The Electron-Neutron Interaction as Deduced from Pseudoscalar Meson Theory

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Pseudoscalar meson theory with pseudoscalar coupling is used to find a value for the electron-neutron interaction. By means of the Feynman-Dyson-Wick techniques, the one-neutron matrix elements of the S matrix are calculated in the usual weak coupling perturbation theory, carried to second order in the meson-nucleon coupling. Expressions for the magnetic moment and the electron-neutron interaction are deduced from this. By using the former to fix the coupling constant, it is found that the potential well representing the latter has a depth of 5380 ev if, as is conventional, this is taken to be square, with a radius of e^2/mc^2. The various results found by previous authors and their relation to the present work are discussed.

I. INTRODUCTION

SEVERAL years after Dee's[1] attempt to detect an interaction between neutrons and electrons, Condon[2] pointed out that the rather large upper limit given by Dee could be reduced by a factor of about one thousand from a consideration of the effect which such an interaction would have on the scattering of slow neutrons. Experiments were carried out in 1947, almost simultaneously, by Havens, Rabi, and Rainwater[3] at Columbia University and by Fermi and Marshall[4] at the University of Chicago, in which the interference between the electronic and nuclear scattering of slow neutrons was studied. Both groups gave values of less than 10 kev for the depth of the interaction potential well, assuming, conventionally, that this is square with a radius equal to e^2/mc^2. Since that time, more precise measurements have been made, typical results being (5300±1000) ev,[5] (4100±1000) ev,[6] and (4200±600) ev.[7]

The calculation of the electron-neutron interaction from meson theory has been carried out by several authors.[8–11] Of the several varieties of meson theory, the pseudoscalar is currently in vogue since the pseudoscalar nature of the pi-meson has been generally accepted. However, even with the same theory—pseudoscalar mesons, pseudoscalar coupling, with the usual weak coupling perturbation theory carried to the second order in the meson nucleon coupling—different methods of calculation led to well depths ranging from 1300 ev[9,11] to 5220 ev.[10] Since it has not been certain whether these discrepancies were due to errors, to different methods of evaluating divergent integrals, or to the use of different formal procedures in carrying out the perturbation theory, a clarification of the theoretical situation seems very desirable.

Moreover, still another result is obtained from Foldy's treatment[12] which makes no use of meson theory. A Pauli term is simply added to the Dirac equation to represent the empirical neutron magnetic moment. As a consequence of the relativistic covariance of the Dirac equation, it then follows that the neutron will also interact with an external electric field. If this interaction is expressed in terms of an electron-neutron well depth, a result of 4080 ev is obtained, which agrees with none of the values which have been calculated from meson theory.

Of course, meson theory weak coupling calculations of such effects cannot be taken too seriously—for instance, they do not even give the correct neutron to proton magnetic moment ratio—but it certainly should be possible to state definitely just what value pseudoscalar meson theory does predict for this effect. In this paper, the electromagnetic properties of neutrons are investigated with the aid of the Feynman-Dyson-Wick techniques. Special attention is given to the interpretation of the terms obtained from field theory so that the relationship of the various results obtained by previous authors can be made quite clear. Naturally, the result of this calculation should not be expected to agree very closely with the experiments, even if field theory is correct, since terms of higher order in the meson nucleon coupling g have not been taken into account and g is by no means small. In fact, the mesonic analog of the fine structure constant must have a value of about 7 in order to fit the neutron magnetic moment [see Eq. (56) below]. The chief purpose of this investigation is to clear away the confusion arising from the number of theoretical values which have been reported. The results obtained are in agreement with those found by Slotnick and Heitler[8] and by Dancoff and Drell.[10]

A similar calculation has been made for the case of pseudovector coupling but no new results were obtained. It is found that the magnetic moment is un-

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changed (provided that, as usual, the coupling constant is multiplied by twice the nucleon to meson mass ratio) as might be expected from the theorems on the "equivalence" of the two couplings.[13] However, the electron-neutron interaction is found to be logarithmically divergent in agreement with Feynman's conclusions.[14]

The field theory calculation of the S matrix is outlined in Sec. II, most of the algebraic details being left for Sec. III. These results are discussed and interpreted in Sec. IV and the status of previous calculations is explained.

II. CALCULATION OF THE S MATRIX

The system to be studied consists of quantized nucleon and pseudoscalar meson fields in presence of an external, c-number electromagnetic field. The charge symmetric formalism will be used according to which the mesons are described by three Hermitian fields, ϕ_1, ϕ_2, ϕ_3, or, alternatively, by the field ϕ_3 and the complex fields

$$\phi=(\phi_1+i\phi_2)/\sqrt{2},\quad \phi^*=(\phi_1-i\phi_2)/\sqrt{2}. \qquad (1)$$

The potentials of the external electromagnetic field will be denoted by A_ν, ($\nu=1, 2, 3, 4$). It will be convenient to use the isotopic spin formalism and combine the proton and neutron spinor fields, ψ_P, ψ_N, into a single, eight component spinor,

$$\psi=\begin{pmatrix}\psi_N\\ \psi_P\end{pmatrix}.$$

The Lagrangian density is

$$\mathcal{L}=\mathcal{L}_0+\mathcal{L}_g+\mathcal{L}_e$$

with

$$\mathcal{L}_0=-\frac{1}{2}\left(\frac{\partial\phi_\alpha}{\partial x_\nu}\frac{\partial\phi_\alpha}{\partial x_\nu}+\mu^2\phi_\alpha\phi_\alpha\right)+i\psi^\dagger\left(\gamma^\nu\frac{\partial}{\partial x_\nu}+M\right)\psi,$$

