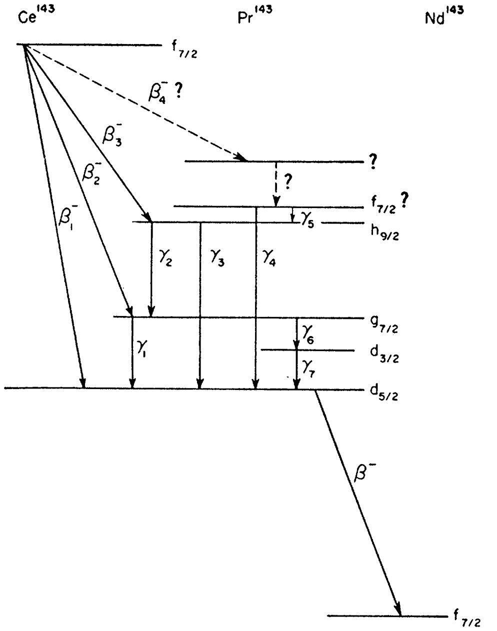

FIG. 5. Proposed decay scheme of Ce^{143}: $\beta_1^- = 1.39^5$ Mev; $\beta_2^- = 1.09^0$ Mev; $\beta_3^- = 0.71$ Mev; $\gamma_1 = 0.290$ Mev; $\gamma_2 = 0.360$ Mev; $\gamma_3 = 0.660$ Mev; $\gamma_4 = 0.720$ Mev; $\gamma_5 = 0.057$ Mev; $\gamma_6 = 0.126$ Mev; $\gamma_7 = \sim 0.160$ Mev.

assigned to $f_{7/2}$, then the beta-ray feeding that level should be an allowed transition on the basis of the assignment of $f_{7/2}$ for the Ce^{143}. Such a transition should produce a beta-group of greater intensity than any of the others. Because the 0.71-Mev group is less intense than the 1.09-Mev group, it seems likely that it is not an unresolved combination of the allowed and forbidden transitions. The assignment $f_{7/2}$ as shown in Fig. 5 is therefore questionable. It also seems unlikely that this level is $f_{7/2}$ because the K/L conversion ratio of the 0.057-Mev gamma as measured by Keller and Cork is less than unity, suggesting an $E2$ (or higher) transition.[21] If the 0.057-Mev transition were an $f_{7/2} \rightarrow h_{9/2}$ ($M1$) the K/L conversion ratio should be much greater than unity.[21] Perhaps the 0.057-Mev gamma-ray is in cascade with an undiscovered gamma which in turn is fed by the fourth lower energy beta-ray group. A more careful examination of the low energy end of the beta-spectrum of Ce^{143} is required to determine the place of a lower energy beta-group in the decay scheme and a more clear picture of the upper-lying levels of Pr^{143}. This must involve obtaining samples of Ce^{143} without serious contamination by Ce^{141}. A further possibility, of course, is that the assignment $h_{9/2}$ is also incorrect and β_3^- may be simply a first-forbidden transition.

The K/L conversion ratio of the 0.290-Mev gamma is approximately[5] 10, and the assignment $g_{7/2} \rightarrow d_{5/2}$ is consistent with this measurement. From our coincidence experiments we cannot, of course, say whether the 0.126-Mev gamma lies above or below the 0.160-Mev gamma in the decay scheme.

The assistance of L. M. Langer and M. Bunker with the β-spectrometer measurements and of J. W. Starner with the scintillation experiments is gratefully acknowledged.

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The Parallel Susceptibility of an Antiferromagnet at Low Temperatures*

JACK R. TESSMAN†
Department of Physics, University of California, Berkeley, California

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The spin wave method of Heller and Kramers is applied to a cubic antiferromagnetic crystal, oriented so that the direction of alignment of the sublattice spins is parallel to an external magnetic field. Taking into account exchange interactions and an anisotropy term, then the parallel susceptibility $\chi_{\parallel}$ is zero at $T = 0$°K, in agreement with the result of the molecular field treatment of the problem by Van Vleck. If nearest neighbor magnetic dipole interactions are included, $\chi_{\parallel}$ is nonzero but negligibly small: $\chi_{\parallel} \sim 10^{-11}$ at $T = 0$°K. For $T > 0$ but much less than the antiferromagnetic Curie temperature, if the dipole interactions are neglected and the anisotropy is small, such that $\zeta = S(24JK)^{\frac{1}{2}}/kT \ll 1$, where K is an anisotropy constant, then $\chi_{\parallel} \propto T^2$. If $\zeta \gg 1$, then $\chi_{\parallel} \propto T^{\frac{1}{2}} \exp[-S(24JK)^{\frac{1}{2}}/kT]$.

1. INTRODUCTION

AN antiferromagnetic crystal does not exhibit any net spontaneous magnetization in the absence of an external magnetic field, because the magnetizations of the ordered sublattices, below the antiferromagnetic Curie temperature, are equal and opposite. In an external field, the magnetization of a two-sublattice antiferromagnetic crystal depends upon the orientation of the external field $\mathbf{H}$ with respect to the directions of alignment $\pm\mathbf{e}$ of the sublattice spins.

Using the modification of the Bloch spin wave method due to Heller and Kramers,[1] Hulthén[2] investigated the

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* This work was assisted in part by the ONR.

† Now at Department of Physics, Pennsylvania State College, State College, Pennsylvania.

[1] G. Heller and H. A. Kramers, Proc. Roy. Acad. Sci. (Amsterdam) 37, 378 (1934).

[2] L. Hulthén, Proc. Roy. Acad. Sci. (Amsterdam) 39, 190 (1936).

susceptibility of a two-sublattice antiferromagnet at low temperatures, when $\mathbf{H}\perp\mathbf{e}$. He found, after quantization, that the perpendicular susceptibility $\chi_\perp \propto (1-bT^2)$, where b is a constant. Hulthén recognized that, for small external fields, it is energetically favorable for the sublattice spins to set themselves approximately perpendicular to the external field if only exchange interactions are considered. Hence, he did not investigate the case when $\mathbf{H}\|\mathbf{e}$. However, if some anisotropy is present and $\mathbf{H}$ is applied parallel to $\mathbf{e}$, the sublattice spins may remain parallel and antiparallel to the field rather than turn perpendicular to it.

