0\gamma)}\right]^{\frac{1}{2}}. \tag{43}$$

Using expression (30) and comparing the expression for $g_q^{(-)}(r)$ for the internal and external region near the edge of the interaction we have

$$\frac{\sin\delta}{q}\frac{[1-|\alpha|r]}{r}=\frac{\sin\delta}{q}f(r), \tag{44}$$

or $f(r)\simeq(1-|\alpha|r)/r$ near the boundary of the potential well. Using this expression and equating Eqs. (40) and (41), we have

$$N_D=Ce^{-\gamma r}/(1-|\alpha|r)\simeq C, \tag{45}$$

since $\gamma\simeq\alpha$ and both are sufficiently small that $e^{-\gamma r}/(1-|\alpha|r)\simeq 1$ at the well boundary. Then from Eq. (27), we have

$$T_{Ba}(\text{deuteron})=C(f(r),\bar{R}), \tag{46}$$

which is to be compared with Eq. (32).

From Eqs. (32) and (46) it is easy to calculate the corresponding cross sections in the center-of-mass system. Let T be the meson kinetic energy, μ its rest mass, M the nucleon mass, and T_0 the maximum kinetic energy of the meson. Then the differential cross section in the forward direction for an unbound n-p state is to within a numerical factor, G,

$$\frac{d\sigma}{d\Omega dT}=\frac{\sqrt{2}}{\pi}\left(\frac{M}{\mu}\right)^{\frac{3}{2}}T^{\frac{1}{2}}(T_0-T)^{\frac{1}{2}}\frac{\sin^2\delta}{q^2}G, \tag{47}$$

where q is the relative momentum of the nucleons and $\mu/2M$ has been neglected compared to unity. The corresponding cross section with the formation of a deuteron is

$$\frac{d\sigma_D}{d\Omega}=\frac{\gamma(p/\mu)^3}{1-r_0\gamma}G, \tag{48}$$

to within the same numerical factor G as that for Eq. (47). Here p is the meson momentum. In these equations, it was assumed[4] that $(f(r),\bar{R})$ contained a factor of p.

The foregoing cross sections have been transformed to the laboratory coordinate system and are compared with the experimental data of Cartwright, Richman, and Whitehead[23] in Fig. 3. Here G was taken to be $3.4(10)^{-30}$ cm^2 (steradian mev)$^{-1}$. For comparison, the monochromatic spectrum from $d\sigma_D/d\Omega$ was "smeared out" over the experimental energy resolution. The agreement with the experimental data seems to be satisfactory, except possibly at the lower meson energies. The agreement with the earlier[4] calculations, which involved integrating the Schrödinger equation for an exponential well, is quite good.

V. FURTHER APPLICATIONS

The applicability of Eqs. (32) and (36) is somewhat greater than was indicated in Sec. III. Let us suppose that the state B contains two particles whose scattering is described by a one-level Breit-Wigner resonance formula at energies for which Eqs. (32) and (36) are still valid. It is also supposed that the primary interaction is sufficiently weak that only the first term in

[21] When some of the produced particles can be in a bound state, Eq. (9) is not so useful [see, for example, H. S. Snyder, Phys. Rev. **83**, 1154 (1951)]. In this case it seems preferable to start with Eq. (17) as the defining equation for T_{Ba}. Then the boundary condition on $\phi_B^{(-)}$ is determined by the time reversal arguments of Appendix (B). When none of the particles are actually in a bound state, the arguments of Appendix (A) still imply the equality of expressions (9) and (17). For the part of $\phi_B^{(-)}$ referring to the bound state, the use of "$+i\epsilon$" is irrelevant.

[22] W. Heisenberg, Z. Naturforsch. **11/12**, 607 (1946).

[23] Cartwright, Richman, and Whitehead (to be published) and W. F. Cartwright, Ph.D. thesis, University of California, Berkeley, California (unpublished).

Eq. (22) need be kept. We write [see Eq. (36)]

$$\tan\delta_L=\tfrac{1}{2}\Gamma_L/(E_L-E).$$

The modification of the phase space factor for only two particles in the state "B" is such as to replace dq by q^{-1} in Eq. (37), neglecting numerical factors. This equation then becomes

$$\sigma\simeq\frac{1}{q^{2L+1}}\frac{\Gamma_L{}^2}{(E-E_L)^2+\frac{1}{4}\Gamma_L{}^2} \tag{37'}$$

except for the factors not explicitly written in Eq. (37).

Near threshold $q^{-(2L+1)}\Gamma_L{}^2=\Gamma_L$, except for a numerical factor. Writing all the neglected factors in the form $(2L+1)\Gamma_a K^{-2}$, we have the cross section for a one-level resonance reaction involving spinless particles:

$$\sigma=\frac{\pi(2L+1)}{K^2}\frac{\Gamma_L\Gamma_a}{(E-E_L)^2+\frac{1}{4}\Gamma_L{}^2}. \tag{49}$$

Here K is the relative momentum of the two incoming particles and $\Gamma_a\ll\Gamma_L$ by the assumption that V is a sufficiently weak interaction. The extension to particles with spin is straightforward.