$$\mathcal{L}_g=-g\psi^\dagger\gamma^5\tau^\alpha\phi_\alpha\psi,$$

$$\mathcal{L}_e=-eA_\nu\left\{\frac{\partial\phi_1}{\partial x_\nu}\phi_2-\frac{\partial\phi_2}{\partial x_\nu}\phi_1-\psi_P{}^\dagger\gamma^\nu\psi_P\right\}-e^2A_\nu A_\nu\frac{\phi_1{}^2+\phi_2{}^2}{2},$$

where τ^1, τ^2, τ^3 are the Pauli isotopic spin matrices which operate on the charge index of ψ; $\psi^\dagger$ and γ^ν are as defined in reference 15; and natural units, $\hbar=c=1$, are used. The usual canonical formalism[15] leads to a Hamiltonian density

$$\mathcal{H}=\mathcal{H}_0+\mathcal{H}_\mathrm{I},$$

where $\mathcal{H}_0$ is the part which remains when e and g are set equal to zero. Going over to the interaction representation, the development in time of the Schrödinger state functional, $F(t)$, is given by

$$F(t)=e^{-iH_0t}G(t),$$

where

$$H_0=\int d^3x\mathcal{H}_0(x)$$

and $G(t)$ satisfies

$$i\partial G(t)/\partial t=H_1(t)G(t)=\left[\int d^3x\mathcal{H}_1(x)\right]G(t).$$

Here

$$\begin{aligned}\mathcal{H}_1(x)&\equiv e^{iH_0t}\mathcal{H}_\mathrm{I}e^{-iH_0t}=g\psi^\dagger\gamma^5\tau^\alpha\phi_\alpha\psi\\&\quad+eA_\nu\left\{\frac{\partial\phi_1}{\partial x_\nu}\phi_2-\frac{\partial\phi_2}{\partial x_\nu}\phi_1-\psi_P{}^\dagger\gamma^\nu\psi_P\right\}\\&\quad+e^2(A_\nu A_\nu-A_4{}^2)\frac{\phi_1{}^2+\phi_2{}^2}{2}\\&=g\sqrt{2}(\psi_P{}^\dagger\gamma^5\psi_N\phi+\psi_N{}^\dagger\gamma^5\psi_P\phi^*)\\&\quad+g\psi^\dagger\gamma^5\tau^3\psi\phi_3-eA_\nu\psi_P{}^\dagger\gamma^\nu\psi_P\\&\quad+ieA_\nu\left(\frac{\partial\phi}{\partial x_\nu}\phi^*-\frac{\partial\phi^*}{\partial x_\nu}\phi\right)+e^2(A_\nu A_\nu-A_4{}^2)\phi^*\phi. \qquad (2)\end{aligned}$$

The commutation relations for the field operators in this interaction representation are:

$$[\phi(x),\phi^*(y)]=[\phi_3(x),\phi_3(y)]=-i\Delta(x-y),$$
$$\{\psi(x),\psi^\dagger(y)\}=\Gamma(x-y),$$

with

$$\Delta(x)=\Delta(\mathbf{x},t)\equiv\frac{1}{(2\pi)^3}\int d^3k e^{i\mathbf{k}\cdot\mathbf{x}}\frac{\sin[(\mathbf{k}^2+\mu^2)^{\frac{1}{2}}t]}{(\mathbf{k}^2+\mu^2)^{\frac{1}{2}}},$$

and

$$\Gamma(x)\equiv\left(M-\gamma^\nu\frac{\partial}{\partial x_\nu}\right)\Delta(x).$$

The actual calculation consists of finding the matrix elements, between two one-neutron states, of the operator S which carries the initial state function into the final one:

$$G(\infty)=SG(-\infty),\quad S=1+\sum_{n=1}^{\infty}S_n,$$

$$S_n=\frac{(-i)^n}{n!}\int d^4x_1\cdots\int d^4x_nT[\mathcal{H}_1(x_1)\cdots\mathcal{H}_1(x_n)],$$

where T denotes the temporal ordering of operators.[16] Applying Wick's formalism to the T-product leads to the Feynman rules, which will be used in what follows.

The one-neutron, time-independent state functionals will be called F_1, F_2, while F_0 will be used to denote the vacuum state (no neutrons or mesons present). It is assumed that F_1 describes a state with one free neutron

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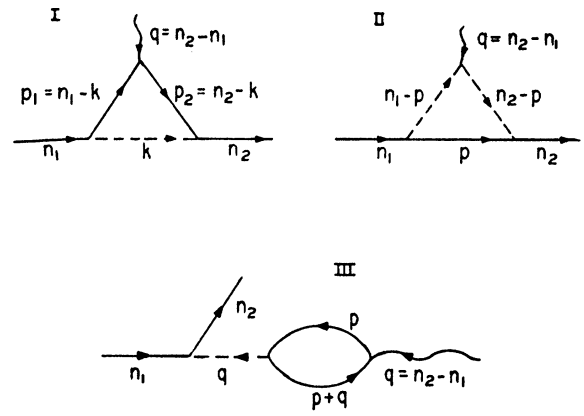

FIG. 1. Feynman diagrams.

of momentum n_1, spinor u_1, i.e., that

$$(F_0, \psi(x)F_1)=u_1 e^{in_1\cdot x},$$

where $u_1 e^{in_1\cdot x}$ is a solution of the unperturbed Dirac equation,

$$\left(\gamma^\nu\frac{\partial}{\partial x_\nu}+M\right)u_1 e^{in_1\cdot x}=0.$$