Generalizing work by Néel[3] and by Bitter,[4] Van Vleck[5] investigated both the perpendicular and parallel susceptibility using a molecular field model. He found that $\chi_\perp$ should be constant between 0°K and the antiferromagnetic Curie temperature, and that $\chi_\parallel$ should be zero at $T=0$.

In this paper we apply the spin wave method used by Hulthén,[2] and the recent clarifications of the method,[6,7] to the case when $\mathbf{H}\|\mathbf{e}$. We investigate the ground state of the system taking into account exchange interactions, an external field, anisotropy, and nearest neighbor magnetic dipole interactions. This will yield $\chi_\parallel$ at $T=0$. Then, omitting the dipole interaction term, we shall obtain expressions for $\chi_\parallel$ at $T>0$ for the cases of small and large anisotropy.

2. $\chi_\parallel$ AT $T=0$°K

A. Neglecting Magnetic Dipole Interactions

Let the lattice structure of our crystal be simple cubic with N atoms and lattice constant a, subdivided into two face-centered cubic sublattices, $\mathbf{m}$ and $\mathbf{n}$, such that the six nearest neighbors of any one atom, of either sublattice, are all members of the other sublattice. We shall assume that, for each of the atoms, $S_{\mathbf{m}}^z \simeq S$ or $S_{\mathbf{n}}^z \simeq -S$, where S is the spin quantum number of the atom. Then, writing $\check{S}^2 = S(S+1)$, since

$$(S_{\mathbf{m}}^z)^2 = \check{S}^2 - [(S_{\mathbf{m}}^x)^2 + (S_{\mathbf{m}}^y)^2] \quad \text{and} \quad (S_{\mathbf{n}}^z)^2 = \check{S}^2 - [(S_{\mathbf{n}}^x)^2 + (S_{\mathbf{n}}^y)^2],$$

therefore

$$\begin{aligned} S_{\mathbf{m}}^z &\simeq \check{S} - [(S_{\mathbf{m}}^x)^2 + (S_{\mathbf{m}}^y)^2]/2\check{S}, \\ S_{\mathbf{n}}^z &\simeq -\check{S} + [(S_{\mathbf{n}}^x)^2 + (S_{\mathbf{n}}^y)^2]/2\check{S}. \end{aligned} \tag{1}$$

The antiferromagnetic exchange interaction energy of the crystal is

$$\mathcal{H}_1 = J \sum_{\langle \mathbf{m},\mathbf{n}\rangle} \mathbf{S}_{\mathbf{m}} \cdot \mathbf{S}_{\mathbf{n}},$$

where J is positive and the summation is over all pairs of nearest neighbors. We note that the sign convention for J is opposite to that usually employed in ferromagnetism.

In the presence of an external magnetic field $\mathbf{H}$ in the $+z$ direction, there is an additional energy,

$$\mathcal{H}_2 = -g\mu_B H(\sum_{\mathbf{m}} S_{\mathbf{m}}^z + \sum_{\mathbf{n}} S_{\mathbf{n}}^z),$$

where $g\mu_B \mathbf{S}_{\mathbf{m}}$ is the magnetic moment operator for atom $\mathbf{m}$.

We assume the presence of an anisotropy energy of uniaxial symmetry which may be approximated by

$$\mathcal{H}_3 = K\{\sum_{\mathbf{m}}[(S_{\mathbf{m}}^x)^2 + (S_{\mathbf{m}}^y)^2] + \sum_{\mathbf{n}}[(S_{\mathbf{n}}^x)^2 + (S_{\mathbf{n}}^y)^2]\}.$$

This term will maintain the $\pm z$ axes as the directions of magnetization of the sublattices. In a magnetic field $\mathbf{H}$, the directions of magnetization of the sublattices would become approximately perpendicular to the field were it not for the anisotropy energy.

Collecting terms, and applying Eqs. (1), the total Hamiltonian, $\mathcal{H} = \mathcal{H}_1 + \mathcal{H}_2 + \mathcal{H}_3$, is

$$\begin{aligned} \mathcal{H} = -3JNS(S+1) + 3J\{&\sum_{\mathbf{m}}[(S_{\mathbf{m}}^x)^2 + (S_{\mathbf{m}}^y)^2] \\ &\times(1+K'-\alpha) + \sum_{\mathbf{n}}[(S_{\mathbf{n}}^x)^2 + (S_{\mathbf{n}}^y)^2](1+K'+\alpha)\} \\ &+ J \sum_{\langle \mathbf{m},\mathbf{n}\rangle} (S_{\mathbf{m}}^x S_{\mathbf{n}}^x + S_{\mathbf{m}}^y S_{\mathbf{n}}^y). \end{aligned} \tag{2}$$

Here,

$$K' = K/3J \quad \text{and} \quad \alpha = Hg\mu_B/6J\check{S}. \tag{3}$$

Consistent with the approximation of (1), we have dropped terms of higher order than the small quantities $(S_{\mathbf{m}}^x)^2/\check{S}^2$ and $(S_{\mathbf{m}}^y)^2/\check{S}^2$.

We introduce, following Anderson,[7] two sets of spin waves through the transformation,

$$\begin{aligned} S_{\mathbf{m}}^x &= (2S/N)^{\frac{1}{2}} \sum_{\mathbf{k}} Q_{1\mathbf{k}} \exp(i\mathbf{k}\cdot\mathbf{m}), \\ S_{\mathbf{m}}^y &= (2S/N)^{\frac{1}{2}} \sum_{\mathbf{k}} P_{1\mathbf{k}} \exp(-i\mathbf{k}\cdot\mathbf{m}), \\ S_{\mathbf{n}}^x &= (2S/N)^{\frac{1}{2}} \sum_{\mathbf{k}} Q_{2\mathbf{k}} \exp(i\mathbf{k}\cdot\mathbf{n}), \\ S_{\mathbf{n}}^y &= -(2S/N)^{\frac{1}{2}} \sum_{\mathbf{k}} P_{2\mathbf{k}} \exp(-i\mathbf{k}\cdot\mathbf{n}), \end{aligned} \tag{4}$$

where $\mathbf{k}$, the spin wave vector, takes on $N/2$ discrete values with $k_{x,y,z}$ going from $-\pi/a$ to $+\pi/a$.