The usefulness of Eq. (49) lies not in its generality (rather restrictive conditions were imposed on the model, such as $\Gamma_a\ll\Gamma_L$) but in its relation of the reaction cross section to the interaction of the two particles in the final state. Indeed, expressions (32) and (36) do not depend upon any specific dependence of δ_L on the energy, but upon being able to write the final state wave function in the general form (31) for small distances (when several partial waves are involved, Eq. (31) is to be replaced by a sum of such terms, one for each partial wave). This suggests a means of establishing phenomenological relationships between various processes—such as between meson-nucleon scattering and photomeson production.

VI. REPULSIVE INTERACTIONS

When the final state interaction between two particles is repulsive, one again expects a correlation in the relative angles and energies of these particles when produced in a given reaction. Since the particles repel each other, one expects a smaller probability that relative angles and energies will be small than if there were no interaction. There does not seem to be a general, simple theory to describe this, however, as there was for the case of the attractive interactions. Also, the correlations seem to be less pronounced for repulsive interactions, leading in general to smooth angular variations which would be difficult to separate from the relatively weak correlation which might be expected to occur in the primary interaction itself.

For these estimates a primary interaction in the form of a very short-range square well was taken. [$\bar{R}$ in Eq. (26).] The repulsive final state interaction was also taken to be a square well and the outgoing particles were considered to be mesons. Taking the repulsive potential to be the negative of a well 100 Mev deep and having a range of $3\hbar/\mu c$, the angular correlation given in Fig. 2 was obtained. The repulsive potential in this case was approximately ten times as strong as was the attractive potential whose correlation is also plotted in Fig. 2.

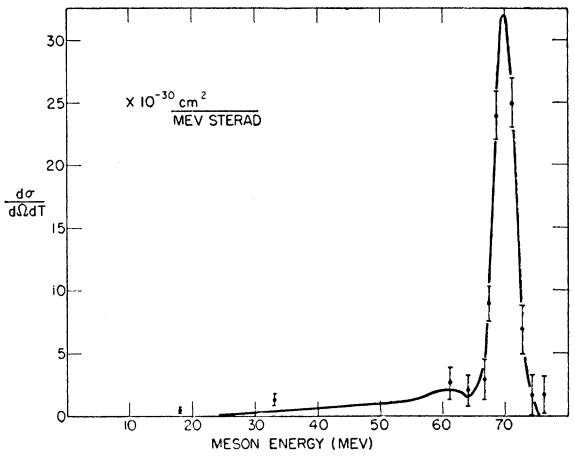

FIG. 3. The energy spectrum of Π^+ mesons produced in the forward direction in p–p collisions as calculated from Eqs. (47) and (48) is compared to the experimental data of Cartwright, Richman, and Whitehead (reference 22). Only the low energy triplet n–p scattering parameters enter into the shape of the curve, which is uniquely determined by the theory.

VII. CONCLUSIONS

To study the effect of final state interactions on a reaction cross section, we have been led to the expression (17) for the transition amplitude. In Eq. (22) the appearance of the final state interaction has been somewhat more explicitly exhibited. In a great many applications the first term only of Eq. (22) need be kept, the condition usually being that the "primary interaction" V can be considered to be a small perturbation (e.g., as in β-decay).

It has been possible to carry the analysis further for the consideration of strong final state interactions and low relative energies, the dependence of the transition amplitude in this case upon the relative momentum of two interacting particles being of the form given in Eq. (32).

In the development of Eq. (32), we have not considered the possibility that the final state interaction of two created particles might be modified in the presence of the primary interaction. This is not expected to change our conclusions in general, however. First, if the primary interaction V is weak or impulsive so that the first term only of Eq. (22) can be used, we need not expect a modification of v, since V is then essentially "turned off" once the particles are created. This condition should cover the photomesonic processes, for instance. Second, for processes such as meson production in nucleon-nucleon collisions, for which the range of V

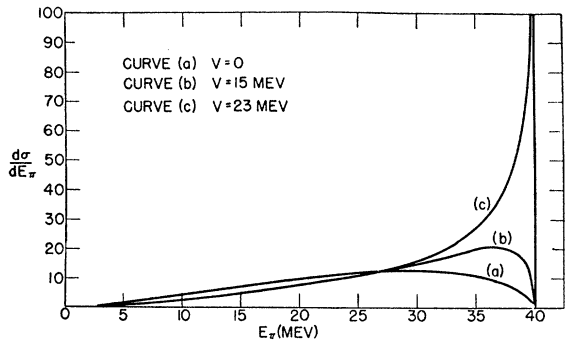

FIG. 4. The energy spectrum of Π-mesons produced in nucleon-nucleon collisions in the center-of-mass coordinate system. The final state nucleon-nucleon interaction was taken to be a square well of range $1.4\,\hbar/\mu c$ with the corresponding well depths indicated in the drawing. Curve (c) corresponds approximately to the n–p singlet force.