This gives

$$(M+i\boldsymbol{n}_1)u_1=0, \tag{3}$$

with

$$\boldsymbol{n}_1\equiv(n_1)_\nu\gamma^\nu.$$

(If f_ν is any four-vector, the symbol $\boldsymbol{f}$ will denote the matrix $\sum_{\nu=1}^{4} f_\nu\gamma^\nu$. Also, $f\cdot g$ will mean the scalar product $\sum_{\nu=1}^{4} f_\nu g_\nu$, and $\mathbf{f}$ will be used for the three-vector with components f_1, f_2, f_3.) Taking the Hermitian conjugate of (3) and multiplying on the right by γ^4 yields

$$u_1^\dagger(M+i\boldsymbol{n}_1)=0.$$

Similarly, the state F_2 has one free neutron with momentum n_2, spinor u_2, and

$$(M+i\boldsymbol{n}_2)u_2=0, \quad u_2^\dagger(M+i\boldsymbol{n}_2)=0.$$

If F_1 and F_2 are different states,

$$(F_2, F_1)=(F_2, S_1F_1)=(F_2, S_2F_1)=0,$$

so the lowest order contribution will come from S_3. It will be proportional to eg^2 if terms of higher order in e or eA are neglected. (In this approximation, the e^2 term of $\mathcal{H}_1$, Eq. [2], may be dropped.) Thus, to order eg^2,

$$(F_2, SF_1)=(F_2, S_3F_1)=M_{\rm I}+M_{\rm II}+M_{\rm III},$$

where $M_{\rm I}$, $M_{\rm II}$, $M_{\rm III}$ are the matrix elements associated with the Feynman diagrams of Fig. 1. In these diagrams solid lines denote nucleons, dotted lines stand for mesons, and the wavy line indicates the external electromagnetic field. Neutron momenta are labeled n, proton momenta, p. The mesons in I and II are charged, while that of III is neutral.

Specifically,

$$M_{\rm I}=\frac{-eg^2}{8\pi^4}\int d^4k u_2^\dagger\gamma^5\frac{1}{M+i(\boldsymbol{n}_2-\boldsymbol{k})}\times\boldsymbol{a}(q)\frac{1}{M+i(\boldsymbol{n}_1-\boldsymbol{k})}\gamma^5 u_1\frac{1}{\mu^2+k^2},$$

$$M_{\rm II}=\frac{-ieg^2}{8\pi^4}\int d^4p u_2^\dagger\gamma^5\frac{1}{M+i\boldsymbol{p}}\times\gamma^5 u_1\frac{(n_1+n_2-2p)\cdot a(q)}{[\mu^2+(n_1-p)^2][\mu^2+(n_2-p)^2]},$$

$$M_{\rm III}=\frac{-eg^2}{16\pi^4}\int d^4p u_2^\dagger\gamma^5 u_1\times \mathrm{Sp}\left\{\gamma^5\frac{1}{M+i\boldsymbol{p}}\boldsymbol{a}(q)\frac{1}{M+i(\boldsymbol{p}+\boldsymbol{q})}\right\}\frac{1}{\mu^2+q^2},$$

where $a_\nu(k)$ is the Fourier transform of the electromagnetic potential

$$A_\nu(x)=\frac{1}{(2\pi)^4}\int d^4k e^{ik\cdot x}a_\nu(k), \tag{4}$$

$$a_\nu(k)=\int d^4x e^{-ik\cdot x}A_\nu(x), \tag{5}$$

and

$$q\equiv n_2-n_1.$$

Hereafter, $a_\nu(q)$ will be written simply as a_ν, so that $\boldsymbol{a}$ stands for the matrix $a_\nu(q)\gamma^\nu$. As usual, the nucleon and meson masses, M and μ, are to be understood to have negative imaginary parts which are set equal to zero after all integrations have been performed.

We consider first the integral $M_{\rm III}$. It is formally quadratically divergent, but the spur of the Dirac matrix in the curly braces vanishes since γ^5, $\gamma^5\gamma^\mu$, $\gamma^5\gamma^\mu\gamma^\nu$, $\gamma^5\gamma^\mu\gamma^\nu\gamma^\lambda$ all have zero spur. Thus, any cut-off or regularization procedure which makes the integral finite will give $M_{\rm III}$ the value zero, so in this sense we may simply set

$$M_{\rm III}=0.$$

The other terms, $M_{\rm I}$ and $M_{\rm II}$, are also divergent, although only logarithmically. The subsequent algebraic manipulations in which the "convergent" parts are separated from the "divergent" parts are therefore meaningful only if we imagine these integrals regularized or cut off in some manner. The terms which appear "convergent" are those which would remain finite if, say, the cut-off momentum were allowed to become infinite, while the "divergent" terms are those which would not approach any finite limit. The interpretation of the latter is discussed in more detail below.

The expressions for $M_{\rm I}$ and $M_{\rm II}$ can be simplified by rationalizing the denominators and bringing the γ^5 factors together. Using also the Dirac equations for u_1 and u_2,

$$(M+in_1)u_1=0, \quad u_2^\dagger(M+in_2)=0 \tag{6}$$

we find

$$M_{\rm I}=\frac{-eg^2}{8\pi^4}\int d^4k \times\frac{u_2^\dagger \boldsymbol{kak}u_1}{[M^2+(n_2-k)^2][M^2+(n_1-k)^2][\mu^2+k^2]}. \tag{7}$$

In terms of the average neutron momentum,

$$n\equiv(n_1+n_2)/2,$$

$M_{\rm II}$ may be written

$$M_{\rm II}=\frac{-eg^2}{8\pi^4}2i\int d^4p \times\frac{u_2^\dagger(M+i\mathbf{p})u_1(n-p)\cdot a}{[M^2+p^2][\mu^2+(n_1-p)^2][\mu^2+(n_2-p)^2]}. \tag{8}$$

Thus, evaluation of $M_{\rm I}$ and $M_{\rm II}$ depends upon a knowledge of integrals of the form

$$\int d^4k\frac{(1,\, k_\mu,\, k_\mu k_\nu)}{(k^2-2n_1\cdot k+\Delta_1)(k^2-2n_2\cdot k+\Delta_2)(k^2+\Delta_3)}. \tag{9}$$