The new operators approximately satisfy the commutation relations

$$[Q_{1\mathbf{k}}, P_{1\mathbf{k}'}] = i\delta_{\mathbf{kk}'}, \quad [Q_{2\mathbf{k}}, P_{2\mathbf{k}}]' = i\delta_{\mathbf{kk}'}.$$

All other commutators are zero. Also,

$$\sum_{\mathbf{m}}[(S_{\mathbf{m}}^x)^2 + (S_{\mathbf{m}}^y)^2] = S\sum_{\mathbf{k}}(Q_{1\mathbf{k}}Q_{1\mathbf{k}}^* + P_{1\mathbf{k}}P_{1\mathbf{k}}^*),$$

$$\sum_{\mathbf{n}}[(S_{\mathbf{n}}^x)^2 + (S_{\mathbf{n}}^y)^2] = S\sum_{\mathbf{k}}(Q_{2\mathbf{k}}Q_{2\mathbf{k}}^* + P_{2\mathbf{k}}P_{2\mathbf{k}}^*),$$

$$\begin{aligned} \sum_{\langle \mathbf{m},\mathbf{n}\rangle} (S_{\mathbf{m}}^x S_{\mathbf{n}}^x + S_{\mathbf{m}}^y S_{\mathbf{n}}^y) = S\sum_{\mathbf{k}}(&Q_{1\mathbf{k}}Q_{2\mathbf{k}}^* + Q_{1\mathbf{k}}^*Q_{2\mathbf{k}} \\ &- P_{1\mathbf{k}}P_{2\mathbf{k}}^* - P_{1\mathbf{k}}^*P_{2\mathbf{k}})(\cos k_x a + \cos k_y a + \cos k_z a). \end{aligned}$$

$Q_{1\mathbf{k}}^*$ is the Hermitian conjugate of $Q_{1\mathbf{k}}$. Henceforth, as a convenient abbreviation, we shall drop the conjugate sign and write Q^2 for QQ^*, etc. None of the succeeding arguments is affected thereby.

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[4] F. Bitter, Phys. Rev. 54, 79 (1937).

[5] J. H. Van Vleck, J. Chem. Phys. 9, 85 (1941).

[6] M. J. Klein and R. S. Smith, Phys. Rev. 80, 1111 (1950).

[7] P. W. Anderson, Phys. Rev. 86, 694 (1952). I should like to express my gratitude to Dr. Anderson for having had the opportunity to study his paper prior to publication.

The Hamiltonian (2) now takes the form

$$\mathcal{H}=-3JNS(S+1)+3JS\sum_{\mathbf{k}}[\mathcal{A}(Q_{1\mathbf{k}}^2+P_{1\mathbf{k}}^2)+\mathcal{B}(Q_{2\mathbf{k}}^2+P_{2\mathbf{k}}^2)+2\mathcal{C}_{\mathbf{k}}(Q_{1\mathbf{k}}Q_{2\mathbf{k}}-P_{1\mathbf{k}}P_{2\mathbf{k}})],\quad(5)$$

where

$$\begin{aligned}\mathcal{A}&=1+K'-\alpha,\\ \mathcal{B}&=1+K'+\alpha,\\ \mathcal{C}_{\mathbf{k}}&=\tfrac{1}{3}(\cos k_xa+\cos k_ya+\cos k_za).\end{aligned}\quad(6)$$

The bracketed expression in (5) has the form

$$\Phi=\mathcal{A}(Q_1^2+P_1^2)+\mathcal{B}(Q_2^2+P_2^2)+2\mathcal{C}(Q_1Q_2-P_1P_2).$$

Let us transform to new variables q, p by the canonical transformation[8]

$$\begin{aligned}Q_1&=(1+w)^{-1}(wq_1+q_2),\quad P_1=(1-w)^{-1}(-wp_1+p_2),\\ Q_2&=(1+w)^{-1}(q_1+wq_2),\quad P_2=(1-w)^{-1}(p_1-wp_2).\end{aligned}$$

The mixed terms in Φ will be eliminated if w satisfies the equation

$$\mathcal{C}w^2+(\mathcal{A}+\mathcal{B})w+\mathcal{C}=0.\quad(7)$$

The resulting expression for Φ is

$$\Phi=(\mathcal{A}w^2+\mathcal{B}+2\mathcal{C}w)[(1+w)^{-2}q_1^2+(1-w)^{-2}p_1^2]+(\mathcal{A}+\mathcal{B}w^2+2\mathcal{C}w)[(1+w)^{-2}q_2^2+(1-w)^{-2}p_2^2].$$

Since $[q_1,p_1]=[q_2,p_2]=i$ and all other commutators vanish, therefore the eigenvalues of Φ are

$$\Phi=(2\mu+1)(\mathcal{A}w^2+\mathcal{B}+2\mathcal{C}w)/(1-w^2)+(2\nu+1)(\mathcal{A}+\mathcal{B}w^2+2\mathcal{C}w)/(1-w^2),\quad \mu,\nu=0,1,2,\cdots.$$

Substituting the values of w and $1/w$ from the solution of Eq. (7), we obtain

$$\Phi=(\mu+\tfrac{1}{2})\{\mathcal{B}-\mathcal{A}+[(\mathcal{A}+\mathcal{B})^2-4\mathcal{C}^2]^{\frac{1}{2}}\}+(\nu+\tfrac{1}{2})\{\mathcal{A}-\mathcal{B}+[(\mathcal{A}+\mathcal{B})^2-4\mathcal{C}^2]^{\frac{1}{2}}\},\ \mu,\nu=0,1,2,\cdots.$$

In terms of K' and α, we have

$$\Phi=(2\mu+1)\{\alpha+[(1+K')^2-\mathcal{C}^2]^{\frac{1}{2}}\}+(2\nu+1)\{-\alpha+[(1+K')^2-\mathcal{C}^2]^{\frac{1}{2}}\}.$$

Only α contains the external field H.