is considerably less than the range of v, there should be no serious modification of v since the effect of the final state interaction comes from regions of space for which $V=0$. Finally, the usefulness of Eq. (32) implies quite a predominantly strong final state interaction, which is not expected to be greatly modified if it is stronger than the other interactions involved. Here the argument is perhaps most clearly expressed by the discussion given in the introduction, which refers to final state interactions whose "effective area of interaction" is greater than the "effective area of the interaction V."

Much less correlation is to be expected when the reaction becomes weak. To see this we give in Fig. 4 the expected energy spectrum of mesons in the center-of-mass system when the final state nucleon-nucleon interaction has a range $1.4\hbar/\mu c$ and (a) a depth of zero Mev, (b) a depth of 15 Mev, (c) a depth of 23 Mev. Case (c) corresponds roughly to the n-p single potential. We have taken the range of the primary interaction $\bar{R}$ to be zero for this example, which is most favorable for obtaining large correlations.

APPENDIX A. THE FINAL STATE BOUNDARY CONDITION

We consider Eq. (15) for T, replacing ϵ by ϵ_0 in Eq. (16) for $\omega^{(-)\dagger}$. We wish to show that this equation is equivalent to Eq. (12) for calculating T_{Ba} as long as ϵ, $\epsilon_0>0$ and approach zero independently after the integrals are evaluated. From Eq. (11) with the use of Eq. (14) we readily find

$$\omega^{(-)\dagger}V\Omega^{(+)}$$
$$=V+(V+v)\frac{1}{E_a+i\epsilon-H_0-V-v}(V+v)+\Delta, \quad \text{(A-1)}$$

where we have used the assumed property of v—that it (or any function of it) has no matrix elements connecting states χ_a and χ_B. The quantity Δ is

$$\Delta=i(\epsilon-\epsilon_0)v\frac{1}{(E_a+i\epsilon_0-H_0-v)}\times\frac{1}{(E_a+i\epsilon-H_0-v-V)}(V+v). \quad \text{(A-2)}$$

Except for Δ, Eq. (A-1) agrees with Eq. (12), the defining equation for T. It is now to be shown in the limit ϵ, $\epsilon_0\to 0$ that Δ gives no contribution to T_{Ba} when, and in general only when, ϵ, $\epsilon_0\to 0$ from positive values. We define

$$t=v\frac{1}{E_a+i\epsilon_0-H_0-v}v+v, \quad \text{(A-3)}$$

a scattering amplitude for particles in states "B." Then we easily obtain with the use of Eq. (14)

$$\Delta=i(\epsilon-\epsilon_0)t\frac{1}{(E_a+i\epsilon_0-H_0)}\frac{1}{(E_a+i\epsilon-H_0)}T, \quad \text{(A-4)}$$

where T is now given by its defining Eq. (12). If we choose a representation in which H_0 is diagonal with eigenvalue E_k, we see that Eq. (A-4) will contain an integral of the form

$$\int\cdots\frac{dE_k}{(E_k-E_a-i\epsilon_0)(E_k-E_a-i\epsilon)}\cdots. \quad \text{(A-5)}$$

In the limit when ϵ, $\epsilon_0\to 0$ the only contribution of the integral (A-5) to Δ will come from the vicinity of $E_k=E_a$. Thus we can factor t and T out of the integral, setting $k=a$ in them. This can be done as the scattering amplitudes are supposed to be well defined on the energy shell (that is, when k is set equal to a). We may also take infinite limits ($-\infty$ to $+\infty$) for the integral (A-5). If ϵ_0 is positive (ϵ is known to be positive), the integral vanishes. If ϵ_0 is negative, the integral is $2\pi/(\epsilon-\epsilon_0)$. Thus

$$\Delta=0, \text{ if } \epsilon,\ \epsilon_0\to 0(+);$$
$$\Delta=2\pi i[t\delta(H_0-E_a)T] \text{ if } \epsilon\to 0(+),\ \epsilon_0\to 0(-1). \quad \text{(A-6)}$$

Had we used a "principal value" definition for $\omega^{(-)\dagger}$, Δ would again have been nonvanishing.

We recall that the role of ϵ and ϵ_0 was just to help keep track of the path of integration and that these could have been set equal to zero at the outset (by properly specifying the contour of integration). In particular, we can go to the limit ϵ, $\epsilon_0=0$ in $\phi_B^{(-)}$ and $\psi_a^{(+)}$ before evaluating Eq. (17).