These will now be computed and the result, Eq. (13), will be substituted into Eqs. (7) and (8) for $M_{\rm I}$ and $M_{\rm II}$. It will be shown that both of these are of the form

$$u_2^\dagger[H(q^2)\boldsymbol{a}+K(q^2)(\boldsymbol{aq}-\boldsymbol{qa})+L(q^2)q^2\boldsymbol{a}]u_1,$$

where H, K, and L are definite integrals containing q^2 as a parameter. If the external fields are slowly varying so that only small values of q^2 are relevant, it will be in order to make an expansion in powers of q. Retaining only terms of order q^2, we have

$$M_{\rm I}+M_{\rm II}=u_2^\dagger(E\boldsymbol{a}-\tfrac{1}{2}F[\boldsymbol{a},\boldsymbol{q}]-Gq^2\boldsymbol{a})u_1, \tag{10}$$

where E, F, and G are calculable constants, or, expressed in terms of the actual electromagnetic potentials and fields,

$$M_{\rm I}+M_{\rm II}=\int d^4x u_2^\dagger e^{-in_2\cdot x}(EA+\tfrac{1}{2}F\mathfrak{F}_{\mu\nu}\sigma_{\mu\nu} +G\Box^2A)u_1e^{in_1\cdot x}, \tag{11}$$

with

$$\mathfrak{F}_{\mu\nu}=\partial A_\nu/\partial x_\mu-\partial A_\mu/\partial x_\nu, \quad \sigma_{\mu\nu}=[\gamma^\mu,\gamma^\nu]/2i. \tag{12}$$

The statements made in this paragraph will be substantiated in Sec. III and the values of E, F, and G will be calculated explicitly. The interpretation of the result (11) will then be discussed in Sec. IV.

III. ALGEBRAIC DETAILS

The integrals (9) can be evaluated by standard methods.[17] We have

$$\int d^4k\frac{(1,\, k_\mu,\, k_\mu k_\nu)}{(k^2-2n_1\cdot k+\Delta_1)(k^2-2n_2\cdot k+\Delta_1)(k^2+\Delta_2)}$$
$$=\int_0^1 dx\int_0^1 dy\int d^4k\frac{2x[1,\,(xn_y)_\mu,\,k^2\delta_{\mu\nu}/4+(xn_y)_\mu(xn_y)_\nu]}{(k^2+\Delta)^3}$$
$$=\frac{\pi^2 i}{2}[A(\Delta),\,2B(\Delta)n_\mu,\,C(\Delta)\delta_{\mu\nu} +D(\Delta)(n_{1\mu}n_{2\nu}+n_{1\nu}n_{2\mu})+E(\Delta)(n_{1\mu}n_{1\nu}+n_{2\nu}n_{2\nu})], \tag{13}$$

where

$$\Delta=x\Delta_1+(1-x)\Delta_2+M^2x^2+q^2x^2y(1-y), \tag{14}$$

$$n_y=n_1y+n_2(1-y), \tag{15}$$

and

$$\left.\begin{aligned}
&A(\Delta)=2\int_0^1 dx\int_0^1 dy\frac{x}{\Delta}, \quad B(\Delta)=2\int_0^1 dx\int_0^1 dy\frac{x^2y}{\Delta},\\
&C(\Delta)=\frac{1}{\pi^2 i}\int_0^1 dx\int_0^1 dy\int d^4k\frac{k^2x}{(k^2+\Delta)^3}\\
&\qquad=\frac{1}{\pi^2 i}\int_0^1 dx\int dy\int d^4k\frac{x}{(k^2+\Delta^2)}-\frac{1}{4},\\
&D(\Delta)=2\int_0^1 dx\int_0^1 dy\frac{x^3y(1-y)}{\Delta},\\
&E(\Delta)=2\int_0^1 dx\int_0^1 dy\frac{x^3y^2}{\Delta}.
\end{aligned}\right\} \tag{16}$$

All of these integrals are "convergent" in the sense explained above except for C, which is logarithmically "divergent." However, it will be seen that in the sum $M_{\rm I}+M_{\rm II}$ the difference of two such terms appears and that this is "convergent."

The result (13) may now be substituted into (7) and (8) to find $M_{\rm I}$ and $M_{\rm II}$. In the case of $M_{\rm I}$, $\Delta_1=0$, since $n_1^2=n_2^2=-M^2$, and $\Delta_2=\mu^2$, so

$$\Delta_{\rm I}=M^2X_{\rm I}+x^2y(1-y)q^2, \tag{17}$$

where

$$X_{\rm I}\equiv x^2-\eta x+\eta, \tag{18}$$

$$\eta\equiv\mu^2/M^2. \tag{19}$$

For $M_{\rm II}$, $\Delta_1=\mu^2-M^2$ and $\Delta_2=M^2$, so

$$\Delta_{\rm II}=M^2X_{\rm II}+x^2y(1-y)q^2, \tag{20}$$

$$X_{\rm II}=x^2+(\eta-2)x+1. \tag{21}$$

[17] See, for instance, the appendix of Feynman's article, reference 14.