The eigenvalues of the Hamiltonian (5) are, therefore,

$$\begin{aligned}\mathcal{H}=&-3JNS(S+1)\\&+3JS\sum_{\mathbf{k}}(2\mu_{\mathbf{k}}+1)\{\alpha+[(1+K')^2-\mathcal{C}_{\mathbf{k}}^2]^{\frac{1}{2}}\}\\&+3JS\sum_{\mathbf{k}}(2\nu_{\mathbf{k}}+1)\{-\alpha+[(1+K')^2-\mathcal{C}_{\mathbf{k}}^2]^{\frac{1}{2}}\}.\end{aligned}\quad(8)$$

In the ground state, $\mu_{\mathbf{k}}=\nu_{\mathbf{k}}=0$, and we have

$$\mathcal{H}_0=-3JNS(S+1)+6JS\sum_{\mathbf{k}}[(1+K')^2-\mathcal{C}_{\mathbf{k}}^2]^{\frac{1}{2}}.\quad(9)$$

The ground state is independent of H. Therefore, at $T=0$, when the antiferromagnet is in its ground state, the magnetization $M_0=-(1/V)(\partial\mathcal{H}_0/\partial H)$ is zero. V is the volume of the crystal. Likewise, the parallel susceptibility $\chi_{\parallel}(T=0)=M_0/H=0$. This agrees with the result obtained by Van Vleck[5] from a Weiss molecular field approach to the problem.

The above analysis is not valid for a sufficiently strong external field, for then the antiferromagnetic direction becomes perpendicular to the applied field, despite the anisotropy term, and the nonzero perpendicular susceptibility allows the energy of the system to decrease. Mathematically, it can be seen that if $\alpha>[(1+K')^2-\mathcal{C}_{\mathbf{k}}^2]^{\frac{1}{2}}$, then, from Eq. (8), the ground state is given by $\nu_{\mathbf{k}}=\infty$. That is, spin waves of infinitely large amplitude are energetically favored. It is no longer true that $S_{\mathbf{m}}^z\simeq S$, and the above analysis is invalid.

B. Including Nearest Neighbor Magnetic Dipole Interactions

Thus far we have considered only the exchange interaction energy. We shall now investigate the effect, on $\chi_{\parallel}$ at $T=0$°K, of including the nearest neighbor magnetic dipole interaction energy. The effect of the dipole interactions on the magnetization of a ferromagnet has been previously investigated by Holstein and Primakoff.[9]

The nearest neighbor magnetic dipole interaction energy adds to the Hamiltonian a term

$$\mathcal{H}_4=G\sum_{\langle\mathbf{m},\mathbf{n}\rangle}\mathbf{S}_{\mathbf{m}}\cdot\mathbf{S}_{\mathbf{n}}-\frac{3G}{a^2}\sum_{\langle\mathbf{m},\mathbf{n}\rangle}(\mathbf{S}_{\mathbf{m}}\cdot\mathbf{r}_{\mathbf{mn}})(\mathbf{S}_{\mathbf{n}}\cdot\mathbf{r}_{\mathbf{mn}}),$$

where $G=g^2\mu_B{}^2/a^3$. The coordinates of the nearest neighbors to a lattice point at the origin are

$$\mathbf{A}=\pm(a,0,0),\quad \mathbf{B}=\pm(0,a,0),\quad \mathbf{C}=\pm(0,0,a).$$

Thus,

$$\begin{aligned}\mathcal{H}_4=&G\sum_{\langle\mathbf{A}\rangle}(-2S_{\mathbf{m}}^xS_{\mathbf{n}}^x+S_{\mathbf{m}}^yS_{\mathbf{n}}^y+S_{\mathbf{m}}^zS_{\mathbf{n}}^z)\\&+G\sum_{\langle\mathbf{B}\rangle}(S_{\mathbf{m}}^xS_{\mathbf{n}}^x-2S_{\mathbf{m}}^yS_{\mathbf{n}}^y+S_{\mathbf{m}}^zS_{\mathbf{n}}^z)\\&+G\sum_{\langle\mathbf{C}\rangle}(S_{\mathbf{m}}^xS_{\mathbf{n}}^x+S_{\mathbf{m}}^yS_{\mathbf{n}}^y-2S_{\mathbf{m}}^zS_{\mathbf{n}}^z),\end{aligned}$$

where the summation over $\langle\mathbf{A}\rangle$ means summation over all pairs of nearest neighbors, **m** and **n**, such that point **n** has the coordinates **A** relative to point **m**.

The approximation (1) reduces $\mathcal{H}_4$ to the terms

$$\begin{aligned}\mathcal{H}_4=&G\sum_{\langle\mathbf{A}\rangle}(-2S_{\mathbf{m}}^xS_{\mathbf{n}}^x+S_{\mathbf{m}}^yS_{\mathbf{n}}^y)\\&+G\sum_{\langle\mathbf{B}\rangle}(S_{\mathbf{m}}^xS_{\mathbf{n}}^x-2S_{\mathbf{m}}^yS_{\mathbf{n}}^y)\\&+G\sum_{\langle\mathbf{C}\rangle}(S_{\mathbf{m}}^xS_{\mathbf{n}}^x+S_{\mathbf{m}}^yS_{\mathbf{n}}^y).\end{aligned}$$

After the transformation (4), we have

$$\mathcal{H}_4=2GS\sum_{\mathbf{k}}[(-2\cos k_xa+\cos k_ya+\cos k_za)Q_{1\mathbf{k}}Q_{2\mathbf{k}}-(\cos k_xa-2\cos k_ya+\cos k_za)P_{1\mathbf{k}}P_{2\mathbf{k}}].$$

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[8] F. Keffer, thesis, University of California, Berkeley, 1952 (unpublished).

[9] T. Holstein and H. Primakoff, Phys. Rev. **58**, 1098 (1940).