APPENDIX B. THE REVERSAL REQUIREMENTS

We wish now to show that the use of $\phi_B^{(-)}$ in Eq. (17) is required by the time reversal invariance of the system. The time reversal operator has been given by

Wigner[17] and is

$$K=UK_0, \tag{B-1}$$

where K_0 is the operation of taking the complex conjugate and U is a unitary matrix. The "time reversed" of a given operator P is

$$P_t\equiv KPK^{-1}=UP^*U^{-1}. \tag{B-2}$$

For instance, time reversal leaves the Hamiltonian invariant but changes the sign of momenta, angular momenta, spins, etc.

In Eqs. (3) and (4) we have introduced the eigenfunctions χ_a of H_0. Let "A" represent a complete set of commuting operators for the χ_a, and let the set "a" represent the corresponding eigenvalues:

$$A\chi_a=a\chi_a. \tag{B-3}$$

Physically, the set "a" represents the momenta, angular momenta, spins, and any internal coordinates of the noninteracting particles of the state "a" (we are not distinguishing between states "a" and "B" here). Now under time reversal the momenta, angular momenta, and spin components change sign, but the internal coordinates are unchanged. Let us denote the eigenvalues for a system in this time reversed state by "$-a$" and the eigenfunctions by χ_{-a}. On the other hand, we denote the wave function which results from applying K to χ_a by χ_a'. Since χ_a' and χ_{-a} describe the same physical situation, they can differ only by a phase factor, say μ_{-a}. That is,

$$\chi_a'\equiv K\chi_a=U\chi_a{}^*=\mu_{-a}\chi_{-a}. \tag{B-4}$$

There is also the set

$$\chi_a''\equiv K\chi_{-a}=U\chi_{-a}{}^*=\mu_a\chi_a. \tag{B-5}$$

We observe in passing that the relation between the $\psi^{(+)}$ and $\psi^{(-)}$ and the $\phi^{(+)}$ and $\phi^{(-)}$ of Eqs. (7) and (8) is

$$\psi_a{}^{(-)}=(1/\mu_a)K\psi_{-a}{}^{(+)},\quad \phi_B{}^{(-)}=(1/\mu_B)K\phi_{-B}{}^{(+)}. \tag{B-6}$$

This relation gives a simple prescription for obtaining the $\phi_B{}^{(-)}$ to be used in Eq. (17) when the usual scattering wave function is known.

The detailed reversibility requirement, which is a direct consequence of time reversal, has been given by Schwinger[24] and Coester.[25] It is

$$T_{-B,-a}=\mu_a{}^*\mu_BT_{aB}, \tag{B-7}$$

where the phases μ_a and μ_B are those of Eqs. (B-4) and (B-5). We have

$$\begin{aligned} T_{-B,-a}&=(\chi_{-B},\,T\chi_{-a})=(T^*\chi_{-a}{}^*,\,\chi_{-B}{}^*)\\ &=(UT^*U^{-1}U\chi_{-a}{}^*,\,U\chi_{-B}{}^*)\\ &=\mu_a{}^*\mu_B(\chi_a,\,[UT^*U^{-1}]^\dagger\chi_B). \end{aligned} \tag{B-8}$$

We have used Eq. (B-5) in the last step. Comparing with Eq. (B-7) we see that

$$UT^*U^{-1}=T^\dagger \tag{B-9}$$

must hold on the energy shell.

Now writing T in the form of Eq. (A-1), we wish to show from Eq. (B-9) that the boundary condition on $\omega^{(-)\dagger}$ is the correct one. That is, we wish to show that

$$(B|\,T-(UT^*U^{-1})^\dagger\,|a)=0, \tag{B-10}$$

only for the "$+i\epsilon$" boundary condition on $\omega^{(-)\dagger}$. Since the first term of Eq. (A-1) obviously satisfies Eq. (B-10), this equation reduces to

$$(B|\,\Delta-(U\Delta^*U^{-1})^\dagger\,|a)=0. \tag{B-11}$$

Now Δ and $(U\Delta^*U^{-1})^\dagger$ have essentially the same structure, so the arguments given in Appendix A for the vanishing of Δ are here valid. (It may be noted that $(U\Delta^*U^{-1})^\dagger$ has the operator v standing on its right, so automatically vanishes if we wish to apply our condition that v vanishes when operating on χ_a.)

Thus, just as in Appendix A, we see that the boundary condition of Eq. (17) is here required.

[24] J. Schwinger, hectographed notes on nuclear physics (unpublished). Relations of the form (B-4) and (B-5) were introduced and used here in the proof of Eq. (B-7).

[25] F. Coester, Phys. Rev. 84, 1259 (1951).