Writing $A_{\rm I}$ for $A(\Delta_{\rm I})$, $A_{\rm II}$ for $A(\Delta_{\rm II})$, etc., Eqs. (6), (7), and (13) give

$$M_{\rm I}=-(ieg^2/16\pi^2)u_2^\dagger\{iM(D_{\rm I}+E_{\rm I})[\boldsymbol{a},\boldsymbol{q}] +D_{\rm I}q^2\boldsymbol{a}-2(M^2E_{\rm I}+M^2D_{\rm I}+C_{\rm I})\boldsymbol{a}\}u_1. \quad (22)$$

Similarly, from (6), (8), and (13) we find

$$M_{\rm II}=(eg^2/8\pi^2)u_2^\dagger\{[2iM^2(D_{\rm II}+E_{\rm II}-2B_{\rm II}) +iM^2A_{\rm II}-iC_{\rm II}]\boldsymbol{a} +\tfrac{1}{2}M(D_{\rm II}+E_{\rm II}-2B_{\rm II}+A_{\rm II}/2)[\boldsymbol{a},\boldsymbol{q}]\}u_1. \quad (23)$$

Adding the two results,

$$M_{\rm I}+M_{\rm II}=(eg^2/16\pi^2)u_2^\dagger\{[4iM^2(D_{\rm II}+E_{\rm II}-2B_{\rm II}) +2iM^2(A_{\rm II}+E_{\rm I}+D_{\rm I})+2i(C_{\rm I}-C_{\rm II})]\boldsymbol{a} -iD_{\rm I}q^2\boldsymbol{a}+M(D_{\rm II}+E_{\rm II}-2B_{\rm II}+A_{\rm II}/2 +D_{\rm I}+E_{\rm I})[\boldsymbol{a},\boldsymbol{q}]\}u_1. \quad (24)$$

Note that the two divergent integrals appear only in the combination

$$C_{\rm I}-C_{\rm II}=\frac{1}{\pi^2 i}\int_0^1 dx\int_0^1 dy\int d^4k\cdot k^2x \times\left(\frac{1}{(k^2+\Delta_{\rm I})^3}-\frac{1}{(k^2+\Delta_{\rm II})^3}\right), \quad (25)$$

which is "convergent" since

$$1/(k^2+\Delta_{\rm I})^3-1/(k^2+\Delta_{\rm II})^3 \quad (26)$$

behaves as $1/(k^2)^4$ for large values of k^2.

As remarked above, an expansion in powers of q^2 is indicated, e.g.,

$$A(q^2)=A(0)+A'(0)q^2+\cdots \quad (27)$$

so that to order q^2, (24) gives

$$M_{\rm I}+M_{\rm II}=\frac{eg^2}{16\pi^2}u_2^\dagger(P\boldsymbol{a}+Q[\boldsymbol{a},\boldsymbol{q}]+Rq^2\boldsymbol{a})u_1, \quad (28)$$

where P, Q, and R are the constants

$$\left.\begin{aligned}
P&=4iM^2[D_{\rm II}(0)+E_{\rm II}(0)-2B_{\rm II}(0)]+2iM^2A_{\rm II}(0)\\
&\quad+iM^2E_{\rm II}(0)+2i[C_{\rm I}(0)-C_{\rm II}(0)],\\
Q&=M[D_{\rm II}(0)+E_{\rm II}(0)-2B_{\rm II}(0)+[A_{\rm II}(0)/2]\\
&\quad+D_{\rm I}(0)+E_{\rm I}(0)],\\
R&=-iD_{\rm I}(0)+4iM^2[D_{\rm II}'(0)+E_{\rm II}'(0)-2B_{\rm II}'(0)]\\
&\quad+2iM^2A_{\rm II}'(0)+2iM^2[E_{\rm I}'(0)+D_{\rm I}'(0)]\\
&\quad+2i[C_{\rm I}'(0)-C_{\rm II}'(0)].
\end{aligned}\right\} \quad (29)$$

The evaluation of Q and R is quite straightforward and involves only elementary quadratures. For instance, by using (16) through (21) we have

$$Q=2M\int_0^1 dx\int_0^1 dy\left\{\frac{x^3y(1-y)+x^3y^2-2x^2y+x/2}{M^2X_{\rm II}} +\frac{x^3y(1-y)+x^3y^2}{M^2X_{\rm I}}\right\}=\frac{1}{M}\int_0^1 dx\frac{x^2}{X_{\rm I}},$$

since $X_{\rm II}\rightarrow X_{\rm I}$ if $x\rightarrow 1-x$. Thus

$$Q=\frac{1}{M}\int_0^1 dx\frac{x^2}{x^2-\eta x+\eta}=\frac{1}{M}f_0(\eta), \quad (30)$$

$$f_0(\eta)=1-\frac{\eta}{2}\log\eta+\frac{\eta^2-2\eta}{(4\eta-\eta^2)^{\frac{1}{2}}}\cos^{-1}\frac{\eta^{\frac{1}{2}}}{2}. \quad (31)$$

Similarly,

$$R=-(2i/M^2)f_1(\eta), \quad (32)$$

$$f_1(\eta)=\frac{13-4\eta}{3(4-\eta)}+\left(\frac{1}{4}-\frac{2\eta}{3}\right)\log\eta +\frac{\eta^{\frac{1}{2}}[-(4/3)\eta^2+(17/2)\eta-(35/3)]}{(4-\eta)^{\frac{3}{2}}}\cos^{-1}\frac{\eta^{\frac{1}{2}}}{2}. \quad (33)$$