The total Hamiltonian, $\mathcal{H}=\mathcal{H}_1+\mathcal{H}_2+\mathcal{H}_3+\mathcal{H}_4$, including exchange interaction, external field, anisotropy term, and nearest neighbor dipole interaction, is now

$$\mathcal{H}=-3JNS(S+1)+3JS\sum_{\mathbf{k}}[\mathcal{A}(Q_{1\mathbf{k}}^2+P_{1\mathbf{k}}^2)+\mathcal{B}(Q_{2\mathbf{k}}^2+P_{2\mathbf{k}}^2)+2\mathcal{D}_{\mathbf{k}}Q_{1\mathbf{k}}Q_{2\mathbf{k}}-2\mathcal{F}_{\mathbf{k}}P_{1\mathbf{k}}P_{2\mathbf{k}}], \tag{10}$$

where $\mathcal{A}$ and $\mathcal{B}$ are defined in (6) and

$$\begin{aligned}\mathcal{D}_{\mathbf{k}}&=\tfrac{1}{3}(\cos k_xa+\cos k_ya+\cos k_za)\\&\quad+(G/3J)(-2\cos k_xa+\cos k_ya+\cos k_za),\\ \mathcal{F}_{\mathbf{k}}&=\tfrac{1}{3}(\cos k_xa+\cos k_ya+\cos k_za)\\&\quad+(G/3J)(\cos k_xa-2\cos k_ya+\cos k_za).\end{aligned} \tag{11}$$

The bracketed expression in (10) has the form

$$\Phi'=\mathcal{A}(Q_1^2+P_1^2)+\mathcal{B}(Q_2^2+P_2^2)+2\mathcal{D}Q_1Q_2-2\mathcal{F}P_1P_2.$$

Transform to new variables q, p by the canonical transformation

$$\begin{aligned}Q_1&=[1+(vw)^{\frac{1}{2}}]^{-1}(vq_1+q_2),\\ P_1&=[1-(vw)^{\frac{1}{2}}]^{-1}(-wp_1+p_2),\\ Q_2&=[1+(vw)^{\frac{1}{2}}]^{-1}(q_1+wq_2),\\ P_2&=[1-(vw)^{\frac{1}{2}}]^{-1}(p_1-vp_2).\end{aligned} \tag{12}$$

If v and w satisfy the simultaneous equations

$$\begin{aligned}\mathcal{A}v+\mathcal{D}w+\mathcal{C}(1+vw)&=0,\\ \mathcal{D}v+\mathcal{A}w+\mathcal{C}(1+vw)&=0,\end{aligned} \tag{13}$$

then there will be no mixed terms in Φ'. We get

$$\Phi'=\frac{\mathcal{A}v^2+\mathcal{B}+2\mathcal{D}v}{[1+(vw)^{\frac{1}{2}}]^2}q_1^2+\frac{\mathcal{A}w^2+\mathcal{B}+2\mathcal{F}w}{[1-(vw)^{\frac{1}{2}}]^2}p_1^2+\frac{\mathcal{A}+\mathcal{B}w^2+2\mathcal{D}w}{[1+(vw)^{\frac{1}{2}}]^2}q_2^2+\frac{\mathcal{A}+\mathcal{B}v^2+2\mathcal{F}v}{[1-(vw)^{\frac{1}{2}}]^2}p_2^2.$$

The commutators $[q_1, p_1]=[q_2, p_2]=i$, and all other commutators vanish. Thus the eigenvalues of Φ' are

$$\begin{aligned}\Phi'&=(2\mu+1)(1-vw)^{-1}(\mathcal{A}v^2+\mathcal{B}+2\mathcal{D}v)^{\frac{1}{2}}(\mathcal{A}w^2+\mathcal{B}+2\mathcal{F}w)^{\frac{1}{2}}\\&\quad+(2\nu+1)(1-vw)^{-1}(\mathcal{A}+\mathcal{B}w^2+2\mathcal{D}w)^{\frac{1}{2}}(\mathcal{A}+\mathcal{B}v^2+2\mathcal{F}v)^{\frac{1}{2}},\\&\qquad\mu,\nu=0,1,2,\cdots.\end{aligned}$$

In the ground state of the Hamiltonian (10), we have $\mu=\nu=0$, yielding

$$\begin{aligned}\mathcal{H}_0&=-3JNS(S+1)+3JS\sum_{\mathbf{k}}(1-vw)^{-1}\\&\quad\times[(\mathcal{A}v^2+\mathcal{B}+2\mathcal{D}_{\mathbf{k}}v)^{\frac{1}{2}}(\mathcal{A}w^2+\mathcal{B}+2\mathcal{F}_{\mathbf{k}}w)^{\frac{1}{2}}\\&\quad+(\mathcal{A}+\mathcal{B}w^2+2\mathcal{D}_{\mathbf{k}}w)^{\frac{1}{2}}(\mathcal{A}+\mathcal{B}v^2+2\mathcal{F}_{\mathbf{k}}v)^{\frac{1}{2}}].\end{aligned} \tag{14}$$

Now, at $T=0$, the magnetization $M_0=-(1/V)(\partial\mathcal{H}_0/\partial H)=-(g\mu_B/6J\check{S}V)(\partial\mathcal{H}_0/\partial\alpha)$. Thus, M_0 can be obtained by differentiating (14) and inserting the values of v and w from (13). A persistent application of the rules and procedures of algebra and differential calculus leads to

$$\begin{aligned}M_0&=\frac{g^2\mu_B{}^2H}{6J(S+1)V}\sum_{\mathbf{k}}\frac{2\mathcal{A}_0}{(\mathcal{D}_{\mathbf{k}}-\mathcal{F}_{\mathbf{k}})^2}\\&\quad\times\left\{\left[1-\frac{\mathcal{D}_{\mathbf{k}}+\mathcal{F}_{\mathbf{k}}}{2\mathcal{A}_0}\right]^2\left[\frac{1+\mathcal{F}_{\mathbf{k}}/\mathcal{A}_0}{1-\mathcal{D}_{\mathbf{k}}/\mathcal{A}_0}\right]^{\frac{1}{2}}+\left[1+\frac{\mathcal{D}_{\mathbf{k}}+\mathcal{F}_{\mathbf{k}}}{2\mathcal{A}_0}\right]^2\left[\frac{1-\mathcal{D}_{\mathbf{k}}/\mathcal{A}_0}{1+\mathcal{F}_{\mathbf{k}}/\mathcal{A}_0}\right]^{\frac{1}{2}}\right.\\&\quad\left.-2[(1-\mathcal{D}_{\mathbf{k}}/\mathcal{A}_0)(1+\mathcal{F}_{\mathbf{k}}/\mathcal{A}_0)]^{\frac{1}{2}}\right\}.\end{aligned} \tag{15}$$

Here $\mathcal{A}_0=1+K'(=\mathcal{A}+\alpha=\mathcal{B}-\alpha)$. Second-order terms and higher in the small quantity, $\alpha=g\mu_BH/6J\check{S}$, have been dropped. In addition, it has been possible to eliminate a number of terms by noticing the symmetry relationships between $\mathcal{D}_{\mathbf{k}}$ and $\mathcal{F}_{\mathbf{k}}$ over the range of $\mathbf{k}$ values.

If we expand the expression in curly brackets, in (15), in powers of $\mathcal{D}_{\mathbf{k}}/\mathcal{A}_0$ and $\mathcal{F}_{\mathbf{k}}/\mathcal{A}_0$, the lowest nonvanishing even power term has the value

$$(1/32\mathcal{A}_0{}^6)(\mathcal{D}_{\mathbf{k}}-\mathcal{F}_{\mathbf{k}})^4(\mathcal{D}_{\mathbf{k}}+\mathcal{F}_{\mathbf{k}})^2.$$

Odd power terms need not be considered since they vanish when $\sum_{\mathbf{k}}$ is performed. ($\mathcal{D}_{\mathbf{k}}{}^{\beta}\mathcal{F}_{\mathbf{k}}{}^{\gamma}$ is an even or odd power term according as $\beta+\gamma$ is even or odd.) Thus,

$$M_0=\frac{g^2\mu_B{}^2H}{96J(S+1)\mathcal{A}_0{}^5}\frac{N}{2V}\langle(\mathcal{D}_{\mathbf{k}}-\mathcal{F}_{\mathbf{k}})^2(\mathcal{D}_{\mathbf{k}}+\mathcal{F}_{\mathbf{k}})^2\rangle_{\mathrm{Av}}.$$

Substituting the values of $\mathcal{D}_{\mathbf{k}}$ and $\mathcal{F}_{\mathbf{k}}$ from (11) and noting that

$$\begin{aligned}\langle\cos k_{x,y,z}a\rangle_{\mathrm{Av}}&=\langle\cos^3k_{x,y,z}a\rangle_{\mathrm{Av}}=0,\\ \langle\cos^2k_{x,y,z}a\rangle_{\mathrm{Av}}&=\tfrac{1}{2},\quad\langle\cos^4k_{x,y,z}a\rangle_{\mathrm{Av}}=\tfrac{3}{8},\end{aligned}$$

we get

$$\langle(\mathcal{D}_{\mathbf{k}}-\mathcal{F}_{\mathbf{k}})^2(\mathcal{D}_{\mathbf{k}}+\mathcal{F}_{\mathbf{k}})^2\rangle_{\mathrm{Av}}=G^2/3J^2+O(G^3/J^3).$$

Therefore,

$$M_0\simeq\frac{H}{576(S+1)}\frac{g^2\mu_B{}^2}{a^3}\frac{G^2}{J^3}=\frac{H}{576(S+1)}\frac{G^3}{J^3}, \tag{16}$$

and

$$\chi_{\shortparallel}(T=0)=M_0/H\sim10^{-3}G^3/J^3\sim10^{-11}. \tag{17}$$

Thus, we see that the dipolar interactions mix into the ground state a negligible amount of states having total $S^z\neq0$.

3. $\chi_{\shortparallel}$ FOR $T>0°$

Let us now determine the behavior of $\chi_{\shortparallel}$ at temperatures above $T=0$°K but at temperatures low compared to the antiferromagnetic Curie temperature, so that we may continue our assumption that $S_m{}^z\simeq S$ and $S_n{}^z\simeq-S$.

The magnetization is given by $M=-(1/V)(\partial F/\partial H)$, where $F=-kT\ln Z$ is the Helmholtz free energy function and $Z=\sum\exp(-H/\mathrm{k}T)$ is the partition function, the summation being over all eigenvalues.

We shall take (5) as the Hamiltonian, that is, we shall include exchange interactions, anisotropy, and an external field, but not dipole interactions. The eigenvalues of this Hamiltonian are given in Eq. (8), which we shall write as

$$\mathcal{H}=-3JNS(S+1)+\sum\nolimits_{\mathbf{k}}(\mu_{\mathbf{k}}+\tfrac{1}{2})\hbar\omega_{\mathbf{k}}^{+}+\sum\nolimits_{\mathbf{k}}(\nu_{\mathbf{k}}+\tfrac{1}{2})\hbar\omega_{\mathbf{k}}^{-},$$

where $\hbar\omega_{\mathbf{k}}^{\pm}=6JS\{\pm\alpha+[(1+K')^2-\mathcal{C}_{\mathbf{k}}^2]^{\frac{1}{2}}\}$.

The partition function is, therefore,

$$\begin{aligned}Z&=e^{-3JNS(S+1)/\mathrm{k}T}\sum_{\mu_{\mathbf{k}},\nu_{\mathbf{k}}=0}^{\infty}\exp\{-\sum_{\mathbf{k}}[(\mu_{\mathbf{k}}+\tfrac{1}{2})\hbar\omega^{+}{}_{\mathbf{k}}\\&\qquad+(\nu_{\mathbf{k}}+\tfrac{1}{2})\hbar\omega_{\mathbf{k}}^{-}]/\mathrm{k}T\}\\&=e^{-3JNS(S+1)/\mathrm{k}T}\prod_{\mathbf{k}}\sum_{\mu_{\mathbf{k}},\nu_{\mathbf{k}}=0}^{\infty}\exp\{-[(\mu_{\mathbf{k}}+\tfrac{1}{2})\hbar\omega_{\mathbf{k}}^{+}\\&\qquad+(\nu_{\mathbf{k}}+\tfrac{1}{2})\hbar\omega_{\mathbf{k}}^{-}]/\mathrm{k}T\}\\&=e^{-3JNS(S+1)/\mathrm{k}T}\prod\nolimits_{\mathbf{k}}\exp[-\hbar(\omega_{\mathbf{k}}^{+}+\omega_{\mathbf{k}}^{-})/2\mathrm{k}T]\\&\qquad\times[1-\exp(-\hbar\omega_{\mathbf{k}}^{+}/\mathrm{k}T)]^{-1}[1-\exp(-\hbar\omega_{\mathbf{k}}^{-}/\mathrm{k}T)]^{-1},\end{aligned}$$