However, some care must be excerised in the case of P, for it contains the difference of two divergent integrals. The C terms which appear in R, i.e., $C_{\rm I}'(0)$ and $C_{\rm II}'(0)$, are individually "convergent" since in the expansion of $C_{\rm I}$ or $C_{\rm II}$ in powers of q^2 the coefficient of q^2 is a "convergent" integral. On the other hand, the terms in these expansions which are independent of q, i.e., the $C_{\rm I}(0)$ and $C_{\rm II}(0)$ which appear in P, are each divergent and only their difference is finite. Therefore, unless the evaluation of P is done rather carefully, almost any result may be found. The procedure adopted here is simply to compute $M_{\rm I}+M_{\rm II}$ directly from (7) and (8) for the case $n_1=n_2$, $u_1=u_2$, using a Lorentz frame in which

$$\mathbf{n}_1=\mathbf{n}_2=0.$$

Then $q=0$ and we can find P from the equation

$$M_{\rm I}+M_{\rm II}=\frac{eg^2}{16\pi^4}u_1^\dagger P\boldsymbol{a}u_1=\frac{ieg^2}{16\pi^4}Pa_4,$$

where only the a_4 term remains because

$$u_1^\dagger\gamma u_1=0,\quad u_1^\dagger\gamma^4u_1=i.$$

A straightforward computation leads to the result

$$M_{\rm I}+M_{\rm II}=\int d^3kf(\mathbf{k}),$$

where $f(\mathbf{k})$ is a sum of terms which, taken separately, would lead to divergent integrals. However,

$$f(\mathbf{k})\equiv 0,$$

so we may conclude that, in this case,

$$M_{\mathrm{I}}+M_{\mathrm{II}}=0, \tag{34}$$

and consequently

$$P=0.$$

It has now been shown that (10) is correct:

$$M_{\mathrm{I}}+M_{\mathrm{II}}=u_2^\dagger(E\mathbf{a}-\tfrac{1}{2}F[\mathbf{a},\mathbf{q}]-Gq^2\mathbf{a})u_1, \tag{10}$$

with

$$E=0,\quad F=\frac{-eg^2}{8\pi^2M}f_0(\eta),\quad G=\frac{eg^2i}{8\pi^2M^2}f_1(\eta). \tag{35}$$

Remembering Eqs. (4) and (5), we have:

$$M_{\mathrm{I}}+M_{\mathrm{II}}=\int d^4x u_2^\dagger e^{-in_2\cdot x}(\tfrac{1}{2}F\sigma_{\mu\nu}\mathfrak{F}_{\mu\nu}+G\Box^2A)u_1e^{in_1\cdot x}. \tag{36}$$

IV. INTERPRETATION

We shall now discuss the interpretation of the result (36) of the field theory calculation for the one-neutron matrix elements of S. Of course, quantities such as the scattering cross section for neutrons in an external electromagnetic field can be obtained directly from (36) in the usual way, but in order to find the neutron magnetic moment and the electron-neutron interaction it is convenient to deduce from (36) an effective Hamiltonian for neutrons moving in an electromagnetic field. Indeed, if the ordinary Dirac equation for the neutron wave function ψ (nonquantized) were

$$(H^0+V)\psi=(m\beta+\boldsymbol{\alpha}\cdot\mathbf{p}+V)\psi=i\dot{\psi},\quad(\mathbf{p}=-i\nabla) \tag{37}$$

with

$$V=-\beta(\tfrac{1}{2}F\sigma_{\mu\nu}\mathfrak{F}_{\mu\nu}+G\Box^2A), \tag{38}$$

then the properties of neutrons deduced from Eq. (37) using the first Born approximation would be the same as those obtained from (36). A plane wave expansion of ψ proves this immediately.

Since the electron-neutron interaction experiments use slow neutrons, it will be in order to consider the two-components Pauli-Schrödinger equation to which (37) reduces in the nonrelativistic limit. This reduction may, for instance, be accomplished by means of a unitary transformation,

$$\psi=e^T\chi,$$

with

$$T=(\boldsymbol{\alpha}\cdot\mathbf{p}/2M)\beta=-T^*.$$

Then the Dirac equation,

$$H\psi=(H^0+V)\psi=i\dot{\psi},$$

becomes

$$e^{-T}He^T\chi=i\dot{\chi}.$$

For the transformed Hamiltonian we have

$$\begin{aligned}H'&=e^{-T}He^T\\&=(1-T+\tfrac{1}{2}T^2+\cdots)(H^0+V)(1+T+\tfrac{1}{2}T^2+\cdots)\\&=M\beta+(p^2/2M)\beta+V+[V,T]+\cdots.\end{aligned} \tag{39}$$

Thus, in the absence of the perturbation V, states χ which have only the first and second components different from zero will be steady states of positive energy, while those with vanishing 1, 2 components will be negative energy states. Treating V as a small perturbation, we may expect that for states of positive energy, χ_3, χ_4 will be negligibly small compared to χ_1, χ_2, so that in a nonrelativistic approximation, odd[18] matrices, i.e., those which connect the upper two components with the lower two, may be dropped. Since

$$V=-\beta F(\mathbf{H}\cdot\boldsymbol{\sigma}-i\mathbf{E}\cdot\boldsymbol{\alpha})+4\pi\beta GJ,$$

where

$$J_\mu=-(1/4\pi)\Box^2A_\mu,$$

and E, H are the external electromagnetic fields, we find

$$V+[V,T]+\cdots=-\beta F\mathbf{H}\cdot\boldsymbol{\sigma}+4\pi GJ_4+iF[\beta\mathbf{E}\cdot\boldsymbol{\alpha},T]+\cdots.$$

Here, odd matrices have been omitted as well as terms which are of higher order in the neutron momentum, p, or which involve second or higher derivatives of the external fields, since we assume the latter to be slowly varying. Evaluating the commutator, we have

$$\frac{iF}{2M}[\beta\mathbf{E}\cdot\boldsymbol{\alpha},\boldsymbol{\alpha}\cdot\mathbf{p}\beta]=-\frac{F}{2M}\,\mathrm{div}\mathbf{E}-\frac{F}{2M}\boldsymbol{\sigma}\cdot(\mathbf{E}\times\mathbf{p}-\mathbf{p}\times\mathbf{E}), \tag{40}$$

so

$$V+[V,T]=-\beta F\mathbf{H}\cdot\boldsymbol{\sigma}-\frac{F}{2M}\,\mathrm{div}\mathbf{E}+4\pi GJ_4-\frac{F}{2M}\boldsymbol{\sigma}\cdot(\mathbf{E}\times\mathbf{p}-\mathbf{p}\times\mathbf{E}). \tag{41}$$