where $\sum_{\mathbf{k}}$ and $\prod_{\mathbf{k}}$ are over the $N/2$ values of $\mathbf{k}$. Thus,

$$\begin{aligned}F=-\mathrm{k}T\ln Z&=3JNS(S+1)+\tfrac{1}{2}\sum\nolimits_{\mathbf{k}}(\hbar\omega_{\mathbf{k}}^{+}+\hbar\omega_{\mathbf{k}}^{-})\\&\quad+\mathrm{k}T\sum\nolimits_{\mathbf{k}}\ln[1-\exp(-\hbar\omega_{\mathbf{k}}^{+}/\mathrm{k}T)]\\&\quad+\mathrm{k}T\sum\nolimits_{\mathbf{k}}\ln[1-\exp(-\hbar\omega_{\mathbf{k}}^{-}/\mathrm{k}T)].\end{aligned}$$

Only the last two terms involve the external magnetic field, and therefore, only they need be considered in obtaining the magnetization $\mathrm{M}=-(1/V)(\partial F/\partial H)$.

Expanding the logarithm in an infinite series, we have

$$\begin{aligned}&\ln[1-\exp(-\hbar\omega_{\mathbf{k}}^{+}/\mathrm{k}T)]+\ln[1-\exp(-\hbar\omega_{\mathbf{k}}^{-}/\mathrm{k}T)]\\&\quad=-\sum_{\rho=1}^{\infty}\rho^{-1}[\exp(-\rho\hbar\omega_{\mathbf{k}}^{+}/\mathrm{k}T)+\exp(-\rho\hbar\omega_{\mathbf{k}}^{-}/\mathrm{k}T)]\\&\quad=-2\sum_{\rho=1}^{\infty}\rho^{-1}\cosh(6\rho JS\alpha/\mathrm{k}T)\\&\qquad\qquad\times\exp\{-6\rho JS[(1+K')^2-\mathcal{C}_{\mathbf{k}}^2]^{\frac{1}{2}}/\mathrm{k}T\}.\end{aligned}$$

Substituting $\alpha=Hg\mu_B/6J\check{S}$, we obtain

$$\begin{aligned}M=(2/V)(g\mu_BS/\check{S})\sum_{\mathbf{k}}\sum_{\rho=1}^{\infty}\sinh(\rho SHg\mu_B/\check{S}\mathrm{k}T)\\\times\exp\{-6\rho JS[(1+K')^2-\mathcal{C}_{\mathbf{k}}^2]^{\frac{1}{2}}/\mathrm{k}T\}.\end{aligned}$$

For large values of ρ, the exponential becomes negligibly small; thus we can write, for $Hg\mu_B\ll\mathrm{k}T$,

$$\sinh(\rho SHg\mu_B/\check{S}\mathrm{k}T)=\rho SHg\mu_B/\check{S}\mathrm{k}T.$$

Then,

$$\begin{aligned}M=\frac{2}{V}\frac{S}{S+1}\frac{Hg^2\mu_B^2}{\mathrm{k}T}\sum_{\rho=1}^{\infty}\sum_{\mathbf{k}}\rho\exp\{-6\rho JS[(1+K')^2\\-\mathcal{C}_{\mathbf{k}}^2]^{\frac{1}{2}}/\mathrm{k}T\}.\quad(18)\end{aligned}$$

The $\sum_{\mathbf{k}}$ can be evaluated as follows:

$$\begin{aligned}\sum\nolimits_{\mathbf{k}}&=(N/2)\langle\rho\exp\{\ \}\rangle_{\mathrm{Av}}\\&\simeq\frac{N}{2}\left(\frac{a}{2\pi}\right)^3\int_{k_{x,y,z}=-\pi/a}^{\pi/a}\rho\exp\{\ \}d\mathbf{k}.\qquad(19)\end{aligned}$$

For $K'\ll1$, we have $(1+K')^2\simeq1+2K'$. The principal contribution to the integral will come from values of $\mathcal{C}_{\mathbf{k}}$ in the neighborhood of 1. From (6), we see that this means $k_{x,y,z}\simeq0$. In this region, $\mathcal{C}_{\mathbf{k}}^2\simeq1-\frac{1}{3}k^2a^2$ and the integral in (19) can be written

$$\begin{aligned}I&=\int_{-\infty}^{\infty}\rho\exp\left\{-\rho\frac{6JS}{\mathrm{k}T}(2K'+\tfrac{1}{3}k^2a^2)^{\frac{1}{2}}d\mathbf{k}\right.\\&=\frac{4\pi\rho}{a^3}\left(\frac{\mathrm{k}T}{2(3)^{\frac{1}{2}}JS}\right)^3\int_0^{\infty}u^2\exp[-\rho(\zeta^2+u^2)^{\frac{1}{2}}]du,\end{aligned}$$

where $\zeta^2=(72J^2S^2/\mathrm{k}^2T^2)K'=24S^2JK/\mathrm{k}^2T^2$, and we have let $k=(\mathrm{k}T/2(3)^{\frac{1}{2}}JSa)u$. ($k$, the magnitude of the wave vector, on the left side, is not to be confused with the Boltzmann k on the right side.)

In the limit of vanishingly small anisotropy constant K, and thus ζ, the integral can be performed readily, and we get

$$I=(\pi/3(3)^{\frac{1}{2}}a^3)(\mathrm{k}T/JS)^3\rho^{-2}.\qquad(20)$$

Combining (18), (19), and (20), and noting that

$$\sum_{\rho=1}^{\infty}\rho^{-2}=\pi^2/6,$$

we obtain[10]

$$\chi_{\shortparallel}=\frac{M}{H}=\frac{1}{144(3)^{\frac{1}{2}}S^2(S+1)}\frac{g^2\mu_B^2}{a^3}\frac{(\mathrm{k}T)^2}{J^3},\qquad(21)$$

for $\zeta=S(24JK)^{\frac{1}{2}}/\mathrm{k}T\ll1$, and $T\ll T_c$.