If χ' denotes the two component spinor composed of the 1,2 components of χ, then in a nonrelativistic approximation, χ' satisfies the Pauli-Schrödinger equation

$$\left(M+\frac{p^2}{2M}+V'\right)\chi'=i\dot{\chi}'; \tag{42}$$

$$V'=-F\mathbf{H}\cdot\boldsymbol{\sigma}-\frac{F}{2M}\,\mathrm{div}\mathbf{E}+4\pi GJ_4-\frac{F}{2M}\cdot(\mathbf{E}\times\mathbf{p}-\mathbf{p}\times\mathbf{E}). \tag{43}$$

If $\mathbf{E}=0$, then $\mathbf{H}$ is static and we may set $J_4=0$, giving

$$V'=-F\mathbf{H}\cdot\boldsymbol{\sigma}.$$

This shows that F is the magnetic moment of the neutron. Since

$$F=-(g^2/4\pi^2)f_0(\eta)(e/2M), \tag{44}$$

the value of this moment in nuclear magnetons is seen

[18] L. L. Foldy and S. A. Wouthuysen, Phys. Rev. 78, 29 (1950).

to be

$$\mu_N = -(g^2/4\pi^2) f_0(\eta), \qquad (45)$$

in agreement with previous calculations.[8,9,11,19]

Consider next the case where $\mathbf{H}=0$ and $\mathbf{E}$ has as source a static charge distribution

$$\mathrm{div}\mathbf{E} = 4\pi\rho.$$

Then

$$J_4 = -\frac{1}{4\pi}\nabla^2 A_4 = \frac{i\,\mathrm{div}\mathbf{E}}{4\pi} = i\rho,$$

and

$$V' = 4\pi\left(iG - \frac{F}{2M}\right)\rho - \frac{F}{2M}(\mathbf{E}\times\mathbf{p} - \mathbf{p}\times\mathbf{E})$$

$$= \frac{eg^2}{2\pi M^2}\left(\frac{f_0(\eta)}{2} - f_1(\eta)\right)\rho - \frac{F}{2M}\boldsymbol{\sigma}\cdot(\mathbf{E}\times\mathbf{p} - \mathbf{p}\times\mathbf{E}). \quad (46)$$

As pointed out by Fermi,[4] the $\boldsymbol{\sigma}\cdot(\mathbf{E}\times\mathbf{p} - \mathbf{p}\times\mathbf{E})$ term has no effect on the usual experiments on the electron-neutron interaction and we shall drop it for the remainder of this discussion. (Since the experiments use unpolarized neutrons, this term, which involves spin flip of the neutron but not of the nucleus, cannot interfere coherently with the nuclear scattering, in which the neutron and nucleus both change their spin states or both remain unchanged.)

The case where the charge cloud represents an electron with fixed location has often been used for stating the results. In this case,

$$\rho = -e\rho_0, \qquad \int d^3x \rho_0 = 1;$$

so

$$V' = \frac{-e^2 g^2}{2\pi M^2}\left(\frac{f_0(\eta)}{2} - f_1(\eta)\right)\rho_0. \qquad (47)$$

The neutron sees, then, a potential given by (47). Since

$$W \equiv \int d^3x V' = \frac{-e^2 g^2}{2\pi M^2}\left(\frac{f_0(\eta)}{2} - f_1(\eta)\right), \qquad (48)$$

a square well of radius e^2/mc^2 with the same volume integral would have a depth

$$V_0 = W \Big/ \left[\frac{4\pi}{3}\left(\frac{e^2}{m}\right)^3\right]$$

$$= \frac{3}{2\pi}\frac{g^2}{4\pi}\left(\frac{m}{M}\right)^2 \frac{m}{e^4}\left[\frac{f_0(\eta)}{2} - f_1(\eta)\right]. \qquad (49)$$

It is now possible to point out the relation between these results and those found by previous authors. Foldy's phenomenological calculation, based on the anomalous magnetic moment, gives the same result as dropping the $f_1(\eta)$ term in (49). His success in thus finding correctly the part of V_0 which involves $f_0(\eta)$ stems from the fact that both it and the magnetic moment originate in the $\mathfrak{F}_{\mu\nu}\sigma_{\mu\nu}$ term of V, (38). Consequently, both are proportional to $g^2 f_0(\eta)$ and since their ratio is determined merely by relativistic invariance, not by meson theory, fixing one empirically—thus determining $g^2 f_0(\eta)$—makes it possible to find the other without the use of field theory. However, an approach of this sort could not predict the part of V_0 which involves $f_1(\eta)$, since even if a term $\Box^2 A$ were added[20] to the Dirac equation Hamiltonian along with $\mathfrak{F}_{\mu\nu}\sigma_{\mu\nu}$, phenomenological considerations would not indicate the ratio of the coefficients of the two terms.

The value of V_0 found by Case[9] and by Borowitz and Kohn[11] corresponds to dropping the $f_0(\eta)$ term in (49), retaining only the part involving $f_1(\eta)$. Since $f_1(\eta)$ has a magnitude about $\frac{1}{3}$ that of $f_0(\eta)/2$, this leads to a low value for V_0.[21]

For comparison with the result of Slotnick and Heitler and Dancoff and Drell, it is convenient to convert (45) and (49) to practical units,

$$V_0 = \frac{3}{2\pi}\frac{g^2}{4\pi\hbar c}\left(\frac{m}{M}\right)^2\left(\frac{\hbar c}{e^2}\right)^2 mc^2\left(\frac{f_0(\eta)}{2} - f_1(\eta)\right)$$

$$= 1.36\frac{g^2}{4\pi\hbar c}\left(\frac{f_0(\eta)}{2} - f_1(\eta)\right)\text{ kev}, \qquad (50)$$

$$\mu_N = \frac{-g^2}{4\pi\hbar c}\frac{f_0(\eta)}{\pi}. \qquad (51)$$

Using for η the value found by Barkas, Smith, and Gardner,[22]

$$\sqrt{\eta} = \mu/M = 0.151, \qquad (52)$$

we find from (31) and (33)

$$f_0(\eta) = 0.820, \quad f_1(\eta) = -0.131; \qquad (53)$$

so

$$V_0 = 0.736 g^2/4\pi\hbar c \text{ kev}, \qquad (54)$$

$$\mu_N = -(0.820/\pi) g^2/4\pi\hbar c. \qquad (55)$$

If (55) is used to fix g^2, with[23]

$$\mu_N = -1.913,$$

then

$$g^2/4\pi\hbar c = 7.33, \qquad (56)$$

and

$$V_0 = 9.97[\tfrac{1}{2}f_0(\eta) - f_1(\eta)]\text{ kev} = 5380\text{ ev}. \qquad (57)$$

The f_0 and f_1 parts ("magnetic" and "electrostatic"

[19] J. M. Luttinger, Helv. Phys. Acta 21, 483 (1948).

[20] L. L. Foldy, Phys. Rev. 87, 688, 693 (1952).

[21] Both of these authors have slight errors in their expressions for $f_1(\eta)$. In Case's Equation (103), the term $(13-4\eta)/3(4-\eta)$ is given as $[(13/3)-4\eta](4-\eta)^{-1}$. In Borowitz and Kohn's Eq. (6.11), the coefficient of the $\cos^{-1}$ contains $17\eta/3$ instead of $17\eta/2$. However, neither of these errors has an appreciable effect on the numerical value of $f_1(\eta)$.

[22] Barkas, Smith, and Gardner, Phys. Rev. 82, 102 (1951).

[23] Bloch, Nicodemus, and Staub, Phys. Rev. 74, 1025 (1948).

parts) are 4080 and 1300 ev, respectively. The total value agrees fairly well with Slotnick and Heitler's figure of 5000 ev and with the 5220 ev[24] obtained by Dancoff and Drell. [These values have been re-computed using the choice (56) for the coupling constant.]

[24] This figure was incorrectly stated as 6100 ev in the author's letter, Phys. Rev. 86, 434 (1952). The error was pointed out by Dr. Drell in a private communication.

The discrepancies between their results and the 5380 ev can probably be attributed to their having used, in their numerical integrations, slightly different values for the meson-nucleon mass ratio.

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Cross Section and Angular Distribution of the D(*d*, *p*)T Reaction*

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The D(*d*, *p*)T reaction cross section has been measured by two methods using D_2O ice targets. For E_d from 206 to 516 kev, a double-focusing magnetic spectrometer was used to obtain the momentum spectrum of the protons and tritons, from which the reaction cross section can be determined. For E_d from 35 to 550 kev, the proton yield from a thick target was differentiated to obtain the cross section. Both thin and thick target methods were used to measure the angular distribution over the energy range E_d from 35 to 550 kev. The angular distribution is expressed in terms of a Legendre polynomial expansion. Various sources of experimental error are considered and the probable error of the total cross section is found to be ±5 percent.

I. INTRODUCTION

Since the discovery by Lawrence, Lewis, and Livingston[1] in 1933 that two deuterons can react with the emission of long-range protons, the reaction $H^2+H^2\rightarrow H^1+H^3+4.032$ Mev has been studied extensively. Although early investigators determined that the yield was large and anisotropic, accurate measurements of the cross section have been possible only in the past few years.[2–6] In attempts to extend the measurements to very low energies, some experimental problems which are minor at higher energies become increasingly troublesome. The use of thin gas targets with differential pumping is subject to uncertainties due to beam neutralization and energy loss. The use of foils introduces straggling in the beam energy and requires an accurate knowledge of the window thickness at each energy at which the reaction is to be studied. The accuracy of the thick solid target measurements has been limited by uncertainties in the stopping cross section, which is needed to obtain the reaction cross section from the thick target yield. Recent measurements in this laboratory[7] of the stopping cross section of D_2O ice for protons of 18–550 kev enable us to measure the D(*d*, *p*)T cross section by the thick target method with higher accuracy than has been attained previously.

At higher energies, E_d from 206 to 516 kev, the cross section was obtained by another method. A double-focusing magnetic spectrometer was used to measure the yield of protons and tritons from a thick target as a function of their momenta. This spectrum of the emitted particles can be used to determine the "thin target" cross section, and the observation of both protons and tritons provides a check on the internal consistency of the experimental method.

The angular distribution of the reaction was measured by making observations at 10° intervals over the range $\theta_{\rm lab}$ from 80° to 150°. Above 200 kev a thin ice target was used, and the yield at each angle relative to a monitor counter at $\theta_{\rm lab}=70°$ gives the angular distribution. For deuteron energies below 200 kev it was necessary to measure the thick target yield at each angle. The differentiated thick target yield gives $d\sigma/d\Omega$ at each angle.

II. CROSS SECTION MEASURED WITH THE MAGNETIC SPECTROMETER

A double focusing magnetic spectrometer was used to observe particles emitted at an angle of 90.3° with respect to the incident deuteron beam. The D_2O ice target was deposited on a copper surface cooled with liquid nitrogen in a target chamber that has been described previously.[7] The aperture of the spectrometer was adjusted to subtend a small solid angle, 0.00127 steradian, at the target, in order to obtain high effective resolution. A scintillation counter was used to detect

* Assisted by the joint program of the ONR and AEC.

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