At low temperatures and for large values of K, we have $\zeta\gg1$, and the integral I can be evaluated by the method of steepest descents. We need only consider $\rho=1$. We get

$$I=\frac{\pi^{\frac{3}{2}}}{3(3)^{\frac{1}{2}}a^3S^3}\left(\frac{\mathrm{k}T}{J}\right)^3\zeta^{\frac{3}{2}}e^{-\zeta}.\qquad(22)$$

Combining (18), (19), and (22) gives

$$\begin{aligned}\chi_{\shortparallel}=\frac{M}{H}=\frac{1}{24(3)^{\frac{1}{2}}\pi^{\frac{3}{2}}S^{\frac{1}{2}}(S+1)}\frac{g^2\mu_B^2}{a^3}\frac{(24JK)^{\frac{3}{4}}(\mathrm{k}T)^{\frac{1}{2}}}{J^3}\\\times\exp[-S(24JK)^{\frac{1}{2}}/\mathrm{k}T].\quad(23)\end{aligned}$$

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[10] After obtaining this relationship, there came to the attention of the author a recent manuscript by T. Nakamura in which this same result is obtained. Nakamura only considers the case of vanishingly small anisotropy. I should like to thank Dr. Nakamura for having had the opportunity to see his manuscript prior to publication.

Note added in proof: Two additional articles have appeared recently in which the method of spin waves is applied to antiferromagnetism: J. M. Ziman, Proc. Phys. Soc. (London) **65A**, 540 (1952) and R. Kubo, Phys. Rev. **87**, 568 (1952).

The temperature dependence is

$$\chi_{\parallel} \propto T^{\frac{1}{2}} \exp[-S(24JK)^{\frac{1}{2}}/\mathrm{k}T], \tag{24}$$

for $\zeta = S(24JK)^{\frac{1}{2}}/\mathrm{k}T \gg 1$.

4. CONCLUSION

There is, as yet, very little information concerning the separate values of $\chi_{\parallel}$ and $\chi_{\perp}$ for an antiferromagnet. Griffel and Stout[11] have obtained values of $\chi_{\parallel}-\chi_{\perp}$ for MnF_2 at various temperatures, which, together with previous work by de Haas, Schultz, and Koolhaus[12] on the powder susceptibility of MnF_2, enable them to plot the values of $\chi_{\parallel}$ and $\chi_{\perp}$. It is quite clear from their work that $\chi_{\perp}$ is not constant below the Curie point but increases as T decreases toward zero. This is the type of behavior predicted by Hulthén, and differs from the prediction of the molecular field theory. $\chi_{\parallel}$ decreases as the temperature is lowered and seems to approach zero at $T=0$. The data are not sufficient to allow the temperature dependence of either $\chi_{\parallel}$ or $\chi_{\perp}$ to be determined very much more specifically.

It is a pleasure to thank Professor C. Kittel for the suggestion of the problem and for many discussions concerning it.

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Ionization Loss and Straggling of Fast Electrons*

E. L. GOLDWASSER, F. E. MILLS,† AND A. O. HANSON
Department of Physics, University of Illinois, Urbana, Illinois

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The most probable energy loss of 9.6- and 15.7-Mev electrons in samples of about one gram per cm^2 of beryllium, polystyrene, aluminum, copper, and gold has been measured. The losses measured were of the order of one Mev, and the resolution of the apparatus made possible an accuracy of 20 kev. The observed distributions of energy losses are found to be in good agreement with the Landau straggling calculations for the light elements. For the heavier elements there is a spreading of the distribution introduced by radiation and K electron effects. Calculations made by Yang and Kennedy for gold, including these effects, check well with the experimental data.

Applying Fermi's correction for the polarization effect at extreme relativistic velocities to Landau's result for the most probable energy loss, one obtains for the predicted loss in Mev $\Delta_{pc} = 0.1537D(\Sigma Z/\Sigma A) \times [19.43 + \ln(D/\rho)]$, where D is the absorber thickness in g/cm^2 and ρ is the absorber density in g/cm^3. Experimental results for the light elements are in excellent agreement with this theory. The heavier elements show losses somewhat smaller than those calculated.

INTRODUCTION

THERE have been a number of measurements of the ionization loss and energy straggling suffered by electrons in passing through matter, and the general processes appear to be well understood. A recent study of the energy loss distributions for electrons having initial energies up to 1 Mev has been reported by Chen and Warshaw.[1] They find that the observed most probable energy loss as well as the energy loss distributions are in good agreement with the calculation of Landau.[2] Further work with electrons having energies up to 5 Mev has been carried out by Paul and Reich[3] and is discussed by Schultz.[4] They find that the energy loss at these energies is less than that given by the theory and attribute the decrease to the effect of polarization in the material.

The present work represents an attempt to measure the energy losses and energy distributions at higher energies using highly monoenergetic electrons from the 22-Mev betatron.

EXPERIMENTAL ARRANGEMENT AND PROCEDURE

The experimental arrangement for the removal of the electrons from the betatron and the focusing of the beam into the chamber has been described in previous work.[5,6] The modifications of the scattering chamber and the detector arrangements for the present experiments are shown in Fig. 1. The electron beam is brought to a focus upon the absorber samples, which were mounted on the remotely controlled sample holder at the center of the chamber. A horizontal slit one inch long and 0.020 inch high was mounted in the center of the chamber directly behind the absorbers. This slit was formed by two pieces of gold $\frac{1}{16}$ inch thick. The adjustable aperture and adjustable collimating slit in front of the ionization chamber were edged with pieces of gold $\frac{1}{8}$ inch thick, as described in the previous work.[6] Gold was chosen as the slit edging with the idea that electrons impinging on the slit edge would be scattered completely out of the beam or be so degraded in energy

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† AEC Predoctoral Fellow.

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